σ-CONTINUITY AND RELATED FORCING NOTIONS

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Abstract. The Steprāns forcing notion arises as quotient of the algebra of Borel sets modulo the ideal of σ-continuity of a certain Borel not σ-continuous function. We give a characterization of this forcing in the language of trees and use this characterization to establish such properties of the forcing as fusion and continuous reading of names. We show that the associated σ-ideal is not generated by closed sets. This has been conjectured by Steprāns in [5]. Eventually, we exhibit a variety of forcing notions which do not have continuous reading of names in any presentation. This gives an answer to a question of Hrušák and Zapletal from [2].

1. Introduction

Many classical forcing notions arise as quotient Boolean algebras of $\mathrm{Bor}(X)$ modulo an ideal I in a Polish space X. Forcings of this form are called idealized forcings (cf. [7]) and are usually denoted by $\mathbb{P}_I$ to indicate the ideal they arise from. In this way Cohen forcing is associated with the ideal of meager sets, Sacks forcing with countable sets and Miller forcing with K_σ sets in the Baire space, to recall just a few examples. The generic extensions given by these forcings are always extensions by a single real, which is called the generic real.

Idealized forcings $\mathbb{P}_I$ are often equivalent to forcings with certain families of trees ordered by inclusion. For instance, Sacks forcing is the forcing with perfect trees and Miller forcing with superperfect trees in $\omega^{<\omega}$.

In examining forcing effects on the real line it is often convenient to have a nice representation of names for reals in the extension. In many examples such a representation is given by functions from the ground model. Namely each real in the extension is the value of a certain function from the ground model at the generic real. Not always, however, can the function be defined globally. If we assume properness of the forcing $\mathbb{P}_I$ then we are provided by a representation in terms of Borel functions:

Theorem 1 (Zapletal, [7]). *If the forcing notion $\mathbb{P}_I$ is proper and $\dot{x}$ is a name for a real then for each $B \in \mathbb{P}_I$ there is a condition $C \leq B$ and* *a Borel function* $f: C \to \mathbb{R}$ *such that*

$$C \Vdash \dot{x} = f(\dot{g})$$

where $\dot{g}$ *is the name for generic real.*

The most desirable situation is when the function can be chosen to be continuous and in many cases it actually happens. The property is called continuous reading of names. One should be aware, however, that it depends (at least formally) on the topology of the space X. How common the continuous reading of names is among idealized forcings, can be partially accounted for by the following theorem.

Theorem 2 (Zapletal, [7]). *If the ideal* I *is generated by closed sets then the associated forcing* $\mathbb{P}_I$ *is proper and has continuous reading of names.*

There is one important example of a forcing notion $\mathbb{P}_I$ which is proper but fails to have continuous reading of names in the natural topology of the space. Let us recall the old problem of Lusin whether there is a Borel function which is not σ-continuous. In [1] a particularly simple example of such a function was given, namely the Pawlikowski function P. The ideal I_P of sets on which P is σ-continuous gives rise to the forcing notion $\mathbb{P}_{I_P}$, usually called (cf. [7]) the Steprāns forcing. In [5] Steprāns introduced this forcing notion and used it to increase the cardinal characteristic $\mathrm{cov}(I_P)$ in a generic extension. The key feature of the forcing $\mathbb{P}_{I_P}$ is that it adds a real which is not contained in any ground model set from I_P.

Since the ideal I_P can be seen as a porosity ideal (cf. [7]), properness of the forcing follows from another general result.

Theorem 3 (Zapletal, [7]). *If* I *is a porosity ideal then the forcing* $\mathbb{P}_I$ *is proper.*

Steprāns forcing has many nice properties, one of them is the fact that compact sets are dense in it. This follows from the following theorem.

Theorem 4 (Zapletal, [6]). *For any Borel not* σ*-continuous function* $f: \omega^\omega \to \omega^\omega$ *and for any Borel set* $B \notin I_f$ *there exists a compact set* $C \subseteq B$ *such that* $C \notin I_f$.

The proof of this theorem introduces a certain Borel game which detects σ-continuity of a given Borel function. The result follows then from determinacy of this game.

Steprāns forcing, however, does not have continuous reading of names because the function P, treated as a name for a real, is itself a counterexample. This single obstacle may be handled by extending the topology to one which has the same Borel sets and makes P continuous. The question is if this would bring about continuous reading of names in the Steprāns forcing.

This has been investigated by the authors of [2] who argued that the ideal associated with Steprāns forcing is generated by closed sets in the extended topology. It should result in continuous reading of names but the argument from [2] is incorrect. The problem whether the ideal is generated by closed sets in this extended topology was also raised in [5].

We will show that the ideal I_P is not generated by closed sets in the extended topology. Nevertheless, we shall prove that the Steprāns forcing has continuous reading of names in this topology. To this end we will establish a description of the forcing in terms of trees and deduce continuous reading of names from properties of these trees. This will also enable us to define fusion in Steprāns forcing.

In light of the above another question arises. Are there any forcing notions of the form $\mathbb{P}_I$ which do not have continuous reading of names in any presentation (i.e. in any Polish topology which gives the same Borel structure)? This question was already posed in [2]. Recently an example has been given by Zapletal in [7], namely he proved the eventually different real forcing has this property. We will present a different example.

In fact, we will show that such forcings are quite common among the idealized forcings. Namely, there is a method of constructing them out of forcings which, as the Steprāns forcing, do not have continuous reading of names in one topology.

2. Definitions and notation

Throughout this paper an ideal will always mean a σ-ideal of subsets of a Polish space.

In a space X a system of sets indexed by a tree $T \subseteq Y^{<\omega}$ (Y is an arbitrary set) is to be understood as a map $T \ni \tau \mapsto D_\tau \subseteq X$ such that if $\tau \subseteq \tau' \in T$ then $D_{\tau'} \subseteq D_\tau$. The system is disjoint if $D_\tau \cap D_{\tau'} = \emptyset$ for $\tau \neq \tau', |\tau| = |\tau'|$.

In a space X^ω, whatever be its topology, for a finite partial function $\tau : \omega \to \mathcal{P}(X)$ we will denote by $[\tau]$ the set $\{t \in X^\omega : \forall n \in \mathrm{dom}(\tau)\ t(n) \in \tau(n)\}$. For a tree $T \subseteq X^{<\omega}$ let its limit, denoted $\lim T$, be the set $\{x \in X^\omega : \forall n \in \omega \quad x \restriction n \in T\}$. For a node $\tau \in T$ the end-extension of T above τ will stand for the subtree $\{\sigma \in T : \tau \subseteq \sigma \vee \sigma \subseteq \tau\}$. For a set $T_0 \subseteq T$ the end-extension of

T above T_0 will be the union of end-extensions of T above elements of T_0.

