Decomposing Borel functions and structure at finite levels of the Baire hierarchy. ☆

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Abstract

We prove that if f is a partial Borel function from one Polish space to another, then either f can be decomposed into countably many partial continuous functions, or else f contains the countable infinite power of a bijection that maps a convergent sequence together with its limit onto a discrete space. This is a generalization of a dichotomy discovered by Solecki for Baire class 1 functions. As an application, we provide a characterization of functions which are countable unions of continuous functions with domains of type $\mathbf{\Pi}^0_n$, for a fixed $n < \omega$. For Baire class 1 functions, this generalizes analogous characterizations proved by Jayne and Rogers for $n = 1$ and Semmes for $n = 2$.

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1. Introduction

In a well-known question Luzin asked if any Borel function is necessarily *countably continuous*, i.e., if its domain can be written as a union of countably many sets (we call them *witnessing sets*), on which the function is continuous. The answer to that question is negative and by now counterexamples have been given by several authors (see Keldiš [11], Sierpiński [17], Adyan and Novikov [1], Cichoń and Morayne [2], Jackson and Mauldin [7], van Mill and Pol [12], Darji [4]).

Several authors (see Jayne and Rogers [8], Solecki [18], Motto Ros and Semmes [6], Kačena, Motto Ros and Semmes [9]) also considered countably continuous functions with closed witnessing sets. All such functions are Baire class 1 and Jayne and Rogers [8] proved the following characterization: a function f is countably continuous with closed witnessing sets if and only if f is $\mathbf{\Pi}^0_2$-measurable, i.e. such that the f-preimages of $\mathbf{\Pi}^0_2$ sets are $\mathbf{\Pi}^0_2$. Recently, Semmes [15] proved an analogous characterization for functions which are countably continuous with $\mathbf{\Pi}^0_2$ witnessing sets.

A particularly simple counterexample to the question of Luzin appeared in [3] and has become known as the function P (see [18, 19]).

Let ω^ω be the countable infinite power of the set of nonnegative integers $\omega = \{0, 1, 2, 3, \dots\}$. Consider two product topological spaces having ω^ω as the underlying set:

- $\mathcal{N}$ – the *Baire space*; the topology arises from the discrete topology on ω.
- $\mathcal{C}$ – the *Cantor space*; the topology on ω arises from $\mathbb{R}$ by identifying

$$0,\ 1,\ 2,\ 3,\ \dots \quad \text{with} \quad 0,\ 2^{-1},\ 2^{-2},\ 2^{-3},\ \dots .$$

Note that $\mathcal{C}$ is homeomorphic to the Cantor space 2^ω, the countable infinite power of the two element discrete space $2 = \{0, 1\}$.

The function P is defined as the identity function from $\mathcal{C}$ to $\mathcal{N}$. It is Baire class 1 and open. In [3] it was shown that P is not countably continuous and Solecki [18, Theorem 4.1] showed that it is the simplest among Baire class 1 functions defined on analytic spaces which are not countably continuous.

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This was later generalized by Zapletal [20, Corollary 2.3.48] to Borel functions defined on Borel subsets of $\mathcal{N}$.

Recall that a *Polish space* is a separable completely metrizable space. An *analytic space* is a separable metrizable space which is a continuous image of the Baire space, equivalently, a space that is homeomorphic to an analytic subset of a Polish space.

In this paper we prove the following theorem, which generalizes the results of Solecki and Zapletal to all Borel functions on analytic spaces.

Theorem 1.1. *Suppose that $\mathcal{D}$ is an analytic space and $\mathcal{R}$ is a separable metrizable space. Let $f : \mathcal{D} \to \mathcal{R}$ be a Borel function. Then either f is countably continous, or else there are topological embeddings $p_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ and $p_{\mathcal{N}} : \mathcal{N} \to \mathcal{R}$ such that the diagram*

$$\begin{array}{ccc} \mathcal{N} & \xrightarrow{p_{\mathcal{N}}} & \mathcal{R} \\ \uparrow P & & \uparrow f \\ \mathcal{C} & \xrightarrow{p_{\mathcal{C}}} & \mathcal{D} \end{array}$$

commutes.

In particular if f is countably continous on all compact subsets of $\mathcal{D}$ then it is countably continous.

It is worth to mention that both proofs of Solecki and Zapletal rely on quite sophisticated methods of mathematical logic. Solecki uses effective descriptive set theory and the Gandy-Harrington topology. Zapletal uses Borel determinacy and the Baire class 1 case of Solecki. Our proof uses only elementary topology and combinatorics. This in particular makes the result available in models of set theory which do not have the full power of the Replacement Axiom.

