FORCING PROPERTIES OF IDEALS OF CLOSED SETS

MARCIN SABOK AND JINDŘICH ZAPLETAL

Abstract. With every σ-ideal I on a Polish space we associate the σ-ideal I^* generated by the closed sets in I. We study the forcing notions of Borel sets modulo the respective σ-ideals I and I^* and find connections between their forcing properties. To this end, we associate to a σ-ideal on a Polish space an ideal on a countable set and show how forcing properties of the forcing depend on combinatorial properties of the ideal.

We also study the *1-1 or constant* property of σ-ideals, saying that every Borel function defined on a Borel positive set can be restricted to a positive Borel set on which it either 1-1 or constant. We prove the following dichotomy: if I is a σ-ideal generated by closed sets, then either the forcing P_I adds a Cohen real, or else I has the 1-1 or constant property.

1. Introduction

This paper is concerned with the study of σ-ideals I on Polish spaces and associated forcing notions P_I of I-positive Borel sets, ordered by inclusion. If I is a σ-ideal on X, then by I^* we denote the σ-ideal generated by the closed subsets of X which belong to I. Clearly, $I^* \subseteq I$ and $I^* = I$ if I is generated by closed sets.

There are natural examples when the forcing P_I is well understood, whereas little is known about P_{I^*}. For instance, if I is the σ-ideal of Lebesgue null sets, then the forcing P_I is the random forcing and I^* is the σ-ideal $\mathcal{E}$. The latter has been studied by Bartoszyński and Shelah [2, 1] but from a slightly different point of view. On the other hand,

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most classical forcing notions, like Cohen, Sacks or Miller forcings fall under the category of P_I for I generated by closed sets.

Some general observations are right on the surface. By the results of [14, Section 4.1] we have that the forcing P_{I^*} is proper and *preserves Baire category* (for a definition see [14, Section 3.5]). In the case when $I \neq I^*$ on Borel sets, the forcing P_{I^*} is not ω^ω-bounding by [14, Theorem 3.3.1], since any condition $B \in P_{I^*}$ with $B \in I$ has no closed I^*-positive subset. It is worth noting here that the forcing P_{I^*} depends not only on the σ-ideal I but also on the topology of the space X.

One of the motivations behind studying the idealized forcing notions P_I is the correspodence between Borel functions and reals added in generic extensions. The well-known property of the Sacks or Miller forcing is that all reals in the extension are either ground model reals, or have the same degree as the generic real. Similar arguments also show that the generic extensions are minimal, in the sense that there are no intermediate models. On the other hand, the Cohen forcing adds continuum many degrees and the structure of the generic extension is very far from minimality. In [14, Theorem 4.1.7] the second author showed that under some large cardinal assumptions the Cohen extension is the only intermediate model which can appear in the P_I generic extension when I is universally Baire σ-ideal generated by closed sets.

Definition 1.1. We say that a σ-ideal I on a Polish space X has the *1-1 or constant* property if for every Borel I-positive set $B \subseteq X$ and every Borel function $f : B \to \omega^\omega$ there is a Borel I-positive subset $C \subseteq B$ such that $f \restriction C$ is either 1-1 or constant.

Note that if P_I is proper and I has the 1-1 or constant property, then the forcing P_I adds one real degree, i.e. for every real x in the generic extension $V[G]$ we have that either $x \in V$, or else $V[x] = V[G]$.

It is well-known that the σ-ideals of countable sets in 2^ω and σ-compact sets in ω^ω both have the 1-1 or constant property. Similarly as with continuous reading of names, the 1-1 or constant property seems to be especially interesting for σ-ideals connected with decomposing certain Borel functions into simpler ones. Suppose f is a Baire class 1 function which cannot be covered by countably many continous functions with closed domains (i.e. f is not *piecewise continuous*; for more on this notion see [4] or [12]) and let I^f be the σ-ideal generated by closed sets on which f is continuous. In this case the σ-ideal I^f has the 1-1 or constant property by [10, Corollary 8.11].

As it turns out, several properties of the σ-ideal I and the forcing P_I determine also the properties of I^* and P_{I^*}. In some cases we need to make definability assumption, namely that I is $\mathbf{\Pi}^1_1$ *on* $\mathbf{\Sigma}^1_1$. For a definition of this notion see [8, 35.9] or [14, Section 3.8]. Note that if I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, then I^* is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ too, by [8, Theorem 35.38]. The following results connect the properties of P_I and P_{I^*}.

Theorem 1.2. *If the forcing P_I is proper and ω^ω-bounding, then the σ-ideal I^* has the 1-1 or constant property.*

Theorem 1.3. *If the forcing P_I is proper and does not add Cohen reals, then the forcing P_{I^*} does not add Cohen reals.*

Theorem 1.4. *If the forcing P_I is proper and does not add independent reals, then the forcing P_{I^*} does not add independent reals.*

Theorem 1.5. *If I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and the forcing P_I is proper and preserves outer Lebesgue measure, then the forcing P_{I^*} preserves outer Lebesgue measure.*

The methods of this paper can be extended without much effort to other cases, for example to show that if P_I is proper and has the weak Laver property, then P_{I^*} inherits this property. As a consequence, by the results of [15, Theorem 1.4] it follows (under some large cardinal assumptions) that if P_I is proper and preserves P-points, then P_{I^*} preserves P-points as well.

To prove the above results we introduce a combinatorial tree forcing notion $Q(J)$ for J which is a hereditary family of subsets of ω. These are relatives of the Miller forcing. To determine forcing properties of $Q(J)$ we study the position of J in the Katětov order, a generalization of the Rudin–Keisler order on ultrafilters. Further, we show that the forcing P_I gives rise to a natural ideal J_I on a countable set and we correlate forcing properties of $Q(J_I)$ with the Katětov properies of J_I. Finally, we prove that the forcing P_{I^*} is, in the nontrivial case, equivalent to $Q(J_I)$. More precisely, we show the following.

Proposition 1.6. *Let I be a σ-ideal on a Polish space X such that the poset P_I is proper. Suppose that P_I is not equivalent to the Cohen forcing under any closed condition. For any $B \in P_{I^*}$*

- *either I^* and I contain the same Borel sets below B,*
- *or there is $C \in P_{I^*}$ below B such that below C the forcing P_{I^*} is equivalent to $Q(J_I)$.*

From Proposition 1.6 we immediately get a representation of P_{I^*} in terms of P_I, the Cohen forcing and $Q(J_I)$ (for notation cf. Section 2).

Corollary. *If I is a σ-ideal on a Polish space X such that P_I is proper, then*

$$P_{I^*} \equiv \bigoplus_{a \in A} Q_a,$$

where each Q_a is either the Cohen forcing, or P_I under some condition, or $Q(J_I)$ under some condition.

It should be noted that the assumption of Proposition 1.6, saying that P_I is not equivalent to the Cohen forcing under any closed condition, means simply that every closed I-positive set contains an I-positive closed (relative) nowhere dense subset. This is of course equivalent to the fact that P_{I^*} is not equivalent to the Cohen forcing under any closed condition, since I and I^* agree on closed sets. For more discussion on this assumption, see Remark 5.3.

It is not difficult to see that the σ-ideal of meager sets has the following maximality property: if I is such that I^* is the σ-ideal of meager sets, then $I = I^*$ on Borel sets. In fact, even if P_{I^*} is equivalent to the Cohen forcing, then $I = I^*$ on Borel sets. Indeed, if the P_{I^*} generic real is a Cohen real, then I^* contains all meager sets. If U is is the union of all basic open sets in I, then $U \in I \cap I^*$ and if F is the complement of U, then on the family of Borel subsets of F the σ-ideals I and I^* are equal to the σ-ideal of (relative) meager subsets of F.

We will show that the same holds for the σ-ideals for the Sacks and Miller forcings.

Proposition 1.7. *If I is a σ-ideal such that $I \neq I^*$ on Borel sets, then P_{I^*} is neither equivalent to the Miller nor to the Sacks forcing.*

Finally, motivated by the examples of the Sacks and Miller forcings, we will prove the following dichotomy.

Theorem 1.8. *If I is a σ-ideal generated by closed sets on a Polish space X, then*

- *either P_I adds a Cohen real,*
- *or else I has the 1-1 or constant property*

This paper is organized as follows. In Section 3 we introduce the tree forcing notions $Q(J)$ and relate their forcing properties with the Katětov properties of J. In Section 4 we show how to associate an ideal J_I to a σ-ideal I and how forcing properties of P_I determine Katětov properties of J. In Section 5 we prove Proposition 1.6. In Section 6 we prove Proposition 1.7. In Section 7 we prove Theorem 1.8.

2. Notation

The notation in this paper follows the set theoretic standard of [5]. Notation concerning idealized forcing follows [14].

For a poset P we write $\mathrm{ro}(P)$ for the Boolean algebra of regular open sets in P. For a Boolean algebra B we write $\mathrm{st}(B)$ for the Stone space of B. If λ is a cardinal, then $\mathrm{Coll}(\omega,\lambda)$ stands for the poset of finite partial functions from ω into λ, ordered by inclusion.

If A is a nonempty set and $(Q_a,\leq_a)$ is a forcing notion for each $a\in A$, then by $\bigoplus_{a\in A} Q_a$ we denote the forcing $(\bigcup_{a\in A} Q_a, \bigcup_{a\in A}\leq_a)$. This means that if 1_a is the biggest element of Q_a, then $\{1_a : a\in A\}$ forms a maximal antichain in $\bigoplus_{a\in A} Q_a$ and below each $a\in A$ this forcing is equal to Q_a. We also use the notaion $P\oplus Q$, understood analogously.