We will say that a Borel function $f : X \to Y$, where X, Y are Polish spaces, is σ-continuous if there exist a countable cover of the space $X = \bigcup_n X_n$ (with arbitrary sets X_n) such that $f \restriction X_n$ is continuous for each n. It follows from the Kuratowski extension theorem that we may require that the sets X_n be Borel. If they can be chosen closed in X then we shall say that f is closed-σ-continuous.

If $f : X \to Y$ is not σ-continuous then I_f denotes the ideal of sets on which f is σ-continuous.

In a metric space (X, d) for $A, B \subseteq X$ we will denote by $\mathrm{dist}(A, B)$ the infimum of $d(a, b)$ for $a \in A$ and $b \in B$. The Hausdorff distance between A and B will be denoted by $h(A, B)$.

The space $(\omega + 1)^\omega$ is endowed with the product topology of order topologies on $\omega + 1$. It is of course homeomorphic to the Cantor space. We also fix a metric ρ on $(\omega + 1)^\omega$ which gives the above topology. For $x, y \in (\omega+1)^\omega$ let $\rho(x, y) = \sum_n \frac{1}{2^n} \rho'(x(n), y(n))$ where ρ' metrizes $\omega+1$ with its order-topology, i.e. $\rho'(n, \omega) = \frac{1}{2^n}$ and $\rho'(n, m) = |\frac{1}{2^n} - \frac{1}{2^m}|$ for $n, m < \omega$. All metric notions on $(\omega + 1)^\omega$, like diameter, distance, etc., will be relative to the metric ρ.

The Pawlikowski function $P : (\omega + 1)^\omega \to \omega^\omega$ is defined as follows:

$$P(x)(n) = \begin{cases} x(n) + 1 & \text{if} \quad x(n) < \omega, \\ 0 & \text{if} \quad x(n) = \omega. \end{cases}$$

It has been shown in [1] that P is not σ-continuous. Hence $I_P = \{A \subseteq (\omega + 1)^\omega : P \restriction A \text{ is } \sigma\text{-continuous}\}$ is a proper ideal. It's subideal I_P^c is defined analogously for closed-σ-continuity.

Note that the smallest topology on $(\omega+1)^\omega$ in which P is continuous is the one with basic clopens of the form $[\sigma]$ for $\sigma \in (\omega + 1)^{<\omega}$. With this topology $(\omega+1)^\omega$ is homeomorphic to the Baire space ω^ω. We will thus refer to the two topologies: the original and the extended one as Cantor and Baire topology, respectively.

Once we have an ideal in $\mathrm{Bor}(X)$ (the family of all Borel sets in X), we consider the associated forcing notion $\mathbb{P}_I$ which is the poset $(\mathrm{Bor}(X) \setminus I, \subseteq)$. Of course, it is equivalent to the Boolean algebra $\mathrm{Bor}(X)/I$. The Steprāns forcing, associated with I_P will be denoted by $\mathbb{S}$ and the forcing associated with I_P^c will be denoted by $\mathbb{S}_c$.

We will say that a forcing $\mathbb{P}_I$ has continuous reading of names in a topology $\mathcal{T}$ of the space X if for any $B \in \mathbb{P}_I$ and any Borel function $f : B \to \omega^\omega$ there exists $\mathbb{P}_I \ni C \subseteq B$ such that $f \restriction C$ is continuous in $\mathcal{T}$.

The general definition of properness of a forcing notion can be found for instance in [3]. Let us recall, however, a characterization formulated by Zapletal in [7]: forcing of the form $\mathbb{P}_I$ is proper if and only if for every countable elementary substructure M of a large enough H_κ and every condition $B \in M \cap \mathbb{P}_I$ the set $\{x \in B : x \text{ is } \mathbb{P}_I\text{-generic over } M\}$ is not in I (this set turns out to be always Borel).

Recall also that a forcing notion $\mathbb{P}$ satisfies Baumgartner's Axiom A if there is a sequence $\leq_n, n < \omega$ of partial orders on $\mathbb{P}$ such that $\leq_0 = \leq$, $\leq_{n+1} \subseteq \leq_n$ and

- if $\mathbb{P} \ni p_n, n < \omega$ are such that $p_{n+1} \leq_n p_n$ there is a $q \in \mathbb{P}$ such that $q \leq_n p_n$ for all n,
- for every $p \in \mathbb{P}$, for every n and for every ordinal name $\dot{x}$ there exist $\mathbb{P} \ni q \leq_n p$ and a countable set B such that $q \Vdash \dot{x} \in B$.

Of course, forcings satisfying Axiom A are proper.

An ideal I in a Polish space X is said to be generated by closed sets if any Borel $B \in I$ has a F_σ superset $C \in I$.

If $(P, \leq)$ is any poset and $p \in P$ then we denote by $P \restriction p$ the poset $\{q \in P : q \leq p\}$ with the ordering $\leq$.

3. Characterization of the Steprāns forcing

Note that for every closed set C in the Cantor topology of $(\omega+1)^\omega$ there exists a tree $T \subseteq (\omega+1)^{<\omega}$ such that $C = \lim T$. Not for every tree, however, its limit is a closed set in the Cantor topology. In general, if $T \subseteq (\omega+1)^{<\omega}$ is a tree then $\lim T$ is a G_δ set in the Cantor topology (since its complement is the union of sets $[\tau]$ for $\tau \notin T$).

As we have already mentioned, many forcings of the form $\mathbb{P}_I$ can be equivalently described as tree forcings. It turns out that Steprāns forcing has similar description in terms of subtrees of $(\omega+1)^{<\omega}$.

In fact, the forcing $\mathbb{SP}$ considered by Steprāns in [5] is actually a forcing with trees. It adds an $\mathbb{S}$-generic but it is not clear that $\mathbb{SP}$ is equivalent to $\mathbb{S}$. But since $\mathbb{SP}$ can be easily seen as a dense subset of $\mathbb{Q}$ (see below), the equivalence follows from Theorem 5.

Before we give a characterization of the Steprāns forcing in terms of trees let us remark that it is not as simple as it may seem to be. It is very tempting to think that a Borel set A is in $\mathbb{P}_I$ iff it contains limit of a tree $T \subseteq (\omega+1)^{<\omega}$ such that every $\tau \in S$ has an extension $\tau' \in S$ which splits into infinitely many immediate successors including $\tau'^\frown\omega$. This is, however, not true, as the following example shows.

Example. We will now construct a tree S with the property that every $\tau \in S$ has an extension $\tau' \in S$ which splits into infinitely many immediate successors including $\tau'^\frown\omega$ but P is σ-continuous on $\lim S$. We will build the tree inductively on its levels. For any node $\tau \in S$ we will also define a set $A_\tau \subseteq \omega$. We begin with $\emptyset$ and put $A_\emptyset = \omega$.