Given an analytic space $\mathcal{D}$, a separable metrizable space $\mathcal{R}$ and a function $f : \mathcal{D} \to \mathcal{R}$, write (id, f) for the function $\mathcal{D} \ni d \mapsto (d, f(d)) \in \mathcal{D} \times \mathcal{R}$. For any separable metrizable space $\mathcal{S}$ and any function $g : \mathcal{D} \to \mathcal{S}$ and $1 \leq \alpha < \omega_1$, let $g^{-1}\mathbf{\Pi}^0_\alpha$ (resp. $g^{-1}\mathbf{\Sigma}^0_\alpha$) be the family of g-preimages of $\mathbf{\Pi}^0_\alpha$ (resp. $\mathbf{\Sigma}^0_\alpha$) subsets of $\mathcal{S}$. Note that $\mathbf{\Pi}^0_\alpha \subseteq (\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_\alpha$ and likewise for $\mathbf{\Sigma}^0_\alpha$.

We use Theorem 1.1 to extend the theorem of Jayne and Rogers in the following way.

Theorem 1.2. *Suppose that $\mathcal{D}$ is an analytic space and $\mathcal{R}$ is a separable metrizable space. Let $f : \mathcal{D} \to \mathcal{R}$ be any function and let $n > 0$ be a natural number. The following are equivalent:*

(1) *f is countably continuous with $\mathbf{\Pi}^0_n$ witnessing sets,*
(2) $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_{n+1} = \mathbf{\Pi}^0_{n+1}$.

Corollary 1.3. *Suppose that $\mathcal{D}$ is an analytic space and $\mathcal{R}$ is a separable metrizable space. Let $n > 0$ be a natural number and let $f : \mathcal{D} \to \mathcal{R}$ be either* 1-1 *and open or Baire class $n-1$ (or Baire class 1 if $n = 1$). Then the following are equivalent:*

(1) *f is countably continuous with $\mathbf{\Pi}^0_n$ witnessing sets,*
(2) $f^{-1}\mathbf{\Pi}^0_{n+1} \subseteq \mathbf{\Pi}^0_{n+1}$.

It should be noted that Theorem 1.2 is false if we put $n = \omega$, and hence this characterization makes sense only at finite levels of the Borel hierarchy. In fact, a counterexample in the case $n = \omega$ is the function P itself. Indeed, P is not countably continuous, so in particular cannot be decomposed into countably many continuous functions with $\mathbf{\Pi}^0_\omega$ domains. But the function P is of Baire class 1, so every set in $\mathbf{\Sigma}^0_\omega(P)$ is $\mathbf{\Sigma}^0_\omega$. This implies that (2)–(4) hold for $f = P$ and $n = \omega$.

Note that if $f^{-1}\mathbf{\Pi}^0_{n+1} \subseteq \mathbf{\Pi}^0_{n+1}$, then f is at most Baire class n. Our proof of the equivalence of (1) and (2) in Corollary 1.3 works for functions of Baire class $n-1$ but the equivalence is probably true also for Baire class n functions. We state it as a conjecture at the end of the paper.

The paper is organized as follows. We prove Theorem 1.1 in Section 3. Section 2 contains preliminary discussion on Borel functions on analytic spaces and on the Hausdorff distance and Luzin schemes, some notions in combinatorics on finite sequences and a method of defining continuous maps that is used later in the proof of Theorem 1.1. The proof of Theorem 1.1 in Section 3 is divided into several, more or less independent parts. In Subsection 3.1 we introduce a notion of a *cylinder* and recall the technique of *unfolding* of analytic sets. In Subsection 3.2 we define *solid* sets, and in in

Subsection 3.3 we define *severing schemes* and state the Crucial Lemma 3.3. Its proof is deferred to Section 4. In Subsection 3.4 we show how to get the functions $p_{\mathcal{C}}$ and $p_{\mathcal{N}}$ from a special kind of a Luzin scheme, which we construct in Subsection 3.5. In Section 5 we gather some lemmas concerning possible complexities of witnessing sets of countably continuous functions. Section 6 contains a proof of Theorem 1.2 and Corollary 1.3 and Section 7 lists some questions that the paper leaves open.

Remark. We have learnt recently that Theorem 1.2 and Corollary 1.3 were independently deduced (with minor variations) from Theorem 1.1 by Luca Motto Ros [14].