If $T\subseteq Y^{<\omega}$ is a tree and $t\in T$ is a node, then we write $T\restriction t$ for the tree $\{s\in T : s\subseteq t \ \vee\ t\subseteq s\}$. For $t\in T$ we denote by $\mathrm{succ}_T(t)$ the set $\{y\in Y : t^\frown y\in T\}$. We say that $t\in T$ is a *splitnode* if $|\mathrm{succ}_T(t)|>1$. The set of all splitnodes of T is denoted by $\mathrm{split}(T)$.

If X is a metric space, $\varepsilon>0$ and $x\in X$, then we write $B_\varepsilon(x)$ for the ball in X centered at x with radius ε.

3. Combinatorial tree forcings

In this section we assume that J is a family of subsets of a countable set $\mathrm{dom}(J)$. We assume that $\mathrm{dom}(J)\notin J$ and that J is *hereditary*, i.e. if $a\subseteq b\subseteq \mathrm{dom}(J)$ and $b\in J$, then $a\in J$. Occasionally, we will require that J is an ideal. We say that $a\subseteq\mathrm{dom}(J)$ is J*-positive* if $a\notin J$. For a J-positive set a we write $J\restriction a$ for the family of all subsets of a which belong to J.

Definition 3.1. The poset $Q(J)$ consists of those trees $T\subseteq\mathrm{dom}(J)^{<\omega}$ for which every node $t\in T$ has an extension $s\in T$ satisfying $\mathrm{succ}_T(s)\notin J$. $Q(J)$ is ordered by inclusion.

Thus the Miller forcing is just $Q(J)$ when J is the Fréchet ideal on ω. $Q(J)$ is a forcing notion adding the generic branch in $\mathrm{dom}(J)^\omega$, which also determines the generic filter. We write $\dot g$ for the canonical name for the generic branch. Basic fusion arguments literally transfer from the Miller forcing case to show that $Q(J)$ is proper and preserves the Baire category. The following proposition is folklore.

Proposition 3.2. *There is a σ-ideal I_J generated by closed sets such that the forcing $Q(J)$ is equivalent to the forcing P_I.*

Proof. To simplify notation assume $\mathrm{dom}(J) = \omega$. Whenever $f : \omega^{<\omega} \to J$ is a function, let $A_f = \{x \in \omega^\omega : \forall n < \omega \ \ x(n) \in f(x \restriction n)\}$. Note that the sets A_f are closed. Let I_J be the σ-ideal generated by all sets of this form.

Lemma 3.3. *An analytic set $A \subseteq \omega^\omega$ is I_J-positive if and only if it contains all branches of a tree in $Q(J)$.*

Proof. For a set $C \subseteq \omega^\omega \times \omega^\omega$ we consider the game $G(C)$ between Players I and II in which at n-th round Player I plays a finite sequence $s_n \in \omega^{<\omega}$ and a number $m_n \in \omega$, and Player II answers with a set $a_n \in J$. The first element of the sequence s_{n+1} must not belong to the set a_n. In the end let x be the concatenation of s_n's and let y be the concatenation of m_n's. Player I wins if $\langle x, y\rangle \in C$.

Claim. *Player II has a winning strategy in $G(C)$ if and only if* $\mathrm{proj}(C) \in I_J$. *If Player I has a winning strategy in $G(C)$, then* $\mathrm{proj}(C)$ *contains all branches of a tree in $Q(J)$.*

The proof of the above Claim is standard (cf. [8, Theorem 21.2]) and we omit it. The clue here is that having a winning strategy for Player II one should consider maximal *good positions* (see [8, proof of Theorem 21.1]) for Player I to show that $\mathrm{proj}(C) \in I_J$. Now, if $C \subseteq \omega^\omega \times \omega^\omega$ is closed such that $\mathrm{proj}(C) = A$, then determinacy of $G(C)$ gives the desired property of A. $\square$

This shows that P_{I_J} has a dense subset isomorphic to $Q(J)$, so the two forcing notions are equivalent. $\square$

If J is coanalytic, then the σ-ideal I_J associated with the poset $Q(J)$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. The further, finer forcing properties of $Q(J)$ depend on the position of J in the Katětov ordering.

Definition 3.4 ([6])**.** Let H and F be hereditary families of subsets of $\mathrm{dom}(H)$ and $\mathrm{dom}(F)$ respectively. H is *Katětov above* F, or $H \geq_K F$, if there is a function $f : \mathrm{dom}(H) \to \mathrm{dom}(F)$ such that $f^{-1}(a) \in H$ for each $a \in F$.

For a more detailed study of this order see [3]. It turns out that for many preservation-type forcing properties ϕ there is a critical hereditary family H_ϕ such that $\phi(Q(J))$ holds if and only if $J \restriction a \not\geq_K H_\phi$ for every $a \notin J$. This section collects several results of this kind.

Definition 3.5. We say that $a \subseteq 2^{<\omega}$ is *nowhere dense* if every finite binary sequence has an extension such that no further extension falls into a. NWD stands for the ideal of all nowhere dense subsets of $2^{<\omega}$.

Theorem 3.6. *$Q(J)$ does not add Cohen reals if and only if $J \restriction a \not\geq_K$ NWD for every J-positive set a.*

Proof. On one hand, suppose that there exists a J-positive set a such that $J \restriction a \geq_K$ NWD as witnessed by a function $f : a \to 2^{<\omega}$. Then, the tree $a^{<\omega}$ forces the concatenation of the f-images of numbers on the generic sequence to be a Cohen real.

On the other hand, suppose that $J \restriction a \not\geq_K$ NWD. Let $T \in Q(J)$ be a condition and $\dot{y}$ be a name for an infinite binary sequence. We must show that $\dot{y}$ is not a name for a Cohen real. That is, we must produce a condition $S \leq T$ and an open dense set $O \subseteq 2^\omega$ such that $S \Vdash \dot{y} \notin \check{O}$.

Strengthening the condition T if necessary we may assume that there is a continuous function $f : [T] \to 2^\omega$ such that $T \Vdash \dot{y} = \dot{f}(\dot{g})$. For every splitnode $t \in T$ and for every $n \in \mathrm{succ}_T(t)$ pick a branch $b_{t,n} \in [T]$ such that $t^\frown n \subseteq b_{t,n}$. Use the Katětov assumption to find a J-positive subset $a_t \subseteq \mathrm{succ}_T(t)$ such that the set $\{f(b_{t,n}) : n \in a_t\} \subseteq 2^\omega$ is nowhere dense.

Consider the countable poset P consisting of pairs $p = \langle s_p, O_p \rangle$ where s_p is a finite set of splitnodes of T, $O_p \subseteq 2^\omega$ is a clopen set, and $O_p \cap \{f(b_{t,n}) : t \in s_p, n \in a_t\} = \emptyset$. The ordering is defined by $q \leq p$ if

- $s_p \subseteq s_q$ and $O_p \subseteq O_q$,
- if $t \in s_q \setminus s_p$, then $f(x) \notin O_p$ for each $x \in [T]$ such that $t \subseteq x$.

Choose $G \subseteq P$, a sufficiently generic filter, and define $O = \bigcup_{p \in G} O_p$ and $S \subseteq T$ to be the downward closure of $\bigcup_{p \in G} s_p$. Simple density arguments show that $O \subseteq 2^\omega$ is open dense. $S \in Q(J)$ since for every node $t \in \bigcup_{p \in G} s_p$ and every $n \in a_t$ we have $t^\frown n \in S$. The definitions show that $f''[S] \cap O = \emptyset$ as desired. $\square$

Definition 3.7. Let $0 < \varepsilon < 1$ be a real number. The ideal S_ε has as its domain all clopen subsets of 2^ω of Lebesgue measure less than ε, and it is generated by those sets a for which $\bigcup a \neq 2^\omega$.

This ideal is closely connected with the Fubini property of ideals on countable sets, as shown below in a theorem of Solecki.

Definition 3.8. If $a \subseteq \mathrm{dom}(J)$ and $D \subseteq a \times 2^\omega$, then we write

$$\int_a D \, dJ = \{y \in 2^\omega : \{j \in a : \langle j, y \rangle \notin D\} \in J\}.$$

J has the *Fubini property* if for every real $\varepsilon > 0$, every J-positive set a and every Borel set $D \subseteq a \times 2^\omega$ with vertical sections of Lebesgue measure at most ε, the set $\int_a D \, dJ$ has outer measure at most ε.

Obviously, the ideals S_ε as well as all families above them in the Katětov ordering fail to have the Fubini property. The following theorem implicitly appears in [13, Theorem 2.1], the formulation below is stated in [3, Theorem 3.13] and proved in [9, Theorem 3.7.1].

Theorem 3.9 (Solecki)**.** *Suppose F is an ideal on a countable set. Then either F has the Fubini property, or else for every (or equivalently, some) $\varepsilon > 0$ there is an F-positive set a such that $F \restriction a \geq_K S_\varepsilon$.*

By μ we denote the outer Lebesgue measure on 2^ω. Recall that a forcing notion P is said to *preserve outer Lebesgue measure* if for any ground model set $A \subseteq 2^\omega$ with $\mu(A) = \delta$ we have that $P \Vdash \mu(\check{A}) = \delta$. For a further discussion on this property see [14, Section 3.6].