Suppose that we have the tree S built up to level k. Now let each node τ split into $\tau^\frown\omega$ as well as $\tau^\frown n$ for $n \in A_\tau$. Define sets $A_{\tau^\frown i}$ for $i \in A_\tau \cup \{\omega\}$ so that they form a partition of A_τ into infinitely many infinite subsets. Note at this point that if $s \in \lim S$ and $s(n) < \omega$ then $s{\restriction}n$ is uniquely determined.

Claim. *The function P is σ-continuous on $\lim S$.*

Proof. Define $X_n = \{s \in \lim S : \forall m \geq n \quad s(m) = \omega\}$ and $X_\infty = \lim S \setminus \bigcup_n X_n$. Note that P is continuous on X_∞. Indeed, take any convergent sequence $s_n \to s$ such that $s_n, s \in X_\infty$ and notice that if $s(m) < \omega$ then $s_n(m) = s(m)$ implies also $s_n{\restriction}m = s{\restriction}m$. Hence, since $[(m, s(m))]$ is a neighborhood of s, there exists $m' < \omega$ such that $s_n{\restriction}m = s{\restriction}m$ for $n > m'$. Thus, if s has infinitely many values $< \omega$ then the sequence s_n eventually stabilizes on each coordinate. This shows that also $P(s_n) \to P(s)$. Since all sets X_n are countable, P is σ-continuous on $\lim S$. □

Throughout the rest of this section all topological notions concerning the space $(\omega+1)^\omega$ will be relative to the Cantor topology.

Definition 1. A tree $T \subseteq (\omega+1)^{<\omega}$ will be called wide if every node $\tau \in T$ has an extension $\tau' \in T$ such that the set $\lim T \cap [\tau']$ is nowhere dense in $\lim T \cap [\tau]$. A subset of $(\omega+1)^\omega$ will be called wide if it is the limit of a wide tree.

Obviously, the node τ' above must be of the form $\tau''^\frown\omega$ for some $\tau'' \supseteq \tau$ which splits in T into infinitely many immediate successors.

Let us denote by $\mathbb{Q}$ the poset of wide trees ordered by inclusion.

Theorem 5. *The Steprāns forcing is equivalent to the forcing $\mathbb{Q}$.*

The idea to consider wide sets comes from the proof of the famous theorem of Solecki.

Theorem 6 (Solecki, [4])**.** *For any Baire class 1 function $f : X \to Y$, where X, Y are Polish spaces, either f is σ-continuous or else there exist topological embeddings φ and ψ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\psi} & Y \\ \uparrow P & & \uparrow f \\ (\omega+1)^\omega & \xrightarrow{\varphi} & X \end{array}$$

Proof of theorem 5. The assertion follows from Proposition 1 and Proposition 2 given below. □

Proposition 1. *Assume $B \subseteq (\omega+1)^\omega$ is Borel such that $P\restriction B$ is not σ-continuous. Then there exists a wide tree T such that $\lim T = D \subseteq B$.*

Proof. Let us begin with a claim and a definition.

Claim. *Let $E \subseteq (\omega+1)^\omega$ be closed such that $P\restriction E$ is not continuous. There exists a sequence of disjoint relative clopens $C_n, n < \omega$ (each of the form $[\tau_n] \cap E$) such that $\bigcup_n C_n$ is dense in E.*

Proof. Note that any relative open set in E contains a relative clopen set of the form $[\sigma] \cap E$, where $\sigma \in (\omega+1)^{<\omega}$. This follows from the fact that any basic open set (in E) of the form $[\tau_1^\frown[n,\omega]^\frown\tau_2] \cap E$ either has a nonempty (in E) clopen subset of the form $[\tau_1^\frown m^\frown\tau_2] \cap E$ for some $m \geq n$ or is equal to the relative clopen $[\tau_1^\frown\omega^\frown\tau_2] \cap E$.

Thus, if we take a maximal antichain of sets of the form $[\tau] \cap E$ then its union will be dense. We will be done if we take an infinite maximal antichain.

Note that E is the limit of a tree which is not finitely-branching (otherwise P would be continuous on E). So let us first pick an infinite antichain given by infinitely many immediate successors (by numbers less than ω) of a node in the tree. And then let us take any extension of this antichain to a maximal antichain. □

Definition 2. A fusion system in $(\omega+1)^\omega$ is a wide tree $T \subseteq (\omega+1)^{<\omega}$ together with a family of trees $T_\tau, \tau \in T$ such that

- each $\lim T_\tau$ is closed,
- T_τ has stem τ,
- for $\tau \subseteq \tau' \in T \;\; T_{\tau'} \subseteq T_\tau$.

By Theorem 4 we may assume that $B = \lim T_\emptyset$ is closed. We may also assume that P is not σ-continuous on any basic clopen set. We will construct a fusion system $T \subseteq (\omega+1)^{<\omega}$, $T_\tau, \tau \in T$ and denote $D_\tau = \lim T_\tau$ and $D = \lim T$.

The construction is carried out inductively (beginning with $T_\emptyset$) in such a way that having constructed τ and T_τ we find infinitely many (pairwise incomparable) extensions of τ and appropriate family of subtrees of T_τ.

Suppose we have constructed a node τ and a tree T_τ. By the above Claim we can find an antichain $\tau_n, n < \omega$ of extensions of τ such that $\{D_\tau \cap [\tau_n] : n < \omega\}$ is a maximal antichain of relative clopens in D_τ. We put T_{τ_n} to be the end-extension above τ_n in T. Let us look now at the closed set $E = D_\tau \setminus \bigcup_n D_{\tau_n}$. In case P is σ-continuous on this set we will extend τ by τ_n's only and call this extension regular. If, however, P is not σ-continuous on E then let us first shrink it to E' by cutting off all relative clopens on which P is σ-continuous. Then take any $\tau_\omega \in (\omega+1)^{<\omega}$ which gives a nonempty relative clopen in E' (of length $> |\tau|$). Now extend τ additionally by τ_ω as well as define T_{τ_ω} as the tree of $E' \cap [\tau_\omega]$. The extension of this form will be called irregular extension and we will refer to τ_ω as to the irregular node. The nodes τ_n will be called regular nodes.

Once the tree has been constructed let us note that each node τ has an irregular extension τ'. Indeed, for otherwise D_τ would be the union of the set $[\tau] \cap D$ and countably many sets on each of which P is σ-continuous. On the set $[\tau] \cap D$, however, P would be continuous, since for $\sigma \in \omega^{<\omega}$ the set $P^{-1}[[\sigma]] \cap [\tau] \cap D$ is open by the assumption that there are no irregular nodes extending τ. This would imply that P is σ-continuous on D_τ, a contradiction.