2. Preliminaries

2.1. Borel functions on analytic spaces. We start by recalling the following standard fact.

Fact 2.1. *Suppose $\mathcal{D}$ and $\mathcal{R}$ are separable metrizable spaces and $f : \mathcal{D} \to \mathcal{R}$. If $\mathcal{D}$ and $\mathcal{R}$ are subspaces of analytic spaces $\mathcal{D}'$ and $\mathcal{R}'$, then the following are equivalent.*

(1) *The graph of f is an analytic subset of $\mathcal{D}' \times \mathcal{R}'$.*
(2) *The preimages of open subsets of $\mathcal{R}'$ are analytic subsets of $\mathcal{D}'$.*
(3) *$\mathcal{D}$ is an analytic space and f is a Borel function from $\mathcal{D}$ into $\mathcal{R}$.*

Note that (1) immediately implies that $\mathcal{R}$, as the projection of the graph of f, is an analytic subset of $\mathcal{R}'$.

Proof: (1) $\Rightarrow$ (2): If $V \subseteq \mathcal{R}'$ is open then $f^{-1}(V)$ is the projection of an analytic subset of $\mathcal{D}' \times \mathcal{R}'$, namely $(\mathcal{D}' \times V) \cap \operatorname{graph}(f)$.

(2) $\Rightarrow$ (3): $\mathcal{D}$ is an analytic subset of $\mathcal{D}'$ as the preimage of $\mathcal{R}'$, which is an open subset of $\mathcal{R}'$. So $\mathcal{D}$ is an analytic space. To see Borel measurability of f let V be an open subset of $\mathcal{R}$ and let V' be an open subset of $\mathcal{R}'$ such that $V = V' \cap \mathcal{R}$. Then $f^{-1}(V) = f^{-1}(V')$ is an analytic subset of $\mathcal{D}'$, thus, being contained in $\mathcal{D}$, $f^{-1}(V)$ is an analytic subset of $\mathcal{D}$. Also, $\mathcal{D} \setminus f^{-1}(V) = f^{-1}(\mathcal{R}' \setminus V')$, and $\mathcal{R}' \setminus V' = \bigcap_{n<\omega} V'_n$ for some open sets $V'_n \subseteq \mathcal{R}'$. It follows that $\mathcal{D} \setminus f^{-1}(V) = \bigcap_n f^{-1}(V'_n)$ is also an analytic subset of $\mathcal{D}$. By Suslin's theorem, in analytic spaces disjoint analytic sets can be separated by a Borel set, so, from $\mathcal{D}$'s point of view, $f^{-1}(V)$ is a Borel set.

(3) $\Rightarrow$ (1): Being Borel as a function from $\mathcal{D}$ into $\mathcal{R}$, f is also Borel as a function from $\mathcal{D}$ into $\mathcal{R}'$. So, the graph of f is a Borel subset of $\mathcal{D} \times \mathcal{R}'$. Since $\mathcal{D} \times \mathcal{R}'$ is an analytic subset of $\mathcal{D}' \times \mathcal{R}'$, the graph of f is an analytic subset of $\mathcal{D}' \times \mathcal{R}'$. □

2.2. Metric notions. Given a metric space $\mathcal{Z}$ with the metric denoted by $|\cdot,\cdot|$ of diameter ≤ 1, a nonempty subset $X \subseteq \mathcal{Z}$ write $|X|_{\mathcal{Z}}$ for the diameter of X and $\operatorname{cl}_{\mathcal{Z}}(X)$ for the closure of X. Given a point $y \in \mathcal{Z}$ and a nonempty subset $X \subseteq \mathcal{Z}$ define the *distance of the point y from the set X* as

$$|y, X|_{\mathcal{Z}} = \inf\{|y, x|_{\mathcal{Z}} : x \in X\}.$$

Given nonempty subsets $X, Y \subseteq (Z)$ define the *Hausdorff distance between X and Y* as

$$|X, Y|_{\mathcal{Z}} = \max(\sup_{x\in X} |x, Y|_{\mathcal{Z}}, \sup_{y\in Y} |y, X|_{\mathcal{Z}}).$$

We also write $|\emptyset, \emptyset|_{\mathcal{Z}} = 0$ and $|X, \emptyset|_{\mathcal{Z}} = 1$ if $X \neq \emptyset$. Given $X \subseteq \mathcal{Z}$ and $\delta > 0$ define the *δ-neighborhood of X* as

$$B_\delta(X) = \{z \in \mathcal{Z} : \exists x \in X \quad |x, z|_{\mathcal{Z}} < \delta\}.$$

Equivalently, the Hausdorff distance between $X, Y \subseteq \mathcal{Z}$ can be defined as

$$|X, Y|_{\mathcal{Z}} = \inf\{\delta > 0 : X \subseteq B_\delta(Y) \wedge Y \subseteq B_\delta(X)\}.$$

It is not difficult to see that $|X, Y|_{\mathcal{Z}} = |\operatorname{cl}_{\mathcal{Z}}(X), \operatorname{cl}_{\mathcal{Z}}(Y)|_{\mathcal{Z}}$. For more about the Hausdorff distance see [10, Section 4F].