Theorem 3.10. *Suppose that J is a universally measurable ideal. $Q(J)$ preserves outer Lebesgue measure if and only if J has the Fubini property.*

Proof. Suppose on one hand that J fails to have the Fubini property. Find a sequence of J-positive sets $\langle b_n : n \in \omega\rangle$ such that $J \restriction b_n \geq_K S_{2^{-n}}$, as witnessed by functions f_n. Consider the tree T of all sequences $t \in \mathrm{dom}(J)^{<\omega}$ such that $t(n) \in b_n$ for each $n \in \mathrm{dom}(t)$. Let $\dot{B}$ be a name for the set $\{z \in 2^\omega : \exists^\infty n \in \omega\ z \in f_n(\dot{g}(n))\}$. T forces that the set $\dot{B}$ has measure zero, and the definition of the ideals S_ε shows that every ground model point in 2^ω is forced to belong to $\dot{B}$. Thus $Q(J)$ fails to preserve Lebesgue outer measure at least below the condition T.

On the other hand, suppose that the ideal J does have the Fubini property. Suppose that $Z \subseteq 2^\omega$ is a set of outer Lebesgue measure δ, $\dot{O}$ is a $Q(J)$-name for an open set of measure less or equal to $\varepsilon < \delta$, and $T \in Q(J)$ is a condition. We must find a point $z \in Z$ and a condition $S \leq T$ forcing $\check{z} \notin \dot{O}$.

By a standard fusion argument, thinning out the tree T if necessary, we may assume that there is a function $h : \mathrm{split}(T) \to \mathcal{O}$ such that

$$T \Vdash \dot{O} = \bigcup\{h(\dot{g} \restriction n+1) : \ \dot{g} \restriction n \in \mathrm{split}(T), n \in \omega\}.$$

Moreover, we can make sure that if $t_n \in T$ is the n-th splitting node, then $T \restriction t_n$ decides a subset of $\dot{O}$ with measure greater than $\varepsilon/2^n$. Hence, if we write $f(t_n) = \varepsilon/2^n$, then for every splitnode $t \in T$ and every $n \in \mathrm{succ}_T(t)$ we have $\mu(h(t^\frown n)) < f(t)$.

Now, for every splitnode $t \in T$ let

$$D_t = \{\langle O, x\rangle : x \in 2^\omega \ \wedge\ O \in \mathrm{succ}_T(t) \ \wedge\ x \in h(t^\frown O)\}.$$

It follows from universal measurability of J that the set $\int_{\mathrm{succ}_T(t)} D_t \, dJ$ is measurable. It has mass not greater than $f(t)$, by the Fubini assumption. Since $\sum_{t\in\mathrm{split}(T)} f(t) < \delta$, we can find

$$z \in Z \setminus \bigcup_{t\in\mathrm{split}(T)} \int_{\mathrm{succ}_T(t)} D_t \, dJ.$$

Let $S \subseteq T$ be the downward closure of those nodes $t^\frown n$ such that $t \in T$ is a splitnode and $n \in \mathrm{succ}_T(t)$ is such that $z \notin h(t^\frown n)$, i.e. $S = \bigcap_{t\in\mathrm{split}(T)}\{s \in T : s \supseteq t^\frown n, n \in \mathrm{succ}_T(t), z \notin h(t^\frown n)\}$. S belongs to $Q(J)$ by the choice of the point z and $S \Vdash \check{z} \notin \dot{O}$, as required. $\square$

An *independent real*[1] is a set x of natural numbers in a generic extension such that both x and the complement of x meet every infinite set of natural numbers from the ground model.

Definition 3.11. SPL is the family of nonsplitting subsets of $2^{<\omega}$, i.e. those $a \subseteq 2^{<\omega}$ for which there is an infinite set $c \subseteq \omega$ such that $t \restriction c$ is constant for every $t \in a$.

Obviously, SPL is an analytic set, but it is not clear whether it is also coanalytic.

Question 3.12. Is SPL a Borel set?

In the following theorem we show that in two quite general cases SPL is critical for the property of adding independent reals.

Note that if J is an ideal, H is hereditary and H' is the ideal generated by H, then $J \geq_K H$ if and only if $J \geq_K H'$. Therefore, in case J is an ideal, $J \geq_K$ SPL is equivalent to J being Katětov above the ideal generated by SPL. The latter is analytic, so in particular it has the Baire property.

Theorem 3.13. *Suppose that J satisfies one of the following*

- *J is a coanalytic hereditary set,*
- *J is an ideal and $J \restriction a$ has the Baire property for each $a \notin J$.*

$Q(J)$ does not add independent reals if and only if $J \restriction a \not\geq_K$ SPL for every J-positive a.

Proof. Again, the left to right direction is easy. If $J \restriction a \geq_K$ SPL for some J-positive set a, as witnessed by a function f, then the condition $a^{<\omega} \in Q(J)$ forces that the concatenation of $\langle f(\dot{g}(n)) : n \in \omega\rangle$ is an independent real.

[1] the term *splitting real* is also commonly used

For the right to left direction, we will need two preliminary general facts. For a set $a \subseteq \omega$ by an *interval in a* we mean a set of the form $[k, l) \cap a$.

First, let $a \subseteq \omega$ be a J-positive set, and let Players I and II play a game $G(a)$, in which they alternate to post consecutive (pairwise disjoint) finite intervals $b_0, c_0, b_1, c_1, \dots$ in the set a. Player II wins if the union of his intervals $\bigcup_{n<\omega} c_n$ is J-positive.

Lemma 3.14. *Player II has a winning strategy in $G(a)$ for any $a \notin J$.*

Proof. In case J is an ideal with the Baire property, this follows immediately from the Talagrand theorem [1, Theorem 4.1.2]. Indeed, if $\{I_k : k < \omega\}$ is a partition of a into finite sets such that each $b \in J$ covers only finitely many of them, then the strategy for II is as follows: at round n pick c_n covering one of the I_k's.

Now we prove the lemma in case J is coanalytic. Consider a related game, more difficult for Player II. Fix a continuous function $f : \omega^\omega \to \mathcal{P}(a)$ such that its range consists exactly of all J-positive sets. The new game $G'(a)$ proceeds just as $G(a)$, except Player II is required additionally to produce sequences $t_n \in \omega^{<\omega}$ of length and all entries at most n, and in the end, Player II wins if $y = \bigcup_{n<\omega} t_n \in \omega^\omega$ and $f(y) \subseteq \bigcup_{n<\omega} c_n$.

Clearly, the game $G'(a)$ is Borel and therefore determined. If Player II has a winning strategy in $G'(a)$, then she has a winning strategy in $G(a)$ and we are done. Thus, we only need to derive a contradiction from the assumption that Player I has a winning strategy in $G'(a)$.

Well, suppose σ is such a winning strategy. We construct a strategy for Player I in $G(a)$ as follows. The first move $b_0 = \sigma(\emptyset)$ does not change. Suppose Player I is going to make her move after the sets $b_0, c_0, \dots, b_n, c_n$ have been chosen. For each possible choice of the sequences t_m for $m < n$ consider a run of $G'(a)$ in which Player I plays according to σ and Player II plays the pairs (b'_m, t_m), where b'_m are the intervals b_m adjusted downward to the previous move of Player I. The next move of Player I is now the union of all finitely many moves the strategy σ dictates against such runs in $G'(a)$. It is not difficult to see that this is a winning strategy for Player I in the original game G. However, Player I cannot have a winning strategy in the game G since Player II could immediately steal it and win herself. □

Second, consider the collection F of those subsets $a \subseteq \omega^{<\omega}$ such that there is no tree $T \in Q(J)$ whose splitnodes all fall into a.

Lemma 3.15. *The collection F is an ideal.*

Proof. The collection F is certainly hereditary. To prove the closure under unions, let $a = a_0 \cup a_1$ be a partition of the set of all splitnodes of a $Q(J)$ tree into two parts. We must show that one part contains all splitnodes of some $Q(J)$ tree. For $i \in 2$ build rank functions $\mathrm{rk}_i : a_i \to \mathrm{Ord} \cup \{\infty\}$ by setting $\mathrm{rk}_i \geq 0$ and $\mathrm{rk}_i(t) \geq \alpha + 1$ if the set $\{n \in \omega : t^\frown n$ has an extension s in a_i such that $\mathrm{rk}_i(s) \geq \alpha\}$ is J-positive. If the rank rk_i of any splitnode is ∞ then the nodes whose rank rk_i is ∞ form a set of splitnodes of a tree in $Q(J)$, contained in a_i. Thus, it is enough to derive a contradiction from the assumption that no node has rank ∞.

Observe that if $t \in a$ is a node with $\mathrm{rk}_i(t) < \infty$, then there is $n \in \omega$ such that a contains nodes extending $t^\frown n$, but all of them either have rank less than $\mathrm{rk}_i(t)$ or do not belong to a_i. Thus, one can build a finite sequence of nodes on which the rank decreases and the last one has no extension in the set a_i. Repeating this procedure twice, we will arrive at a node of the set a which belongs to neither of the sets a_0 or a_1, reaching a contradiction. □

Now suppose that $J \restriction a \not\geq_K \mathrm{SPL}$ for every J-positive set a. Let $T \in Q(J)$ be a condition and $\dot{y}$ be a $Q(J)$-name for a subset of ω. We must prove that $\dot{y}$ is not a name for an independent real. That is, we must find an infinite set $b \subseteq \omega$ as well as a condition $S \leq T$ forcing $\dot{y} \restriction \check{b}$ to be constant. The construction proceeds in several steps.