Hence what is left is to show that if τ' is an irregular extension of τ then the set $[\tau'] \cap D$ is nowhere dense in $[\tau] \cap D$. Let S be the tree formed by all σ such that

- $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$,
- for each σ', if $\tau \subseteq \sigma' \subseteq \sigma$ then σ' is a regular node.

Then $\lim S$ is an intersection of a sequence of dense open sets in $[\tau] \cap D$ (namely the unions of levels of S). By the Baire category theorem $\lim S$ is a dense G_δ set in $[\tau] \cap D$ (recall that $[\tau] \cap D$ is a G_δ set). Moreover, $\lim S$ is disjoint from $[\tau'] \cap D$ and hence the latter set has empty interior in $[\tau] \cap D$. Since $[\tau'] \cap D$ is obviously closed in $[\tau] \cap D$, we have that it is nowhere dense in $[\tau] \cap D$. □

Remark 1. The fusion method from the above proof will be further used to establish Axiom A and continuous reading of names. We would like to mention, however, that Proposition 1 can be also proved without fusion, using the method of Cantor-Bendixson analysis instead: Call a tree $T \subseteq (\omega+1)^{<\omega}$ small if for each $\tau \in T$ the set $[\tau^\frown\omega]$ is relatively open in $\lim T$. It is easy to see that if T is small then P is continuous on $\lim T$. Then use a procedure in the fashion of the Cantor-Bendixson analysis to cut off from T all nodes (and their extensions) such that the end-extension above them is small. Then the remaining tree will be obviously wide. It is maybe more clear now that the wide set which remains is not in the ideal I_P. But we do not need this because Proposition 2 says that actually any wide set is I_P-positive.

Remark 2. Yet another way of proving Proposition 1 is to apply Theorem 6 together with Theorem 4 and the fact that P is of Baire class 1. But the arguments given above are much simpler than the proof of Theorem 6. Nevertheless, one of the ideas from Solecki's proof of Theorem 6 will be used in the proof of Proposition 2.

Proposition 2. *Assume $D \subseteq (\omega+1)^\omega$ is a wide set. Then there are topological embeddings φ and ψ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\ \psi\ } & P[D] \\ \big\uparrow{\scriptstyle P} & & \big\uparrow{\scriptstyle P\restriction D} \\ (\omega+1)^\omega & \xrightarrow{\ \varphi\ } & D \end{array}$$

Proof. Let us suppose $D = \lim T$ is wide in $(\omega+1)^\omega$. We will say that a subtree $S \subseteq T$ is an end-subtree if there is a finite set of nodes of T such that S is the end-extension of this set. Let $\mathbb{C}$ be the forcing with end-subtrees of T ordered by inclusion. $\mathbb{C}$ is of course equivalent to the Cohen forcing. Let M be a countable elementary submodel of a large enough H_κ such that $P, \mathbb{C}, D \in M$. Let $\mathcal{D}_n, n < \omega$ enumerate all dense subsets of $\mathbb{C}$ in M. For a dense set $\mathcal{D} \subseteq \mathbb{C}$ let $\mathcal{D}^*$ denote the set of all finite unions of elements from $\mathcal{D}$.

We are going to construct only the embedding $\varphi : (\omega+1)^\omega \to D$ since it already determines the function ψ. To this end we will define a disjoint system of wide G_δ sets $D_\tau \subseteq D, \tau \in (\omega+1)^{<\omega}$ given as limits of trees $T_\tau \in \mathbb{C}$, such that the branches of the system (i.e. $\{T_\tau : \tau \subseteq t\}$ for $t \in (\omega+1)^\omega$) will generate $\mathbb{C}$-generic filters over M.

The trees T_τ will be constructed by induction on $|\tau|$ and will satisfy the following conditions:

(i) $T_\tau \in \mathcal{D}^*_{|\tau|}$,
(ii) $\operatorname{diam}(D_\tau) < 1/|\tau|$,
(iii) for each n the map $(\omega+1)^n \ni \tau \mapsto D_\tau$ is h-continuous.

Notice that (i) and (ii) implies that each branch generates a generic filter over M. Indeed, because by (i) any extension to an ultrafilter must be generic and by (ii) there is precisely one such extension, since it is determined by an appropriate generic real.

Note at this point that after we have the sets D_τ constructed for all $\tau \in (\omega+1)^{<\omega}$ it is easy to finish the proof. For $t \in (\omega+1)^\omega$ we define $\varphi(t)$ to be the generic real given by the generic filter along the branch t. Thanks to (iii) φ is continuous. Because of disjointness of the system, φ is injective, and hence a topological embedding. On the other hand, ψ is open because the system is disjoint and $P[D_\tau]$ is open in $P[D]$ (since T_τ is an end-subtree). To see that ψ is continuous we use genericity: a formula of the form $P(\varphi(t))(m) = n$ is absolute and if it holds for t which is generic over M then it must be forced by some condition in the generic filter.

Before we go on and construct the sets D_τ, let us introduce some notation. For a fixed n and $0 \leq k \leq n$ let S^n_k be the set of points in $(\omega+1)^n$ of Cantor-Bendixson rank $\geq n-k$. For each $n < \omega$ and $1 \leq k \leq n$ let us pick a function $\pi^n_k : S^n_k \to S^n_{k-1}$ such that

- on S^n_{k-1} π^n_k is the identity,
- if $\tau \in S^n_k \setminus S^n_{k-1}$ then we pick one $i \in n$ such that $\tau(i) < \omega$ and $\tau(i)$ is maximal such and define

$$\pi^n_k(\tau)(i) = \omega, \quad \pi^n_k(\tau)(j) = \tau(j) \quad \text{for } j \neq i.$$

This definition clearly depends on the choice of the index i above. The functions π^n_k will be called projections. The key feature of these functions is that they are continuous, no matter which values we have picked.

Lemma 1. *For each n and $1 \leq k \leq n$ the projection $\pi^n_k : S^n_k \to S^n_{k-1}$ is continuous.*

Proof. Note that any point in S^n_k except $(\omega, \ldots, \omega)$ (k times ω) has a neighborhood in which projection is unambigous and hence continuous. But it is easy to see that at the point $(\omega, \ldots, \omega)$ any projection is continuous. □

In the construction we will use the following lemma which holds in M.