Lemma 2.2. *Suppose $\mathcal{Z}$ is a metric space of diameter ≤ 1. Given nonempty subsets $X, Y, Z \subseteq \mathcal{Z}$, $x \in X$, $y \in Y$ and $\delta > 0$, we have*

(1) $|x, y|_{\mathcal{Z}} \leq |X, Y|_{\mathcal{Z}} + |Y|_{\mathcal{Z}}$;

(2) $|X|_{\mathcal{Z}} \leq 2|X,Y|_{\mathcal{Z}} + |Y|_{\mathcal{Z}}$;
(3) *if* $|X,Z|_{\mathcal{Z}} < \delta$, *then* $||X,Y|_{\mathcal{Z}} - |Z,Y|_{\mathcal{Z}}| < \delta$.

Proof: (1). Given $\epsilon > 0$, pick a point $y' \in Y$ such that $|x,y'|_{\mathcal{Z}} \leq |X,Y|_{\mathcal{Z}} + \epsilon$. Then $|x,y|_{\mathcal{Z}} \leq |x,y'|_{\mathcal{Z}} + |y',y|_{\mathcal{Z}} \leq |X,Y|_{\mathcal{Z}} + \epsilon + |Y|_{\mathcal{Z}}$. As $\epsilon > 0$ was arbitrary, this proves (1).

(2). Fix $\epsilon > 0$. Given $x, x' \in X$ find $y, y' \in Y$ with $|x,y|_{\mathcal{Z}} \leq |X,Y|_{\mathcal{Z}} + \epsilon$ and $|x',y'|_{\mathcal{Z}} \leq |X,Y|_{\mathcal{Z}} + \epsilon$. Then

$$|x,x'|_{\mathcal{Z}} \leq |x,y|_{\mathcal{Z}} + |y,y'|_{\mathcal{Z}} + |y',x'|_{\mathcal{Z}} \leq 2|X,Y|_{\mathcal{Z}} + 2\epsilon + |Y|_{\mathcal{Z}}.$$

As x, x' were arbitrary, this implies that $|X|_{\mathcal{Z}} \leq 2|X,Y|_{\mathcal{Z}} + |Y|_{\mathcal{Z}} + 2\epsilon$. As $\epsilon > 0$ was arbitrary, this proves (2).

(3). It is enough to show that $|X,Y|_{\mathcal{Z}} < |Z,Y|_{\mathcal{Z}} + \delta$ as by symmetry we get also the other inequality. Let $|X,Z|_{\mathcal{Z}} = \delta' < \delta$ and $|Z,Y|_{\mathcal{Z}} = \epsilon$. To show $|X,Y|_{\mathcal{Z}} \leq \epsilon + \delta' < \epsilon + \delta$ we need to prove $X \subseteq B_{\epsilon+\delta'}(Y)$ and $Y \subseteq B_{\epsilon+\delta'}(X)$. The first inclusion follows from the fact that $X \subseteq B_{\delta'}(Z)$ and $Z \subseteq B_{\epsilon}(Y)$. The second follows from $Y \subseteq B_{\epsilon}(Z)$ and $Z \subseteq B_{\delta'}(X)$. This ends the proof. □

Let $0 < N \leq \omega$. Write $|\cdot,\cdot|$ to denote

- in ω, the discrete zero-one metric;
- in ω^N, the product of the above discrete metric, i.e.,

$$|r,s| = \sum_{n<N} |r(n), s(n)| \cdot 2^{-n-1};$$

  the diameter of ω^N is $\sum_{n<N} 2^{-n-1}$;
  the case of $N = \omega$ gives the metric for the space $\mathcal{N}$, the diameter is 1;
- in $\mathcal{N}^N$ and the product of the above metrics for $\mathcal{N}$,

Also, write $\|\cdot,\cdot\|$ to denote

- in ω, the metric obtained by doubling the standard metric of $\mathbb{R}$ when using the identification

$$0 \longmapsto 0, \quad 1 \longmapsto 2^{-1}, \quad 2 \longmapsto 2^{-2}, \quad 3 \longmapsto 2^{-3}, \ \ldots\ ;$$

  the diameter is 1;
- in ω^N, the product of the above metric (defined similarly to the product metric for $|\cdot,\cdot|$); the case of $N = \omega$ gives the metric for the space $\mathcal{C}$, the diameter is 1.