Step 1. First, construct a tree $T' \subseteq T$ and an infinite set $b \subseteq \omega$ such that for every splitnode $t \in T'$ there is a bit $c_t \in 2$ such that for all but finitely many $n \in b$, for all but finitely many immediate successors s of t in T' we have

$$T' \restriction s \Vdash \dot{y}(n) = c_t.$$

To do this, enumerate $\omega^{<\omega}$ as $\langle t_i : i \in \omega \rangle$, respecting the initial segment relation, and by induction on $i \in \omega$ construct a descending sequence of trees $T_i \subseteq T$, sets $b_i \subseteq \omega$, and bits $c_{t_i} \in 2$ as follows:

- if t_i is not a splitnode of T_i, then do nothing and let $T_{i+1} = T_i$, $b_{i+1} = b_i$ and $c_{t_i} = 0$;
- if t_i is a splitnode of T_i, then for each $j \in \mathrm{succ}_{T_i}(t_i)$ find a tree $S_j \leq T_i \restriction t_i^\frown j$ deciding $\dot{y} \restriction j$, and use the Katětov assumption to find a J-positive set $a \subseteq \mathrm{succ}_{T_i}(t_i)$, a bit $c_{t_i} \in 2$, and an infinite set $b_{i+1} \subseteq b_i$ such that whenever $j \in a$ and $n \in b_{i+1} \cap j$ then $S_j \Vdash \dot{y}(n) = c_{t_i}$. Let $T_{i+1} = T_i$, except below t_i replace $T_i \restriction t_i$ with $\bigcup_{j \in a} S_j$.

In the end, let $T' = \bigcap_{i<\omega} T_i$ and let b be any diagonal intersection of the sets b_i.

Step 2. The second step uses Lemma 3.15 to stabilize the bit c_t. Find a condition $T'' \subseteq T'$ such that for every splitnode $t \in T''$, c_t is the same value, say 0.

Step 3. The last step contains a fusion argument. For every splitnode $t \in T''$ fix a winning strategy σ_t for Player II in the game $G(\mathrm{succ}_{T''}(t))$. By induction on $i \in \omega$ build sets $S_i \subseteq T''$, functions f_i on S_i, and numbers $n_i \in b$ so that

- $S_0 \subseteq S_1 \subseteq \dots$, and in fact S_{i+1} contains no initial segments of nodes in S_i that would not be included in S_i already. The final condition will be a tree S whose set of splitnodes is $\bigcup_{i<\omega} S_i$;
- for every node $s \in S_i$, the value $f_i(s)$ is a finite run of the game $G(\mathrm{succ}_{T''}(s))$ according to the strategy σ_s, in which the union of the moves of the second player equals $\{j \in \omega : \exists t \in S_i\ s^\frown j \subseteq t\}$. Moreover, $f_i(s) \subseteq f_{i+1}(s) \subseteq \dots$. This will ensure that every node in $\bigcup_{i<\omega} S_i$ in fact splits into J-positively many immediate successors in the tree S;
- whenever $s \in S_i$ and $j \in \omega$ is the least such that $s \in S_j$, then $T'' \restriction s \Vdash \forall k \in j\ \dot{y}(n_k) = 0$. This will ensure that in the end we have $S \Vdash \forall i < \omega\ \ \dot{y}(n_i) = 0$.

The induction step is easy to perform. Suppose that S_i, f_i and n_j have been found for $j < i$. Let $n_i \in b$ be a number such that for all $s \in S_i$ for all but finitely many $n \in \mathrm{succ}_{T''}(s)$ we have

$$T'' \restriction s^\frown n \Vdash \dot{y}(n_i) = 0.$$

For every node $s \in S_i$, let d_s be a finite set such that for all $n \in \mathrm{succ}_{T''}(s) \setminus d_s$ and for all $j \leq i$

- $T'' \restriction s^\frown n \Vdash \dot{y}(n_j) = 0$
- and $s^\frown n$ is not an initial segment of any node in S_i.

Extend the run $f_i(s)$ to $f_{i+1}(s)$ such that the new moves by Player II contain no numbers in the set d_s.

Put into S_{i+1} all nodes from S_i as well as every t which is the smallest splitnode of T'' above some $s^\frown j$ where j is one of the new numbers in the set answered by Player II in $f_{i+1}(s)$.

In the end put $S = \bigcup_{i<\omega} S_i$. It follows from the construction that $S \Vdash \forall i < \omega\ \ \dot{y}(n_i) = 0$, as desired.

□

We finish this section with an observation about degrees of reals in $Q(J)$ generic extensions.

Definition 3.16. We say that J has the *discrete set property* if for every J-positive set a and every function $f : a \to X$ into a Polish space, there is a J-positive set $b \subseteq a$ such that the set $f''b$ is discrete.

Obviously, the discrete set property is equivalent to being not Katětov above the family of discrete subsets of $\mathbb{Q}$. It is not difficult to show that it also equivalent to being not above the ideal of those subsets of the ordinal ω^ω which do not contain a topological copy of the ordinal ω^ω.

Recall that I_J is the σ-ideal associated to J as in Proposition 3.2.

Proposition 3.17. *If J has the discrete set property, then the σ-ideal I_J has the 1-1 or constant property*

Proof. Let T be a condition in $Q(J)$ and $f : [T] \to \omega^\omega$ be a continuous function. It is enough to find a tree $S \in Q(J)$, $S \leq T$ such that on $[S]$ the function f is either constant, or is a topological embedding. Suppose that f is not constant on any such $[S]$. By an easy fusion argument we build $S \subseteq T$, $S \in Q(J)$ such that for any splitnode s of S there are pairwise disjoint open sets U_i for $i \in \mathrm{succ}_S(s)$ such that $f''[S \restriction s^\frown i] \subseteq U_i$ for each $i \in \mathrm{succ}_S(s)$. This implies that f is a topological embedding on $[S]$. □

4. Closure ideals

From now on we assume that X is a Polish space with a complete metric, I is a σ-ideal on X and $\mathcal{O}$ a countable topology basis for the space X.

Definition 4.1. For a set $a \subseteq \mathcal{O}$, we define

$$\mathrm{cl}(a) = \{x \in X : \forall \varepsilon > 0\ \exists O \in a \quad O \subseteq B_\varepsilon(x)\}.$$

We write

$$J_I = \{a \subseteq \mathcal{O} : \mathrm{cl}(a) \in I\}.$$

It is immediate that the collection J_I is an ideal and that J_I is *dense*[2], i.e. every infinite set in $\mathcal{O}$ contains an infinite subset in J_I. If the σ-ideal I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, then J_I is coanalytic. On the other hand, if X is compact and J_I is analytic, then it follows from the Kechris–Louveau–Woodin theorem [7, Theorem 11] that J_I is $\mathbf{F}_{\sigma\delta}$.

Definition 4.2. An ideal J on a countable set is *weakly selective* if for every J-positive set a, any function on a is either constant or 1-1 on a positive subset of a.

[2]also the term *tall* is commonly used.

Obviously, this is just a restatement of the fact that the ideal is not Katětov above the ideal on $\omega\times\omega$ generated by vertical lines and graphs of functions.

Proposition 4.3. *J_I is weakly selective.*

Proof. Take a J_I-positive set a and $f : a \to \omega$. Suppose that f is not constant on any J_I-positive subset of a. We must find $b \subseteq a$ such that f is 1-1 on b. Write Y for $\mathrm{cl}(a)$ shrunk by the union of all basic open sets U such that $\mathrm{cl}(a) \cap U \in I$. Enumerate all basic open sets which have nonempty intersection with Y into a sequence $\langle U_n : n < \omega\rangle$. Inductively pick a sequence $\langle O_n \in a : n < \omega\rangle$ such that $O_n \subseteq U_n$ and $f(O_n) \neq f(O_i)$ for $i < n$. Suppose that O_i are chosen for $i < n$. Let $Y_n = Y \cap U_n$. This is an I-positive set and hence $a_n = \{O \in a : O \subseteq U_n\}$ is J_I-positive. Note that f assumes infinitely many values on a_n since otherwise we could find J_I-positive $b \subseteq a_n$ on which f is constant. Pick any $O_n \in a_n$ such that $f(O_n) \notin \{f(O_i) : i < n\}$. Now, the set $b = \{O_n : n < \omega\}$ is J_I-positive since $\mathrm{cl}(b)$ contains Y. □

Not every ideal on a countable set can be represented as J_I for a σ-ideal I on a Polish space. It would be interesting, however, to find out what "internal" properties of an ideal can witness existence of this "external" σ-ideal. The following seems to be a natural conjecture.

Conjecture 4.4. If J is a dense $\mathbf{F}_{\sigma\delta}$ weakly selective ideal on ω, then there exists a Polish space with a countable base $\mathcal{O}$, a σ-ideal I on X and a bijection $\rho : \mathcal{O} \to \omega$ such that $J = \{a \subseteq \omega : \rho^{-1}(a) \in J_I\}$.

We will now verify several Katětov properties of the ideal J_I depending on the forcing properties of P_I.