Lemma 2. *Let S, S' be end-subtrees of T, $D = \lim S, D' = \lim S'$, $\delta > 0$ and $k < \omega$.*

1. *There is a sequence $S_i, i \in \omega+1$ of subtrees of S such that $S_i \in \mathbb{C}$, the sets $D_i = \lim S_i$ are disjoint, the map $\omega+1 \ni i \mapsto D_i$ is h-continuous and for each i diam$(D_i) < \delta$ and $D_i \in \mathcal{D}^*_k$.*
2. *If S_i are as above then there is a sequence $S'_i, i \in \omega+1$ of subtrees of S' such that $S'_i \in \mathbb{C}$, the sets $D'_i = \lim S'_i$ are disjoint, the map $\omega+1 \ni i \mapsto D'_i$ is h-continuous, for each i $D'_i \in \mathcal{D}^*_k$, diam$(D'_i) < 3\delta$ and $h(D_i, D'_i) \leq 3h(D, D')$.*

Proof. (1) Let us pick any node $\tau^\omega \in S$ such that $|\tau^\omega| > k$, $\lim S \cap [\tau^\omega]$ has diameter $< \delta/3$ and is nowhere dense in $\lim S$ (let $\tau^\omega = \tau^\frown\omega$). Put S_ω equal to the end-subtree of S above τ^ω. Notice that for any $\varepsilon > 0$ there exists a finite set $\tau_i, i \leq n$ of extensions of τ^ω such that $\mathrm{diam}(D_\omega \cap [\tau_i]) < \varepsilon$ for each $i \leq n$ and $h(D_\omega, \bigcup_{i \leq n} D_\omega \cap [\tau_i]) < \varepsilon$. Using the fact that $\lim S \cap [\tau^\omega]$ is nowhere dense in $\lim S$ we can find nodes τ'_i (being extensions of nodes $\tau^\frown n_i$ for some $n_i < \omega$) such that $\mathrm{diam}(D \cap [\tau'_i]) < \varepsilon$ and $\mathrm{dist}(D \cap [\tau'_i], D_\omega \cap [\tau_i]) < \varepsilon$ hold for each $i \leq n$. Now it easily follows that $h(D_\omega, \bigcup_{i \leq n} D \cap [\tau'_i]) < 3\varepsilon$ (so in particular the first set has diameter $< \delta$ if ε is small enough). We may of course shrink each $D \cap [\tau_i']$ so that it is the limit of a tree in $\mathcal{D}_k$. This ends the first part of the proof.

(2) Let $\gamma = h(D, D')$. First we claim that there is a finite set of nodes $\tau_i', i \le n$ in S' such that

- $\operatorname{diam}(D' \cap [\tau_i']) < \gamma$ for each $i \le n$,
- $h(D_\omega, \bigcup_{i \le n} D' \cap [\tau_i']) < 2\gamma$,
- $\operatorname{diam}(\bigcup_{i \le n} D' \cap [\tau_i']) < 3\delta$.

Indeed, if $\gamma < \delta$ then we may first find finitely many nodes $\tau_i, i \le n$ in S_ω and then appropriate nodes $\tau_i', i \le n$ in S' such that $h(D_\omega, \bigcup_{i \le n} D_\omega \cap [\tau_i]) < 2\gamma$, both $\operatorname{diam}(D_\omega \cap [\tau_i]), \operatorname{diam}(D' \cap [\tau_i']) < \delta$ and also $\operatorname{dist}(D_\omega \cap [\tau_i], D' \cap [\tau_i']) < \delta$ for $i \le n$. Then by the triangle inequality $\operatorname{diam}(\bigcup_{i \le n} D' \cap [\tau_i']) < 3\delta$. If $\gamma \ge \delta$ then we will do by picking in a similar manner just one node τ and τ' in S, S' respectively.

Now above each τ_i' choose a node $\tau_i^{\omega\prime}$ such that $D' \cap [\tau_i^{\omega\prime}]$ is nowhere dense in D'. Then $\bigcup_{i \le n} D' \cap [\tau_i^{\omega\prime}]$ is also nowhere dense and has diameter $< 3\delta$. We may now find conditions in $\mathcal{D}_k$ which are stronger than the end-extensions in S' above the nodes $\tau_i^{\omega\prime}$. And put S_ω' to be the union of these, $D_\omega' = \lim S_\omega'$. It is clear that $h(D_\omega, D_\omega') < 3\gamma$. Similarly as in (1) we can find a sequence of disjoint subtrees $S_n' \in \mathcal{D}_k^*$ such that if we put $D_n' = \lim S_n'$ then $h(D_n', D_\omega) < 1/n$. Now it follows from the triangle inequality that $h(D_n, D_n') < 3\gamma$ as well as $\operatorname{diam}(D_n') < 3\delta$ holds for n big enough. But we may change those finitely many S_n's (in the same way we have found S_ω', possibly shrinking the existing sets) to ensure that this holds for all n. ☐

Now we proceed as follows. Let $D_\emptyset = D$. Having defined D_τ for all $\tau \in (\omega+1)^n$ we define it for $\tau \in (\omega+1)^{n+1}$. This in turn is done by another induction on the sets $S_k^n \times (\omega+1)$ for $0 \le k \le n$. That is, we first define D_τ for $\tau \in S_0^n \times (\omega+1)$ and then show how to extend the definition from $S_k^n \times (\omega+1)$ to $S_{k+1}^n \times (\omega+1)$. During this construction we take care that for each k and $\tau \in S_k^n$

$$\operatorname{diam}(D_\tau) < 1/(3^{n-k}(n+1)) \tag{1}$$

and the map $S_k \times (\omega+1) \ni \tau \mapsto D_\tau$ is h-continuous.

To start with we use Lemma 2(1). Suppose we have D_τ defined for $\tau \in S_k \times (\omega+1)$. Let us abbreviate $\pi_k^n : S_{k+1}^n \to S_k^n$ by π for a moment. For each $\tau \in S_{k+1}^n$ we use Lemma 2(2) for D_τ and $D_{\pi(\tau)}$ to find sets $D_{\tau^\frown i}$ for $i \in \omega+1$ such that

$$h(D_{\tau^\frown i}, D_{\pi(\tau)^\frown i}) \le 3h(D_\tau, D_{\pi(\tau)}), \tag{2}$$

$\operatorname{diam}(D_{\tau^\frown i}) < 3\operatorname{diam}(D_\tau)$ and $D_{\tau^\frown i} \in \mathcal{D}^*_{|\tau|}$. Now (1) follows from the inductive assumption. To see h-continuity notice that if $(\tau_n, i_n) \to (\tau, i)$ is a convergent sequence in $S^n_{k+1} \times (\omega+1)$ then either τ_n is eventually constant or $\tau \in S^n_k$ and then $\pi(\tau_n) \to \tau$ thanks to continuity of π. But then the assertion follows from the induction assumption, (2) and the triangle inequality:

$$h(D_{\tau_n^\frown i_n}, D_{\tau^\frown i}) \le h(D_{\tau_n^\frown i_n}, D_{\pi(\tau_n)^\frown i_n}) + h(D_{\pi(\tau_n)^\frown i_n}, D_{\tau^\frown i_n}) + h(D_{\tau^\frown i_n}, D_{\tau^\frown i}).$$

In this way we have constructed the sets D_τ and finished the proof. □

Propositions 5 and 2 have the following corollaries.