We further overload $|\cdots|$ and in any of the above metric spaces we write

- $|X|$, for the diameter of the set X; $|\varnothing| = 0$;
- $|x,Y|$, for the distance of the point x from the set Y; $|x,\varnothing| = 1$;
- $|X,Y|$, for the Hausdorff distance between the sets X and Y; $|\varnothing,\varnothing| = 0$, $|X,\varnothing| = 1$ if $X \neq \varnothing$.

2.3. Trees and maps. Consider a set Λ with the discrete topology. For a finite sequence $\tau \in \Lambda^{<\omega}$ write

- $\operatorname{lh}\tau$ for the length of τ;
- $[\tau]$ for the basic clopen set $\{t \in \Lambda^\omega : \tau \subseteq t\}$ determined by τ in the product space Λ^ω.

A *tree on* Λ is a subset $T \subseteq \Lambda^{<\omega}$ that is closed under initial segments, i.e., $\tau{\restriction}n \in T$ if $\tau \in T$ and $n < \operatorname{lh}\tau$. A tree T is finitely branching if every non-terminal node has finitely many immediate successors. The *n-th level* of T is the set $\{\tau \in T : \operatorname{lh}\tau = n\}$, an *antichain* in T is a subset of T in which neither node is an initial segment of another.

An *infinite branch* of T is any $t \in \Lambda^\omega$ such that for all $n \in \omega$ we have $t{\restriction}n \in T$. The *body* of T, denoted by $[T]$, is the set of all its infinite branches. This is a closed subset of Λ^ω and it is endowed with the subspace topology.

We follow the standard practice of identifying a sequence $\rho \in (\omega \times \omega)^N$, $N \leq \omega$, with the pair of sequences $(\sigma, \tau) \in \omega^N \times \omega^N$ such that $\rho(n) = (\sigma(n), \tau(n))$. If T is a tree on $\omega \times \omega$, $\sigma \in \omega^{<\omega}$, and $s \in \omega^\omega$, let

$$T_\sigma = \{\tau \in \omega^{<\omega} : (\sigma,\tau) \in T\}, \text{ and}$$

$$T_s = \bigcup_n T_{s{\restriction}n}.$$

Let also

$$\operatorname{pr} T = \{\sigma \in \omega^{<\omega} : T_\sigma \neq \varnothing\}.$$

Lemma 2.3. *Suppose that T is a tree on $\omega \times \omega$ such that for each $s \in \omega^\omega$ the tree T_s is finitely branching and has exactly one infinite branch $\varphi(s)$. Then the function $s \mapsto \varphi(s)$ is continuous, and thus the function $s \mapsto (s, \varphi(s))$ is a homeomorphism from $\mathcal{N}$ onto $[T]$.*

Proof: Given $s \in \mathcal{N}$ and $n < \omega$, we need to find $m \geq n$ such that $\varphi([s\restriction m]) \subseteq [\varphi(s)\restriction n]$. Consider the tree

$$T' = \{\tau \in T_s : \tau \not\supseteq \varphi(s)\restriction n\}.$$

Suppose towards a contradiction that for each $m \geq n$ there is $s_m \supseteq s\restriction m$ such that $\varphi(s_m) \not\supseteq \varphi(s)\restriction n$. Then $\varphi(s_m)\restriction m \in T_s$, by $(s_m\restriction m, \varphi(s_m)\restriction m) \in T$ and $s\restriction m = s_m\restriction m$, hence $\varphi(s_m)\restriction m \in T'$. So T' is a finitely branching infinite tree. By the König lemma there is $t \in [T']$. Then $t \in [T_s]$ and $t \neq \varphi(s)$, which contradicts our assumption that $[T_s] = \{\varphi(s)\}$. ☐

2.4. Luzin schemes. Fix a metric space $\mathcal{Z}$. Given a tree T on some set Λ, we say that a family $\{Z_\tau\}_{\tau \in T}$ of nonempty subsets of $\mathcal{Z}$ is a *Luzin scheme* if

- $Z_\tau \subseteq Z_{\tau^*}$,
- $\tau_0 \neq \tau_1 \wedge \tau_0^* = \tau_1^* \Rightarrow Z_{\tau_0} \cap Z_{\tau_1} = \varnothing$.