Proposition 4.5. *Suppose that P_I is a proper and ω^ω-bounding notion of forcing. Then the ideal J_I has the discrete set property.*

Proof. Take a J_I-positive set a and a function $f : a \to \mathbb{Q}$. Let $B = \mathrm{cl}(a)$. Let $\langle \dot{O}_n : n \in \omega\rangle$ be a sequence of P_I-names for open sets in a such that $\dot{O}_n$ is forced to be wholly contained in the 2^{-n}-neighborhood of the P_I-generic point in B. Passing to a subsequence and a subset of a if necessary, we may assume that the sets $\dot{O}_n$ are pairwise distinct.
Case 1. Assume the values $\{f(\dot{O}_n) : n \in \omega\}$ are forced not to have any point in the range of f as a limit point. Use the ω^ω-bounding property of the forcing P_I to find a condition $B' \subseteq B$, a sequence of finite sets $\langle a_n : n \in \omega\rangle$ and numbers $\varepsilon_n > 0$ such that

- $B' \Vdash \forall m < \omega\, \exists n < \omega\ \dot{O}_m \in \check{a}_n$;

- the collection $\{B_{\varepsilon_n}(f(O)) : O \in a_n, n \in \omega\}$ consists of pairwise disjoint open balls.

To see how this is possible, note that B forces that for every point $y \in f''a$ there is an $\varepsilon > 0$ such that all but finitely many points of the sequence $\langle f(O_m) : m \in \omega\rangle$ have distance greater than ε from y.

Now let $b = \bigcup_{n<\omega} a_n$. Let M be a countable elementary submodel of a large enough structure and let $B'' \subseteq B'$ be a Borel I-positive set consisting only of generic points over M. Note that $B'' \subseteq \mathrm{cl}(b)$ and therefore the set b is as required.

Case 2. If the values $\{f(\dot{O}_n) : n \in \omega\}$ can be forced to have a point in the range of f as a limit point, then, possibly shrinking the set a we can force the sequence $\langle f(\dot{O}_n) : n \in \omega\rangle$ to be convergent and not eventually constant, hence discrete. Similarly as in Case 1, we find $b \subseteq a$ such that $f''b$ is discrete. □

Proposition 4.6. *Suppose that P_I is a proper and outer Lebesgue measure preserving notion of forcing. Then J_I has the Fubini property.*

Proof. Suppose that $\varepsilon > 0$ is a real number, $a \subseteq \mathcal{O}$ is a J_I-positive set, and $D \subseteq a \times 2^\omega$ is a Borel set with vertical sections of measure at most ε. Assume for contradiction that the outer measure of the set $\int_a D\, dJ$ is greater than ε. Let $B = \mathrm{cl}(a)$. This condition forces that there is a sequence $\langle \dot{O}_n : n \in \omega\rangle$ of sets in a such that O_n is wholly contained in in the 2^{-n}-neighborhood of the generic point. Let $\dot{C}$ be a name for the set $\{z \in 2^\omega : \exists^\infty n < \omega\ z \notin \dot{O}_n\}$. This is a Borel set of measure greater than or equal to $1-\varepsilon$. Since the forcing P_I preserves the outer Lebesgue measure, there must be a condition $B' \subseteq B$ and a point $z \in \int_a DdJ$ such that $B' \Vdash \check{z} \in \dot{C}$. Consider the set $b = \{O \in a : z \notin O\}$. The set $\mathrm{cl}(b)$ must be I-positive, since the condition B' forces the generic point to belong to it. This, however, contradicts the assumption that $z \in \int_a D\, dJ$. □

Now we examine the property of adding Cohen reals.

Proposition 4.7. *Suppose that P_I is proper and does not add Cohen reals. Then $Q(J_I)$ does not add Cohen reals.*

Proof. We begin with a lemma.

Lemma 4.8. *Let $a \subseteq \mathcal{O}$ be J_I-positive and $f : a \to 2^\omega$ be a function. There is a J_I-positive set $b \subseteq a$, a closed nowhere dense set $N \subseteq 2^\omega$ and an I-positive Borel set $A \subseteq X$ such that for each $\varepsilon > 0$ we have*

$$A \subseteq \mathrm{cl}(\{O \in b : f(O) \in B_\varepsilon(N)\}).$$

Proof. Write $C = \mathrm{cl}(a)$. We pick a sequence of names $\langle \dot{O}_n : n < \omega \rangle$ for elements of a such that C forces (in P_I) that

- $\dot{O}_n$ is contained in the 2^{-n}-neighborhood of the generic point,
- the sequence $\langle f(\dot{O}_n) : n < \omega \rangle$ is convergent in 2^ω.

Let $\dot{z}$ be a name for $\lim_{n\to\infty} f(\dot{O}_n)$. Since P_I does not add Cohen reals, there is a closed nowhere dense set $N \subseteq 2^\omega$ and a Borel I-positive $A \subseteq C$ such that $A \Vdash \dot{z} \in \check{N}$. Without loss of generality assume that A consists only of generic points over a sufficiently big countable elementary submodel $M \prec H_\kappa$. Let b be the set of all $O \in a$ such that O is equal to $\dot{O}_n$ evaluated in $M[g]$ for some $n < \omega$ and some $g \in A$. Now, if $\varepsilon > 0$, then clearly $A \subseteq \mathrm{cl}(b_\varepsilon)$ since all but finitely many $f(\dot{O}_n)$ are forced into $B_\varepsilon(N)$. □

Suppose $T \in Q(J_I)$ is a condition and $\dot{x}$ is a name for a real. Without loss of generality assume that $T \Vdash \dot{x} = f(\dot{g})$ for some continuous function $f : [T] \to 2^\omega$. For each $t \in \mathrm{split}(T)$ and each $O \in \mathrm{succ}_T(t)$ pick a branch $b_{t,O} \in [T]$ extending $t^\frown O$. By Lemma 4.8 we can assume that for each splitnode $t \in T$ we have $\mathrm{succ}_T(t) \notin J_I$ and there is a closed nowhere dense set $N_t \subseteq 2^\omega$ and an I-positive Borel set $A_t \subseteq X$ such that for each $\varepsilon > 0$ we have

$$A_t \subseteq \mathrm{cl}(\{O \in \mathrm{succ}_T(t) : f(b_{t,O}) \in B_\varepsilon(N_t)\}).$$

For each $t \in \mathrm{split}(T)$ fix an enumeration $\langle V_t^n : n < \omega \rangle$ of all basic open sets which have nonempty intersection with A_t. Enumerate all nonempty basic open subsets of 2^ω into a sequence $\langle U_n : n < \omega \rangle$.

By induction on $n < \omega$, we build increasing finite sets $S_n \subseteq \mathrm{split}(T)$, decreasing trees $T_n \le T$ with $S_n \subseteq T_n$ and nonempty clopen sets $U'_n \subseteq U_n$ such that for each $n < \omega$ the following hold:

(i) for each $s \in \mathrm{split}(T_n)$ there is $\varepsilon > 0$ such that
$$\{O \in \mathrm{succ}_T(s) : f(O) \in B_\varepsilon(N)\} \subseteq \mathrm{succ}_{T_n}(s)$$
and hence $A_s \subseteq \mathrm{cl}(\mathrm{succ}_{T_n}(s))$,

(ii) for each $s \in S_n$ there is $t \supseteq s$ in S_{n+1} such that $t(|s|) \subseteq V_s^n$,

(iii) for each $x \in [T_n]$ we have $f(x) \notin \bigcup_{k<n} U'_k$.

Let $S_0 = \emptyset$ and $T_0 = T$. Suppose S_n and T_n are constructed. For each $s \in S_n$ find $\varepsilon_s^n > 0$ such that

$$W = U_{n+1} \setminus \bigcup_{s \in S_n} B_{\varepsilon_s^n}(N_s) \neq \emptyset.$$

Next, let $a_s = \{O \in \mathrm{succ}_{T_n}(s) : f(b_{O,s}) \in B_{\varepsilon_s^n}(N_s)\}$. Note that, by (i), we still have $A_s \subseteq \mathrm{cl}(a_s)$. Now, for $s \in S_n$ find $O_s \in a_s$ such that O_s is contained in V^n_s. For each $s \in S_n$ and $O \in a_s$ find $k_{s,O} < \omega$ such that

$$f''[T_n \restriction (b_{s,O} \restriction k_{s,O})] \subseteq B_{\varepsilon^n_s}(N_s).$$

To obtain the tree T_{n+1}, extend each $s \in S_n$ by $T_n \restriction (b_{s,O} \restriction k_{s,O})$ above each $s^\frown O$ for each $O \in \{O_s\} \cup a_s$. Put into S_{n+1} all nodes from S_n as well as the first splitnodes of T_{n+1} above each $s^\frown O_s$ for $s \in S_n$. This ends the construction.

Now note that $T = \bigcap_{n<\omega} T_n$ is a condition in $Q(J_I)$ with the set of splitnodes $S = \bigcup_{n<\omega} S_n$, since for each $s \in S$ we have $A_s \subseteq \mathrm{cl}(\mathrm{succ}_T(s))$ by (ii). The set $U = \bigcup_{n<\omega} U'_n$ is dense open and by (iii) we have that $T \Vdash \dot{x} \notin U$, which implies that $\dot{x}$ is not a name for a Cohen real. This ends the proof.

□

Corollary. *If I is such that P_I is proper and does not add Cohen reals, then $J_I \restriction a \not\geq_K$ NWD for any J_I-positive set a.*

Finally, we examine the property of adding independent reals.

Proposition 4.9. *Suppose that P_I is proper and does not add independent reals. Then $Q(J_I)$ does not add independent reals.*

Proof. Suppose $T \in Q(J_I)$ is a condition and $\dot{x}$ is a name for a real. We proceed similarly as in Theorem 3.13.