Corollary 1. *If $B \subseteq (\omega+1)^\omega$ is a Borel set such that $P \restriction B$ is not σ-continuous then there exists a closed wide set $D \subseteq B$.*

Proof. This follows from Proposition 2 since the image of φ is a wide closed set. □

The second corollary is a particular case of Theorem 6 when f is the restriction of P to a I_P-large set.

Corollary 2. *If $D \subseteq (\omega+1)^\omega$ is Borel then either $P\restriction D$ is σ-continuous or else there are topological embeddings φ and ψ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\psi} & P[D] \\ \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle P\restriction D} \\ (\omega+1)^\omega & \xrightarrow{\varphi} & D \end{array}$$

4. Continuous reading of names

Let us recall now that Steprāns forcing does not have continuous reading of names in the Cantor topology of $(\omega+1)^\omega$. We will now show that it has continous reading of names in the Baire topology.

Theorem 7. *The forcing notion $\mathbb{S}$ has continuous reading of names in the Baire topology on $(\omega+1)^\omega$.*

Recall that all metric notions on $(\omega+1)^\omega$ (like diameter, distance, etc.) are relative to the metric ρ (see Section 2) on $(\omega+1)^\omega$.

Proof. Let B be any Borel I_P-positive set in $(\omega+1)^\omega$ and $\dot{x}$ be a $\mathbb{S}$-name for a real. By Proposition 1 we may assume B is a limit of a wide tree. Continuous reading of names will result from the following claim.

Claim. *Let T be a tree and $\sigma \in T$ be such that $[\sigma^\frown\omega] \cap \lim T$ is nowhere dense in $[\sigma] \cap \lim T$. Then for each $\tau \in T$ such that $\sigma^\frown\omega \subseteq \tau$, any $\varepsilon > 0$ and $n < \omega$ there is $m > n$ and $\sigma^\frown m \subseteq \tau' \in T$ such that*

- *$[\tau'] \cap \lim T$ is a relative clopen,*
- *$diam([\tau'] \cap \lim T) < \varepsilon/2$,*
- *$dist([\tau] \cap \lim T, [\tau'] \cap \lim T) < \varepsilon/2$.*

Proof. Consider the family of relative clopen sets of the form $[\tau'] \cap \lim T$ with τ' extending some $\sigma^\frown m, m > n$ and having diameters $< \varepsilon/2$. Put also $\delta = \inf_{i \leq n} \mathrm{dist}([\tau], [\sigma^\frown i])$. If the assertion of this lemma were false, then the open ball around $[\tau] \cap \lim T$ with radius $\min\{\delta, \varepsilon/2\}$ would exhibit that $\sigma^\frown\omega$ has nonempty interior. □

Now let us finish the proof of the theorem. We will say that a set of nodes of a tree S is a spanning set if S is the smallest tree containing those nodes. We will find a wide subtree $S \subseteq T$ and a spanning set $\{\tau_\sigma : \sigma \in \omega^{<\omega}\}$ (with $\sigma \mapsto \tau_\sigma$ being order isomorphism) of nodes of S together with a set of natural numbers k_τ such that for τ in the spanning set

$$(\lim S \cap [\tau]) \Vdash_{\mathbb{S}} \dot{x}(|\tau|) = k_\tau.$$

This will clearly show that on $\lim S$ the name $\dot{x}$ is read by a function continuous in the Baire topology relativized to $\lim S$.

The construction is conducted inductively on $|\tau|$ in a fusion manner, that is we define additionally subtrees T_τ with stems τ, respectively. Suppose we have found everything up to the level n. We will show how to extend a single node. First find an extension $\tau \subseteq \tau' \in T_\tau$ such that $[\tau'^\frown\omega] \cap T_\tau$ is nowhere dense in $[\tau] \cap T_\tau$. Without loss of generality let us assume that $\{n < \omega : \tau'^\frown n \in T_\tau\}$ is the whole ω. Now find a forcing extension of $[\tau'^\frown\omega] \cap T_\tau$ to a limit of a wide tree $T_{\tau^\frown 0}$ such that for some $k_{\tau^\frown 0}$

$$\lim T_{\tau^\frown 0} \Vdash_{\mathbb{S}} \dot{x}(|\tau| + 1) = k_{\tau^\frown 0}.$$

Next using the above lemma and some bookkeeping find extensions $\tau'^\frown n \subseteq \tau'_n, n \in \omega$ so that for any $\sigma \in T_{\tau^\frown 0}$ and $n < \omega$ there is $m \in \omega$ such that $\mathrm{dist}(\lim T_\tau \cap [\tau'_m], \lim T_{\tau^\frown 0} \cap [\sigma]) < 1/n$ and $\mathrm{diam}(\lim T_\tau \cap [\tau'_m]) < 1/n$. Then extend the forcing conditions $[\tau'_m] \cap \lim T_\tau$ to limits of wide trees $T_{\tau^\frown m+1}$ such that for some natural numbers $k_{\tau^\frown m+1}$

$$\lim T_{\tau^\frown m+1} \Vdash_{\mathbb{S}} \dot{x}(|\tau| + 1) = k_{\tau^\frown m+1}.$$

Notice that if

$$\mathrm{diam}(\lim T_\tau \cap [\tau'_m]),\ \mathrm{dist}(\lim T_\tau \cap [\tau'_m], \lim T_{\tau^\frown 0} \cap [\sigma]) < 1/2n$$

then

$$\mathrm{dist}(\lim T_{\tau^\frown m+1}, \lim T_{\tau^\frown 0} \cap [\sigma]) < 1/n,$$

so the interior of $T_{\tau^\frown 0}$ remains empty. Moreover, it will remain empty even when we pass to the fusion tree S. Thus after the fusion we get an I_P-positive set and numbers k_τ which define a continuous function in the Baire topology. $\square$

5. The fusion

It has been established by Zapletal both in [7] and in [6] that Steprāns forcing is proper. The fusion method used in Theorem 5 and Theorem 7 suggests, however, that Axiom A can be deduced quite easily once we have the notion of a wide tree. Axiom A was also established by Steprāns for the forcing $\mathbb{SP}$ considered in [5]. We present a different proof that seems more natural for the forcing of wide trees.