The scheme has *vanishing diameters* if for each $t \in [T]$ the diameters of the sets $Z_{t\restriction n}$ converge to 0. In such a case the *associated injection* Φ is defined on $\{t \in [T] : \bigcap_n Z_{t\restriction n} \neq \varnothing\}$ by

$$\Phi(t) = \text{ the unique element of } \bigcap_n Z_{t\restriction n}.$$

The map Φ is continuous when $\operatorname{dom} \Phi$ bears the topology inherited from $[T]$. If $\mathcal{Z}$ is complete and the scheme consists of closed sets, then $\operatorname{dom} \Phi = [T]$.

An *antichain*, respectively, the *n-th level of the scheme* $\{Z_\tau\}_{\tau \in T}$ is a family of the form $\{Z_\tau : \tau \in A\}$, where A is an antichain, respectively, the n-th level, in T. We say that a family of pairwise disjoint subsets of $\mathcal{Z}$ is *relatively discrete* if each of the sets is relatively open in the union of all of them. The following lemma is straightforward.

Lemma 2.4. *In a Luzin scheme,*

1. *if the diameters vanish and the levels are relatively discrete, then the associated map is a homeomorphism*;
2. *if the levels are relatively discrete, then so are all antichains.*

2.5. Combinatorics of finite sequences. Now we introduce on $\omega^{<\omega}$ operations $\sigma \mapsto \sigma^\circ$, $\sigma \mapsto \sigma^*$ and a well-ordering.

For $\sigma \in \omega^{<\omega}$ write

$$\max \sigma = \max\{\sigma(n) : n < \operatorname{lh} \sigma\}, \text{ and}$$

$$\min\max \sigma = \min\{n < \operatorname{lh} \sigma : \sigma(n) = \max \sigma\},$$

where $\max \varnothing = 0$ and $\min \varnothing = 0$. If $\max \sigma \leq \operatorname{lh} \sigma$, let $\sigma^\circ = \sigma$, otherwise let σ° be the sequence obtained from σ by changing to 0 the value of σ at $\min\max \sigma$, i.e., $\operatorname{lh} \sigma^\circ = \operatorname{lh} \sigma$ and

$$\sigma^\circ(n) = \begin{cases} 0 & \text{if } n = \min\max \sigma \text{ and } \max \sigma > \operatorname{lh} \sigma, \\ \sigma(n) & \text{otherwise.} \end{cases}$$

Note that $\sigma \neq \sigma^\circ$ if and only if $\max \sigma > \operatorname{lh} \sigma$. Write $\sigma^{(0)} = \sigma$, $\sigma^{(k+1)} = (\sigma^{(k)})^\circ$.

Next, define σ^* as the sequence obtained from σ by removing the last term, if possible; i.e. let $\varnothing^* = \varnothing$.

Choose a well-ordering $\preceq$ of $\omega^{<\omega}$ into type ω so that $\sigma^*, \sigma^\circ \preceq \sigma$, and let $\#\sigma$ be the number indicating the position of σ in $\preceq$. Note that $\#\varnothing = 0$, $\#\{0\} = 1$, $\operatorname{lh} \sigma \leq \#\sigma$, and let by convention $\#\varnothing - 1 = 0$.

Lemma 2.5. *Let $\sigma \in \omega^{<\omega}$, $l = \operatorname{lh}\sigma$, and $m = \max\sigma$.*

(1) *If $\sigma \neq \sigma^\circ$ and the change occurs at the n-th place, then*
$$\|\sigma, \sigma^\circ\| \leq 2^{-n-1} \cdot 2^{-l}.$$

(2) $\|\sigma, \sigma^{\circ\circ}\| = \|\sigma, \sigma^\circ\| + \|\sigma^\circ, \sigma^{\circ\circ}\|$; *more generally,*
$$\|\sigma, \sigma^{(k)}\| = \|\sigma^{(0)}, \sigma^{(1)}\| + \cdots + \|\sigma^{(k-1)}, \sigma^{(k)}\|.$$

(3) $(\sigma^{(l)})^\circ = \sigma^{(l)}$ *and* $\|\sigma, \sigma^{(l)}\| < 2 \cdot 2^{-l}$.