Step 1. Without loss of generality assume that that for each $t \in \mathrm{split}(T)$ and $O \in \mathrm{succ}_T(t)$ there is $\rho_{t,O} \in 2^{<\omega}$ such that

- $T \restriction t^\frown O \Vdash \subseteq \rho_{t,O} \subseteq \dot{x}$,
- for each $k < \omega$ the set $\{O \in \mathrm{succ}_T(t) : |\rho_{t,O}| \leq k\}$ is finite.

Write $r_{t,O}$ for $\rho_{t,O}$ extended by a sequence of zeros. Fix also a well-ordering $\preceq$ on $\omega^{<\omega}$ preserving the initial segment relation.

Lemma 4.10. *Let $a \subseteq \mathcal{O}$ be J_I-positive, $f : a \to 2^\omega$ be a function and $b \subseteq \omega$ be an infinite set. There is an I-positive Borel set $B \subseteq \mathrm{cl}(a)$, an infinite set $b' \subseteq b$ and $c \in 2$ such that for each $k \in \omega$ we have*

$$B \subseteq \mathrm{cl}(\{O \in a : \forall i \in b' \cap k \quad f(O)(i) = c\}).$$

Proof. Write $C = \mathrm{cl}(a)$. Pick a sequence of names $\langle \dot{O}_n : n < \omega \rangle$ for elements of a such that C forces (in P_I) that

- $\dot{O}_n$ is contained in the 2^{-n}-neighborhood of the generic point,
- the sequence $\langle f(\dot{O}_n) : n < \omega \rangle$ is convergent in 2^ω.

Let $\dot{z}$ be a name for $\lim_{n\to\infty} f(\dot{O}_n)$. Since P_I does not add independent reals, there is $C' \in P_I$, $C' \subseteq C$, an infinite set $b' \subseteq b$ and $c \in 2$ such that

$$C' \Vdash_{P_I} \forall i \in b' \quad \dot{z}(i) = c.$$

Let $M \prec H_\kappa$ be a sufficiently big countable elementary submodel and $B \subseteq C'$ be Borel I-positive set whose elements are P_I-generic over M. Now it is routine to check that B, b' and c are as needed. $\square$

Using Lemma 4.10 and $\preceq$-induction on $\mathrm{split}(T)$, for each $t \in \mathrm{split}(T)$ we pick a Borel I-positive set $B_t \subseteq \mathrm{cl}(\mathrm{succ}_T(t))$, an infinite set $b_t \subseteq \omega$ and $c_t \in 2$ such that for each $s, t \in \mathrm{split}(T)$ we have

- if $s \preceq t$, then $b_t \subseteq b_s$,
- for each $k \in \omega$ we have
$$B_t \subseteq \mathrm{cl}(\{O \in \mathrm{succ}_T(t) : \forall i \in b_t \cap k \quad r_{t,O}(i) = c_t)\}.$$

Step 2. We find a subtree $T' \subseteq T$ in $Q(J_I)$ such that for each $t \in \mathrm{split}(T')$ we have $\mathrm{succ}_{T'}(t) = \mathrm{succ}_T(t)$ and all c_t for $t \in \mathrm{split}(T')$ have the same value, say 1. To see how this is done, note that either there is $t \in \mathrm{split}(T)$ such that for each $s \in \mathrm{split}(T)$ extending t we have $c_s = 0$, or for each $t \in \mathrm{split}(T)$ there is $s \in \mathrm{split}(T)$ extending t with $c_s = 1$. In the first case pick such $t \in \mathrm{split}(T)$ and take $T' = T \restriction t$. In the second case we construct T' by fusion.
Step 3. Let $b \subseteq \omega$ be any pseudointersection of the sets b_t for $t \in \mathrm{split}(T')$. We will find an infinite subset $b' \subseteq b$ and a subtree $T'' \subseteq T'$ in $Q(J_I)$ such that
$$T'' \Vdash \forall k \in b' \quad \dot{x}(k) = 1.$$
This will show that $\dot{x}$ is not a name for an independent real.

For each $t \in \mathrm{split}(T')$ enumerate as $\langle V^n_t : n < \omega\rangle$ all nonempty open sets which have nonempty intersection with B_t. We construct a descending sequence of trees $T_n \leq T$ in $Q(J_I)$, increasing sequence of finite sets $S_n \subseteq \mathrm{split}(T_n)$ and an increasing sequence of natural numbers $m_n \in b'$ such that

(i) for each $t \in S_n$ we have
$$\{O \in \mathrm{succ}_T(t) : |\rho_{t,O}| > m_n \wedge \forall i \in b_t \cap (m_n+1)\ r_{t,O}(i) = 1)\} \subseteq \mathrm{succ}_{T_n}(t)$$
and hence $B_t \subseteq \mathrm{cl}(\mathrm{succ}_{T_n}(t))$,
(ii) for each $s \in S_n$ there is $t \supseteq s$ such that $t \in S_{n+1}$ and $t(|s|) \subseteq V^n_s$,
(iii) $T_n \Vdash \dot{x}(m_n) = 1$.

At the end let $T'' = \bigcap_{n<\omega} T_n$. T'' is in $Q(J_I)$ with the set of splitnodes $S = \bigcup_{n<\omega} S_n$, since for each $s \in S$ we have $B_s \subseteq \mathrm{succ}_{T''}(s)$ by (ii). Put $b' = \{m_n : n < \omega\}$. Note that by (iii) we have
$$T'' \Vdash \forall k \in b'\ \dot{x}(k) = 1.$$

Let $S_0 = \emptyset$ and $T_0 = T$. Suppose S_n, T_n and $\langle m_i : i \leq n\rangle$ have been constructed. Find $m_{n+1} \in b$ which belongs to all b_s for $s \in S_n$. For each $s \in S_n$ let

$$a_s = \{O \in \mathrm{succ}_T(t) : |\rho_{t,O}| > m_n \wedge \forall i \in b_t \cap (m_n+1)\ r_{t,O}(i) = 1)\}.$$

Note that by Lemma 4.10 we still have $B_s \subseteq \mathrm{cl}(a_s)$. For $s \in S_n$ find $O_s \in a_s$ such that $O_s \subseteq V_s^n$.

Let S_{n+1} be the union of S_n together with the first splitnodes of T_n above each $s^\frown O_s$, for $s \in S_n$. Let T_{n+1} be the subtree of T_n generated by the nodes O_s for $s \in S_n$ and $O \in a_s$.

This ends the construction and the whole proof. □

Corollary. *If I is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ such that P_I is proper and does not add independent reals, then $J_I \restriction a \not\geq_K$ SPL for any J_I-positive set a.*

5. Tree representation

In this section we show that under suitable assumptions the forcing P_{I^*} is equivalent to the tree forcing $Q(J_I)$.

Definition 5.1. Let J be an ideal on $\mathcal{O}$ and $T \in Q(J)$. We say that T is *Luzin* if

- the sets on the n-th level have diameter less than 2^{-n},
- for each $x \in [T]$ we have $\overline{x(n+1)} \subseteq x(n)$ for each $n < \omega$
- and for each $t \in T$ the immediate successors of t in T have pairwise disjoint closures and their diameters vanish to 0.

If T is Luzin, then we write $\pi[T]$ for $\{x \in X : \exists y \in [T]\ x \in \bigcap_{n<\omega} y(n)\}$.

Proposition 5.2. *Let I be a σ-ideal on a Polish space X. If $T \in Q(J_I)$ is Luzin, then $\pi[T] \in P_{I^*}$.*

Proof. The set $\pi[T]$ is a 1-1 continuous image of $[T]$, which is a Polish space, so $\pi[T]$ is Borel. To see that $\pi[T]$ is I^*-positive consider the function $\varphi : [T] \to X$ which assings to any $x \in [T]$ the single point in $\bigcap_{n<\omega} x(n)$. Note that φ is continuous since the diameters of open sets on T vanish to 0. Now if $\pi[T] \subseteq \bigcup_{n<\omega} E_n$ where each E_n is closed and belongs to I, then $\varphi^{-1}(E_n)$ are closed sets covering the space $[T]$. By the Baire category theorem, one of them must have nonempty interior in $[T]$. So there is $n < \omega$ and $t \in T$ such that $\varphi(x) \in E_n$ for every $x \in [T]$ such that $t \subseteq x$. Now for each $u \in \mathrm{succ}_T(t)$ we have $u \cap E_n \neq \emptyset$, which implies that $\mathrm{cl}(\mathrm{succ}_T(t)) \subseteq E_n$ and contradicts the fact that $\mathrm{cl}(\mathrm{succ}_T(t))$ is I-positive. □

Now we prove Proposition 1.6. Combined with the propositions proved in the previous section, this gives Theorems 1.2, 1.3, 1.4 and 1.5 from the introduction (recall that the Cohen forcing adds an independent real and does not preserve outer Lebesgue measure).

Proof of Proposition 1.6. Suppose that $B \subseteq X$ is a Borel set which belongs to I but not to I^*. Assume also that $\overline{B}$ forces that the generic point is not a Cohen real. By the Solecki theorem [11, Theorem 1], we may assume that B is a $\mathbf{G}_\delta$ set and for every open set $O \subseteq X$, if $B \cap O \neq \emptyset$, then $B \cap O \notin I^*$. Represent B as a decreasing intersection $\bigcap_{n<\omega} O_n$ of open sets.