Theorem 8. *The Steprāns forcing notion satisfies Axiom A.*

Proof. Let $\mathbb{W}'$ be a forcing with trees T satisfying the following conditions:

(1) each $\tau \in T$ either has only one immediate successor or is such that $\tau^\frown\omega \in T$ and $[\tau^\frown \lim T] \cap \lim T$ is nowhere dense in $\lim T$,
(2) whenever $\tau \in T$ is such that $\tau^\frown\omega \in T$ and $[\tau^\frown \lim T] \cap \lim T$ is nowhere dense in $\lim T$, we have the following. For each $n < \omega$ such that $\tau^\frown n \in T$ denote the stem of the tree T above $\tau^\frown n$ by τ_n. For each clopen C intersecting $\lim T \cap [\tau^\frown\omega]$ and any $\varepsilon > 0$ there is $n < \omega$ such that $\mathrm{diam}[\tau_n] < \varepsilon$ and $\mathrm{dist}([\tau_n], C) < \varepsilon$.

It is easy to see that $\mathbb{W}'$ is dense in $\mathbb{W}$. So it is enough to show that $\mathbb{W}'$ satisfies Axiom A.

Let $\prec$ be a linear order on $(\omega+1)^{<\omega}$ such that each τ occurs later than its initial segments. For a tree $T \in \mathbb{W}'$ let $w(T)$ be the set of those nodes of T which have more than one immediate successor. For $\tau \in w(T)$ the set of its *immediate successors in* $w(T)$ stands for the set $\{\tau' \in w(T) : \neg\exists \tau'' \in w(T) \quad \tau \subsetneq \tau'' \subsetneq \tau'\}$. Now let us denote by $w_n(T)$ the set of n first (with respect to $\prec$) elements of $w(T)$ together with theirs immediate successors in $w(T)$.

For $T, S \in \mathbb{W}'$ let $T \leq_n S$ if $T \leq S$ and $w_n(S) \subseteq T$. It is now easy to see that with these orderings $\mathbb{W}'$ satisfies Axiom A. $\square$

6. Generating by closed sets

Both properness and continuous reading of names could be deduced more easily if only we knew that I_P were generated by closed sets in the Baire topology. A typical mistake that may lead to such a conclusion is the conviction that if $A \subseteq (\omega+1)^\omega$ is such that $P\restriction A$ is continuous in the Baire topology then so it is on the closure of A in the Baire topology. This is not true, as observed by Pawlikowski (in a private conversation).

The next proposition says that continuous reading of names can hold even when the ideal is not generated by closed sets.

Proposition 3. *The ideal I_P is not generated by closed sets in the Baire topology.*

Throughout this proof let $\overline{X}$ (for $X \subseteq (\omega+1)^\omega$) denote the closure of X in the Baire topology.

Proof. Let us first consider the following set $A = \{\alpha_n, \beta_n : n < \omega\} \subseteq (\omega+1)^\omega$, where

$$\alpha_n(0) = n, \quad \alpha_n(k) = \omega \text{ for } k > 0,$$

$$\beta_n(n) = 0, \quad \beta_n(k) = \omega \text{ for } k \neq n.$$

Note that $P \restriction A$ is continuous. On the other hand, $\alpha = (\omega, \omega, \ldots) \in \overline{A}$, because $\beta_n \to \alpha$ in the Baire topology. In the Cantor topology, however, $\alpha_n \to \alpha$, whereas $P(\alpha_n) \not\to P(\alpha)$. This implies that $P \restriction \overline{A}$ is not continuous.

Using a bijection from ω to $\omega \times \omega$ we may identify $(\omega+1)^\omega$ with $(\omega+1)^{\omega\times\omega} \simeq ((\omega+1)^\omega)^\omega$. Under this identification P becomes $\prod_{n<\omega} P$, which we will denote by P^ω. First note that P^ω is continuous on A^ω as a product of continuous functions, so $A^\omega \in I_{P^\omega}$. We will prove, however, that A^ω cannot be covered by countably many sets $F_n, n < \omega$ closed in the Baire topology, with each $F_n \in I_{P^\omega}$.

Suppose that $A^\omega \subseteq \bigcup_n F_n$ and F_n are closed in the Baire topology. As A is a discrete set in $(\omega+1)^\omega$ (in both topologies), the relative topology (with respect to any of these two) on A^ω is that of the Baire space. $F_n \cap A^\omega$ are relatively closed, so by the Baire category theorem, one of them has nonempty interior. This means that there is $n < \omega$, $k < \omega$ and $\alpha \in A^k$ such that $\alpha^\frown A^{\omega\setminus k} \subseteq F_n$. Without loss of generality $k = 0$ and $\overline{A^\omega} \subseteq F_n$. But $\overline{A^\omega} = (\overline{A})^\omega$ and $\overline{A}$ contains a convergent sequence $\alpha_n \to \alpha$ such that $P(\alpha_n) \not\to P(\alpha)$. So if $A' = \{\alpha, \alpha_n : n < \omega\}$ then $P[A']$ is a discrete set and $(A')^\omega \subseteq F_n$. Notice, however, that $P^\omega \restriction (A')^\omega = (P \restriction A')^\omega$ is a copy of the function P and it is not σ-continuous. Hence $F_n \notin I_{P^\omega}$, which ends the proof. □

7. Connections with the Miller forcing

A natural question that arises after realizing that Steprāns forcing is described in terms of wide trees is whether this forcing is equivalent to the Miller forcing. A negative answer follows for instance from Proposition 5 below. It turns out, however, that Miller forcing is very close to σ-continuity, namely it is isomorphic to the forcing associated to the ideal of closed-σ-continuity of P.

Proposition 4. *The forcing notion $\mathbb{S}_c$ is equivalent to the Miller forcing notion.*

Proof. The Miller forcing is equivalent to the forcing $\mathrm{Bor}(\omega^\omega)/K_\sigma$ and $\mathbb{S}_c$ is equivalent to $\mathrm{Bor}((\omega+1)^\omega)/I_P^c$. The isomorphism is given by the function P itself (as it gives rise to a Borel isomorphism of the spaces). The only thing to realize is that for $A \subseteq (\omega+1)^\omega$ $P[A]$ is compact if and only if A is a closed set on which P is continuous. But this is the case since a continuous image of a compact set is compact, P^{-1} is continuous and a continuous bijection defined on a compact set is a homeomorphism. □

The following definition is a natural generalization of well known notions like Cohen real, Miller real, etc.

Definition 3. Let M be a transitive countable model. We say that $s \in (\omega+1)^\omega$ is a Steprāns real over M if $s \notin A$ for any $A \subseteq (\omega+1)^\omega$ such that $A \in I_P$ and A is coded in M.