(4) *If $\rho \in \omega^l$, then*
$$\|\rho, \sigma\| < 4^{-\max(l,m)} \quad \Rightarrow \quad \exists\, k \leq l\ \rho^{(k)} = \sigma.$$

Proof: (1) $\|\sigma, \sigma^\circ\| = 2^{-n-1} \cdot 2 \cdot 2^{-\sigma(n)} \leq 2^{-n} \cdot 2 \cdot 2^{-l-1}$ as $\sigma(n) \geq l+1$ if the change occurs at the n-th place.

(2) The changes occur at different places.

(3) $\|\sigma, \sigma^{(l)}\| < (2^{-0} + 2^{-1} + 2^{-2} + \cdots + 2^{-(l-1)}) \cdot 2^{-l}$.

(4) Let ρ be as postulated. Note that for all $n < l$ we have
$$\rho(n) \neq \sigma(n) \quad \Rightarrow \quad \sigma(n) = 0 \ \wedge\ \rho(n) > \max(l, m). \tag{$\bigstar$}$$

Indeed, otherwise,
$$\|\sigma(n), \rho(n)\| \geq \begin{cases} 2 \cdot (2^{-(m-1)} - 2^{-m}) = 2 \cdot 2^{-m} & \text{if } \sigma(n) \neq 0, \\ 2 \cdot (2^{-\max(l,m)} - 0) = 2 \cdot 2^{-\max(l,m)} & \text{if } \sigma(n) = 0, \end{cases}$$

so, in any case
$$\begin{aligned} \|\rho, \sigma\| &\geq 2^{-(n+1)} \cdot \|\rho(n), \sigma(n)\| \\ &\geq 2^{-n} \cdot 2^{-\max(l,m)} > 2^{-l} \cdot 2^{-\max(l,m)} > 4^{-\max(l,m)}, \end{aligned}$$

contradicting our assumption.

Suppose that $\rho \neq \sigma$ and let $n = \min\max\rho$. By ($\bigstar$)
$$\rho(n) > \rho^\circ(n) = 0 = \sigma(n),$$

so
$$\|\rho^\circ, \sigma\| < \|\rho, \sigma\| < 4^{-\max(l,m)}.$$

and ρ° agrees with σ more than ρ does, viz. n. If $\rho^\circ \neq \sigma$, continue as above to $\rho^{\circ\circ}$, etc. At some $k \leq \operatorname{lh}\sigma$, $\rho^{(k)}$ agrees with σ completely. $\square$

3. The Solecki dichotomy

In this section we prove Theorem 1.1.

Proof of Theorem 1.1: Fix $\mathcal{D}$,$\mathcal{R}$ and f and suppose that f is not countably continuous. Assume without loss of generality that f is onto. We will show that there are there are topological embeddings $p_{\mathcal{C}} : \mathcal{C} \to \mathcal{D}$ and $p_{\mathcal{N}} : \mathcal{N} \to \mathcal{R}$ such that the diagram in Theorem 1.1 commutes.

Let $\overline{\mathcal{D}}$ and $\overline{\mathcal{R}}$ be metric compactifications of $\mathcal{D}$ and $\mathcal{R}$, respectively, with metrics of diameters ≤ 1. We consider the space $\overline{\mathcal{D}} \times \overline{\mathcal{R}} \times \mathcal{N}^N$, with the metric being the the product of the metrics of $\mathcal{N}$, $\overline{\mathcal{D}}$, and $\overline{\mathcal{R}}$. The diameter is again ≤ 1.

We repeat our convention from Subsection 2.2 and in any of the above metric spaces we write

- $|x, y|$ for the distance between two points x and y;
- $|X|$, for the diameter of the set X; $|\varnothing| = 0$;
- $|x, Y|$, for the distance of the point x from the set Y; $|x, \varnothing| = 1$;
- $|X, Y|$, for the Hausdorff distance between the sets X and Y; $|\varnothing, \varnothing| = 0$, $|X, \varnothing| = 1$ if $X \neq \varnothing$.

For the definition of the Hausdorff distance see Section 2.2.

3.1. Subsets of the graph of f and cylinders. In this section we introduce the notion of a *cylinder* and the notation behind the tilded letters $\widetilde{X}$. The notation of this section will be used to construct Luzin schemes of subsets of the graph of the function f. This in turn will be used to define the functions $p_{\mathcal{C}}$ and $p_{\mathcal{N}}$. Cylinders will be used to guarantee that the intersections of sets on the branches of the Luzin scheme are nonempty.