We build a Luzin scheme T of basic open sets U_t for $t \in \omega^{<\omega}$ satisfying the following demands:

- $U_t \subseteq O_{|t|}$ and $U_t \cap B \neq \emptyset$,
- the sets in $\mathrm{succ}_T(t)$ have pairwise disjoint closures and are disjoint from $\mathrm{cl}(\mathrm{succ}_T(t))$, which is an I-positive set.

To see how this is done, suppose that U_t are built for $t \in \omega^{\leq n}$ and take any $t \in \omega^n$. The set $\overline{B \cap U_t}$ is I-positive, and since the P_I-generic real is not forced to be a Cohen real, there is a closed nowhere dense I-positive subset C of $\overline{B \cap U_t}$. Find a discrete set $D = \{d_n : n < \omega\}$ such that $D \subseteq B \cap U_t$ and $C \subseteq \overline{D}$. For each $n < \omega$ find a basic open neighborhood V_n of d_n such that the closure of V_n is contained in $U_t \cap O_{|t|+1}$, the closures of the sets V_n are pairwise disjoint, disjoint from C and $C \subseteq \mathrm{cl}(\{V_n : n < \omega\})$. Put $U_{t^\frown n} = V_n$.

Let $T \in Q(J)$ be the Luzin scheme constructed above. Clearly, T is Luzin, as well as each $S \in Q(J_I)$ such that $S \leq T$. For each $S \in Q(J)$ if $S \leq T$, then the set $\pi[S] \subseteq \pi[T]$ is Borel and I^*-positive by Proposition 5.2. We will complete the proof by showing that the range of π is a dense subset of P_{I^*} below the condition $\pi[T]$.

For $C \subseteq B$ which is an I^*-positive set we must produce a tree $S \in Q(J), S \subseteq T$, such that $\pi[S] \subseteq C$. Again, by the Solecki theorem [11, Theorem 1] we may assume that the set C is $\mathbf{G}_\delta$, a decreasing intersection $\bigcap_{n<\omega} W_n$ of open sets and for every open set $O \subseteq X$ if $O \cap C \neq \emptyset$, then $O \cap C \notin I$.

By tree induction build a tree $S \subseteq T$ such that for every sequence on the n-th splitting level, the last set on the sequence is a subset of W_n, and still has nonempty intersection with the set C. In the end, the tree $S \subseteq T$ will be as required.

Now suppose that immediate successors of nodes on the n-th splitting level have been constructed. Let t be one of these successors. Find its extension $s \in T$ such that the last set O on it is a subset of W_{n+1} and still has nonempty intersection with C. Note that

$$\overline{\pi[T]} \subseteq \pi[T] \cup \bigcup_{u \in T} \mathrm{cl}(\mathrm{succ}_T(u)),$$

because T is Luzin. Since $\overline{C \cap O} \notin I$ and $\pi[T] \subseteq B \in I$, this means that there must be an extension u of s such that $\overline{C \cap O} \cap \mathrm{cl}(\mathrm{succ}_T(u)) \notin I$. This can only happen if the set $b = \{V \in \mathrm{succ}_T(u) : V \cap C \neq 0\}$ is J-positive, since $\overline{C \cap O} \cap \mathrm{cl}(\mathrm{succ}_T(u)) \subseteq \mathrm{cl}(b)$. Put all nodes $\{u^\frown V : V \in b\}$ into the tree S and continue the construction. □

We finish this section with a remark concerning the assumption of Proposition 1.6.

Remark 5.3. *For any σ-ideal I on a Polish space X there is a σ-ideal J on X such that*

$$P_I \equiv C \oplus P_J,$$

where C is the Cohen forcing or the trivial forcing and the forcing P_J is not equivalent to the Cohen forcing under any condition.

Proof. Let $\mathcal{F}$ be the family of all Borel sets B such that P_I below B is equivalent to the Cohen forcing. Let $\mathcal{A}$ be a maximal antichain in P_I of sets in $\mathcal{F}$ and let $\mathcal{B}$ be such that $\mathcal{A} \cup \mathcal{B}$ is a maximal antichain in P_I. Of course

$$P_I \equiv \bigoplus_{A \in \mathcal{A}} P_I \restriction A \;\oplus\; \bigoplus_{B \in \mathcal{B}} P_I \restriction B$$

and $\bigoplus_{A \in \mathcal{A}} P_I \restriction A$ is equivalent to the Cohen forcing or the trivial forcing (if $\mathcal{F}$ is empty). Let J be the σ-ideal generated by I and $\mathcal{F}$.

Claim. *$\mathcal{B}$ is a maximal antichain in P_J.*

Proof. Let C be a Borel J-positive set. Suppose that $C \cap B \in J$ for each $B \in \mathcal{B}$. Then for each $B \in \mathcal{B}$ the forcing $P_I \restriction (C \cap B)$ is equivalent to the Cohen forcing. On the other hand $P_I \restriction (C \cap A)$ is also equivalent to the Cohen forcing for each $A \in \mathcal{A}$. This shows that $P_I \restriction C$ is equivalent to the Cohen forcing, so $C \in J$. □

By the above Claim we get that $\bigoplus_{B \in \mathcal{B}} P_I \restriction B$ is equivalent to P_J. Now we need to show that P_J is not equivalent to the Cohen forcing under any condition. Suppose that $P_J \restriction C$ is equivalent to the Cohen forcing for some J-positive Borel set C.

Claim. *For each $B \in \mathcal{B}$ we have $C \cap B \in I$.*

Proof. Suppose $B \cap C$ is I-positive for some $B \in \mathcal{B}$. Note that under B the σ-ideals I and J agree on Borel sets, so $P_I \restriction B \cap C \equiv P_J \restriction B \cap C$. The latter forcing is, however, equivalent to the Cohen forcing, which implies that $B \cap C \in \mathcal{F}$. This contradicts the fact that $\mathcal{A}$ was a maximal antichain in $\mathcal{F}$. □

Now, by the above Claim we have that $P_I \restriction C$ is equivalent to $\bigoplus_{A\in\mathcal{A}} P_I \restriction (A\cap C)$, which is equivalent to the Cohen forcing. This implies that $C \in J$ and gives a contradiction. $\square$

6. The cases of Miller and Sacks

In this section we prove Proposition 1.7. This depends on a key property of the Miller and Sacks forcings.

Lemma 6.1. *Suppose X is a Polish space, $B \subseteq X$ is a Borel set, T is a Miller or a Sacks tree and $\dot{x}$ is a Miller or Sacks name for an element of the set B. Then there is $S \subseteq T$ and a closed set $C \subseteq X$ such that $C \setminus B$ is countable, and $S \Vdash \dot{x} \in \check{C}$.*

Proof. For the Sacks forcing it is obvious and we can even require that $C \subseteq B$. Let us focus on the Miller forcing.

Strengthening the tree T if necessary, we may assume that there is a continuous function $f : [T] \to B$ such that $T \Vdash \dot{x} = f(\dot{g})$. The problem of course is that the set $f''[T]$ may not be closed, and its closure may contain many points which do not belong to the set B.

For every splitnode $t \in T$ and for every $n \in \mathrm{succ}_T(t)$ pick a branch $b_{t,n} \in [T]$ such that $t^\frown n \subseteq b_{t,n}$. Next, find an infinite set $a_t \subseteq \mathrm{succ}_T(t)$ such that the points $\{f(b_{t,n}) : n \in a_t\}$ form a discrete set with at most one accumulation point x_t. For $n \in a_t$ find numbers $m_{t,n} \in \omega$ and open sets $O_{t,n}$ with pairwise disjoint closures such that

- $f''[T \restriction (b_{t,n} \restriction m_{t,n})] \subseteq O_{t,n}$,
- $\mathrm{diam}(O_{t,n}) < 2^{-|t|}$ and $\lim_{n\to\infty} \mathrm{diam}(O_{t,n}) = 0$.

Find a subtree $S \subseteq T$ such that for every splitnode $t \in S$, if $t^\frown n \in S$, then $n \in a_t$ and the next splitnode of S past $t^\frown n$ extends the sequence $b_{t,n} \restriction m_{t,n}$.

It is not difficult to see that $\overline{f''[S]} \subseteq f''[S] \cup \{x_t : t \in \omega^{<\omega}\}$, and therefore the tree S and the closed set $C = \overline{f''[S]}$ are as needed. $\square$

Of course, in the previous lemma, B may be in any sufficiently absolute pointclass, like $\mathbf{\Sigma}^1_2$. Now we prove Proposition 1.7.

Proof of Proposition 1.7. If the σ-ideal I does not contain the same Borel sets as I^*, then any condition $B \in I \setminus I^*$ forces in P_{I^*} the generic point into B but outside of every closed set in the σ-ideal I. However, by Lemma 6.1 we have that if the Miller or the Sacks forcing forces a point into a Borel set in a σ-ideal I (containing all singletons), then it forces that point into a closed set in I. Thus, P_{I^*} cannot be in the forcing sense equivalent neither to Miller nor to Sacks forcing in the case that $I \neq I^*$. $\square$

7. The degrees dichotomy

In this section we prove Theorem 1.8. To this end, we need to learn how to turn Borel functions into functions which are continuous and open.

Lemma 7.1. *Suppose I is a σ-ideal on a Polish space X such that P_I is proper. Let B be Borel I-positive, and $f : B \to \omega^\omega$ be Borel. For any countable elementary submodel $M \prec H_\kappa$ the set*

$$f''\{x \in B : x \text{ is } P_I\text{-generic over } M\}$$

is Borel.