It is obvious that the generic real for Steprāns forcing is a Steprāns real over the ground model. In order to distinguish Steprāns forcing from Miller forcing let us prove that there are no Steprāns reals in extensions by a single Miller real.

Proposition 5. *Miller forcing does not add Steprāns real.*

Proof. Let us denote the Miller forcing notion by $\mathbb{M}$. Suppose, towards a contradiction, that $\dot{s}$ is a $\mathbb{M}$-name for a Steprāns real. Since $(\omega+1)^\omega \simeq 2^\omega \subseteq \omega^\omega$, $\dot{s}$ is a name for an element of ω^ω. Since Miller forcing has continuous reading of names we have a forcing condition $B \subseteq \omega^\omega$ and a continuous function $f : B \to \omega^\omega$ such that

$$D \Vdash \dot{s} = f(\dot{m}),$$

where $\dot{m}$ is the name for the $\mathbb{M}$-generic real. By another well known property of Miller forcing there exists a stronger condition $D \subseteq B$ such that either $f \restriction D$ is constant or $f \restriction D$ is a topological embedding. We can exclude the first possibility. Let us denote $E = f[D]$ and note that since $P \restriction E$ is Borel, there is a dense G_δ set $G \subseteq E$ such that $P \restriction G$ is continuous. But then $f^{-1}[G]$ is comeager in D and hence is a condition in Miller forcing. But

$$f^{-1}[G] \Vdash \dot{s} \in G$$

and $G \in I_P$ which gives a contradition. □

8. A forcing without continuous reading of names in any presentation

The Steprāns forcing notion does not have continuous reading of names in one presentation and has it in another. Let us now show how to use Steprāns forcing to produce a forcing $\mathbb{P}_I$ which is proper and does not have continuous reading of names in any presentation.

Theorem 9. *There exist an ideal $I \subseteq Bor(\omega^\omega)$ such that the forcing $\mathbb{P}_I$ is proper but it does not have continuous reading of names in any presentation.*

Proof. First notice that any presentation of a Polish space X is given by a Borel isomorphism with another Polish space Y and the latter can be assumed to be a G_δ subset of $[0,1]^\omega$. Instead of ω^ω let us consider $X = (\omega^\omega)^2$ with its product topology. Note that each G_δ set G in $[0,1]^\omega$ as well as a Borel isomorphism from G to X can be coded by a real. Let $x \in \omega^\omega$ code a pair (G_x, f_x) that defines a presentation of X as above, i.e. $f_x : G_x \to X$. For $x \in \omega^\omega$ $f_x^{-1}[X_x]$ (X_x denotes the vertical section of X at x) is an uncountable Borel set in G_x and contains a copy C_x of $(\omega+1)^\omega$. Let I_x be the transported ideal I_P from C_x to X_x. We define an ideal I on $\mathrm{Bor}(X)$ as follows:

$$I = \{A \in \mathrm{Bor}(X) : \forall x \in \omega^\omega \ A_x \in I_x\}.$$

$\mathbb{P}_I$ does not have continuous reading of names in any presentation for if (G_x, f_x) defines a presentation then $(P \circ f_x^{-1})\restriction(f_x[C_x])$ is a counterexample to continuous reading of names in its topology. Let us show that $\mathbb{P}_I$ is proper. In fact we will prove that the forcing $\mathbb{P}_I$ is equivalent to the Steprāns forcing. This implies that $\mathbb{P}_I$ is proper since the Steprāns forcing is proper. Let us notice that the family $\{X_x : x \in \omega^\omega\}$ forms a maximal antichain in $\mathbb{P}_I$. For each $x \in \omega^\omega$ the forcing $\mathbb{P}_I \restriction X_x$ is equivalent to the Steprāns forcing $\mathbb{S}$. On the other hand, the Steprāns forcing has the same property, i.e. there is a maximal antichain A of cardinality continuum such that for each $a \in A$ we have $\mathbb{S}\restriction a$ is isomorphic to $\mathbb{S}$ (if we see $(\omega+1)^\omega$ as $(\omega+1)^\omega \times (\omega+1)^\omega$ then the antichain can be taken as $\{((\omega+1)^\omega \times (\omega+1)^\omega)_x : x \in (\omega+1)^\omega\}$). This proves that $\mathbb{P}_I$ is equivalent to $\mathbb{S}$. □

Remark 3. The ideal constructed in the previous proof is not intended to be definable. With a little amount of work, however, we can make it such. Below we give a sketch of a construction.

In the previous proof we coded presentations of X by Borel isomorphisms with G_δ sets in $[0,1]^\omega$. Notice, however, that we need only to code Polish topologies which extend the original one. Any such topology on X having the same Borel sets as the original one can be coded by a sequence of Borel sets (a clopen subbase of the extended topology). And vice versa: for any sequence B_n of Borel sets in X there is a Polish topology with the same Borel structure, for which the sets B_n form a clopen subbase.

Recall the usual coding of Borel sets by a $\mathbf{\Delta}^1_1$ formula over a $\mathbf{\Pi}^1_1$ set of codes. In the same way we may choose a $\mathbf{\Pi}^1_1$ set of codes $C \subseteq \omega^\omega$ coding all sequences of Borel sets. For $z \in C$ we will refer to the Borel sets in the coded sequence as to $B_n(z)$.

Now for each $z \in P$ we would like to choose compact subset of X_z which will be a copy of the Cantor set in the topology coded by z. Moreover, we would like to do this effectively.

Let $K(\omega^\omega)$ denote the hyperspace of compact subsets of ω^ω. The elements of $K(\omega^\omega)$ can be seen as subtrees of $\omega^{<\omega}$. The space $P \subseteq K(\omega^\omega)$ of all copies of 2^ω is a G_δ set. Let us consider the set Q of all pairs $(z, K) \in \omega^\omega \times K(X)$ such that $z \in C$, $K \in P$ and $K \subseteq X_z$ is a copy of the Cantor set in the topology coded by z. Notice that Q is a $\mathbf{\Pi}^1_1$ set since the condition saying that K is a copy of the Cantor set in the topology coded by z can be written as:

$$\forall n \in \omega\ \exists U \text{ clopen set in } K\ \forall x \in \omega^\omega \quad (x \in B_n(z) \Leftrightarrow x \in U).$$

Therefore, by the Kondô theorem we may find a $\mathbf{\Pi}^1_1$ uniformization of Q. Let us denote it by k.

As in the previous proof, we define the ideal I in the following way:

$$A \in I \quad \text{iff} \quad \forall z\ (z \in C \Rightarrow A_z \cap k(z) \in I_z),$$

where I_z is the ideal I_P computed on $k(z)$. The same argument as before shows that it satisfies the conclusion of Theorem 9.

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