Proof. Without loss of generality assume that $B = X$. Write $C = \{x \in X : x \text{ is } P_I\text{-generic over } M\}$. Let $\dot{y}$ be a P_I-name for $f(\dot{g})$, where $\dot{g}$ is the canonical name for the generic real for P_I. Take $R \subseteq \mathrm{ro}(P_I)$ the complete subalgebra generated by $\dot{y}$. Notice that for each $y \in \omega^\omega$ we have

$$y \in f''C \quad \text{iff} \quad y \text{ is } R\text{-generic over } M.$$

Hence, it is enough to prove that $C' = \{y \in \omega^\omega : y \text{ is } R\text{-generic over } M\}$ is Borel. C' is a 1-1 Borel image of the set of ultrafilters on $R \cap M$ which are generic over M. The latter set is $\mathbf{G}_\delta$, so C' is Borel. $\square$

Now we prove an "open mapping theorem": we show that Borel functions can be turned into continuous and open functions after restriction their domain and some extension of topology. If Y is a Polish space and I is a σ-ideal on Y, then we say that Y is *I-perfect* if I does not contain any nonempty open subset of Y.

Lemma 7.2. *Suppose I is a σ-ideal on a Polish space X such that P_I is proper. Let $B \subseteq X$ be I-positive, and $f : B \to \omega^\omega$ be Borel. There are Borel sets $Y \subseteq B$ and $Z \subseteq \omega^\omega$ such that Y is I-positive, $f''Y = Z$ and*

- *Y and Z carry Polish zero-dimensional topologies which extend the original ones, preserve the Borel structures and the topology on Y is I-perfect.*
- *the function $f \restriction Y : Y \to Z$ is continuous and open in the extended topologies.*

Proof. Fix κ big enough and let $M \prec H_\kappa$ be a countable elementary submodel coding B and f. Let $Y = \{x \in B : x \text{ is } P_I\text{-generic over } M\}$. By Lemma 7.1 we have that $Z = f''Y$ is Borel.

Note that if a $\mathbf{\Sigma}^0_\alpha$ set A is coded in M, then there are sets A_n coded in M, $A_n \in \mathbf{\Pi}^0_{<\alpha}$ such that $A = \bigcup_n A_n$. Therefore, we can perform the construction from [8, Theorem 13.1] and construct a Polish zero-dimensional topology on X which contains all Borel sets coded in M. It follows that the Borel sets coded in M form a basis for this topology. Note that Y is homeomorphic to the set of ultrafilters in $\mathrm{st}(P_I \cap M)$ which are generic over M. So Y is a $\mathbf{G}_\delta$ set in the extended topology. Let τ be the restriction of this extended topology to Y. The fact that τ is I-perfect on Y follows directly from properness of P_I.

Let σ be the topology on Z generated by the sets $f''(Y \cap A)$ and their complements, for all $A \subseteq B$ which are Borel and coded in M.

Now we prove that $f \restriction Y$ is a continuous open map from (Y, τ) to (Z, σ). The fact that f is open follows directly from the definitions. Now we prove that f is continuous. Fix a cardinal λ greater than $2^{2^{|P_I|}}$ and a Borel set A coded in M.

Sublemma 7.3. *Given $x \in Y$ we have*

- *$f(x) \in f''(A \cap Y)$ if and only if*

$$M[x] \models \mathrm{Coll}(\omega, \lambda) \Vdash \exists x' \, P_I\text{-generic over } M \ [x' \in A \ \wedge \ f(x) = f(x')],$$

- *$f(x) \notin f''(A \cap Y)$ if and only if*

$$M[x] \models \mathrm{Coll}(\omega, \lambda) \Vdash \forall x' \, P_I\text{-generic over } M \ [x' \in A \ \Rightarrow \ f(x) \neq f(x')].$$

Proof. We prove only the first part. Note that in M there is a surjection from λ onto the family of all dense sets in P_I as well as sujections from λ onto each dense set in P_I. Therefore, if $x \in Y$ and $g \subseteq \mathrm{Coll}(\omega, \lambda)$ is generic over $M[x]$, then in $M[x][g]$ the formula

$$\exists x' \, P_I\text{-generic over } M \ [x' \in A \ \wedge \ f(x) = f(x')]$$

is analytic with parameters A, f and a real which encodes the family $\{D \cap M : D \in M \text{ is dense in } P_I\}$ and therefore it is absolute between $M[x][g]$ and V. Hence

$$M[x][g] \models \exists x' \, P_I\text{-generic over } M \ [x' \in A \ \wedge \ f(x) = f(x')]$$

if and only if $f(x) \in f^{-1}(f''(A \cap Y))$. □

Now it follows from from Sublemma 7.3 and the forcing theorem that both sets $Y \cap f^{-1}(f''(A \cap Y))$ and $Y \cap f^{-1}(Z \setminus f''(A \cap Y))$ are in τ. This proves that f is continuous.

We need to show that Z with the topology σ is Polish. Note that it is a second-countable Hausdorff zero-dimensional space, so in particular metrizable. As a continuous open image of a Polish space, Z is Polish by the Sierpiński theorem [8, Theorem 8.19].

The fact that σ has the same Borel structure as the original one follows directly from Lemma 7.1.

□

Now we prove Theorem 1.8.

Proof of Theorem 1.8. It is enough to show that if P_I does not add Cohen reals, then I has the 1-1 or constant property. Suppose $B \subseteq X$ is Borel I-positive and $f : B \to \omega^\omega$ is Borel. Assume that f is not constant on any I-positive Borel subset of B. Let $\dot{x}$ be the name for the value of f at the generic point. Let $C \subseteq B$ be Borel I-positive such that

$$C \Vdash \dot{x} \text{ is not a Cohen real and } \dot{x} \notin V.$$

Without loss of generality assume that $C = X$ and f is continuous.

Find Polish spaces $Y \subseteq X$ and $Z \subseteq \omega^\omega$ as in Lemma 7.2. For notation simplicity of the rest of the argument assume that $Y = X$, $Z = \omega^\omega$ and the extended topologies are the original ones (note that I is still generated by closed sets in any extended topology).

We will construct $T \in Q(J_I)$ such that T is Luzin and $f \restriction \pi[T]$ is a topological embedding. To this end we build two Luzin schemes $U_t \subseteq X$ and $C_t \subseteq \omega^\omega$ (for $t \in \omega^{<\omega}$), both with the vanishing diameter property and such that

- U_t and C_t are clopen sets,
- $f''U_t \subseteq C_t$
- for each $t \in \omega^{<\omega}$ the set $\{U_{t^\frown k} : k < \omega\}$ is J_I-positive.

We put $U_\emptyset = X$ and $C_\emptyset = \omega^\omega$. Suppose U_t and C_t are built for all $t \in \omega^{<n}$. Pick $t \in \omega^{n-1}$. Now $f''U_t$ is an open set. Let K be the perfect kernel of $f''U_t$. K is nonempty since $\dot{x}$ is forced not to be in V. Hence K is a perfect Polish space and $U_t \Vdash \dot{x} \in K$. Note that there is a closed nowhere dense $N \subseteq K$ such that $f^{-1}(N)$ is I-positive, since otherwise

$$U_t \Vdash \dot{x} \text{ is a Cohen real in } K.$$

Pick such an N and let $M = f^{-1}(N)$. N is closed nowhere dense also in $f''U_t$, so M is closed nowhere dense in U_t because f is continuous and open.

Enumerate all basic open sets in U_t having nonempty intersection with M into a sequence $\langle V_k : k < \omega\rangle$. Inductively pick clopen sets $W_k \subseteq U_t$ and $C_k \subseteq \omega^\omega$ such that

- $W_k \subseteq f^{-1}(C_k) \cap V_k$ is basic open,
- C_k are pairwise disjoint,
- $f^{-1}(C_k)$ are disjoint from M.

Do this as follows. Suppose that W_i and C_i are chosen for $i < k$. Since $f^{-1}(C_i)$ are disjoint from M and $V_k \cap M \neq \emptyset$, the set $V_k \setminus \bigcup_{i<k} f^{-1}(C_i)$ is a nonempty clopen set. Pick $x_k \in V_k \setminus \big(\bigcup_{i<k} f^{-1}(C_i) \cup M\big)$. Since $f(x_k) \notin N \cup \bigcup_{i<k} C_i$, there is a clopen neighborhood C_k of $f(x_k)$ which

is disjoint from $N \cup \bigcup_{i<k} C_i$. Let W_k be a basic neighborhood of x_k contained in $f^{-1}(C_k) \cap V_k$. Put $U_{t^\frown k} = W_k$ and $C_{t^\frown k} = C_k$, for each $k < \omega$. Now $M \subseteq \mathrm{cl}(\{W_k : k < \omega\})$, so $\{U_{t^\frown k} : k < \omega\}$ is J_I-positive, as needed.

This ends the construction of $T \in Q(J_I)$ and the whole proof. $\square$

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Marcin Sabok, Instytut Matematyczny Uniwersytetu Wrocławskiego, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland and Institut Mittag-Leffler, Auravägen 17 SE-182 60 Djursholm Sweden
E-mail address: `sabok@math.uni.wroc.pl`

Jindřich Zapletal, Institute of Mathematics of the Academy of Sciences of the Czech Republic, Žitná 25, CZ - 115 67 Praha 1, Czech Republic and Department of Mathematics University of Florida, 358 Little Hall PO Box 118105 Gainesville, FL 32611-8105, USA
E-mail address: zapletal@math.cas.cz