Definition. Say that a closed set $\widetilde{X} \subseteq \overline{\mathcal{D}} \times \overline{\mathcal{R}} \times \mathcal{N}^\omega$ is a *cylinder* if

- the projection of $\widetilde{X}$ into $\overline{\mathcal{D}} \times \overline{\mathcal{R}}$ is contained in the graph of the function f;
- there exist $N < \omega$ and closed $C \subseteq \overline{\mathcal{D}} \times \overline{\mathcal{R}} \times \mathcal{N}^N$ such that $\widetilde{X} = C \times \mathcal{N}^{\omega \setminus N}$.

We follow the convention that if a tilded letter $\widetilde{X}$ denotes a cylinder, then the letter itself, X in this case, denotes the projection of $\widetilde{X}$ into $\mathcal{D}$. We call X the *base* of $\widetilde{X}$ and say that X *unfolds to* $\widetilde{X}$. Note that

- the base of a cylinder is an analytic subset of $\mathcal{D}$;
- if N and C witness that $\widetilde{X}$ is a cylinder, so do $N' > N$ and $C' = C \times \mathcal{N}^{N' \setminus N}$;
- a finite union of cylinders is again a cylinder.

By Fact 2.1, the graph of f is an analytic subset of $\overline{\mathcal{D}} \times \overline{\mathcal{R}}$, so it is the projection of a closed set $C \subseteq \overline{\mathcal{D}} \times \overline{\mathcal{R}} \times \mathcal{N}$. The set $\widetilde{\mathcal{D}} = C \times \mathcal{N}^{\omega \setminus 1}$ is then a cylinder with base $\mathcal{D}$.

3.2. Solid sets. Say that a subset $X \subseteq \mathcal{D}$ is *simple* if it is of the form $U \cap f^{-1}(V)$ for a basic open set $U \subseteq \mathcal{D}$ and a basic open set $V \subseteq \mathcal{R}$. Call $X \subseteq \mathcal{D}$ *small* if X can be covered with countably many analytic sets on which f is continuous. Call X *positive* if it is not small. Note that $\mathcal{D}$ is positive and if X is positive then $f(X)$ is uncountable.

Define

$$\ker X = X \setminus \bigcup \{Y \cap X : Y \text{ is simple and } Y \cap X \text{ is small}\}.$$

Say that $X \subseteq \mathcal{D}$ is *solid* if $X = \ker X \neq \emptyset$.

Lemma 3.1. *For $X \subseteq \mathcal{D}$,*

1. $X \setminus \ker X$ *is small and* $\ker \ker X = \ker X$;
2. *if X is positive, then* $\ker X$ *is solid; if X is analytic and positive, then* $\ker X$ *is analytic solid;*
3. *if X is solid, then all nonempty intersections of X with simple sets are solid;*
4. *any union of countably many solid analytic sets is solid analytic;*
5. *if X is solid and $U \subseteq \mathcal{R}$ is open such that $X \cap f^{-1}(V) \neq \emptyset$, then $X \cap f^{-1}(U)$ is solid;*
6. *if X is solid and $U \subseteq \mathcal{D}$ is open such that $X \cap U \neq \emptyset$, then $X \cap U$ is solid.*

Proof: $\mathcal{D}$ and $\mathcal{R}$ have countable bases, which implies that $\ker X$ is obtained by removing from X countably many small sets. This implies (1), (3) and (4). These sets are relatively Borel if X is analytic, which implies (2). The items (5) and (6) follow from (4) and (3). □

We say that $\widetilde{X}$ is positive or solid if X is such. Note that by (2) and (3) of Lemma 3.1, $\ker \mathcal{D}$ is solid and analytic. Let $\ker \mathcal{D}$ unfold to $\widetilde{\ker \mathcal{D}}$ inside $\widetilde{\mathcal{D}}$.

Lemma 3.2. *Suppose that $\widetilde{X}$ is a cylinder.*

1. *Any analytic set $Y \subseteq X$ can be unfolded to a cylinder $\widetilde{Y} \subseteq \widetilde{X}$;*
2. *If X is positive then for any $\epsilon > 0$ there exists a positive cylinder $\widetilde{Y} \subseteq \widetilde{X}$ with $|\widetilde{Y}| < \epsilon$.*
3. $|X| \leq 2 \cdot |\widetilde{X}|$ *and* $|f(X)| \leq 4 \cdot |\widetilde{X}|$

Proof: (1) Let a closed set $C_Y \subseteq \overline{\mathcal{D}} \times \mathcal{N}$ project onto Y. Define $\widetilde{Y}$ by

$$(x, v, s) \in \widetilde{Y} \quad \text{iff} \quad (x, v, s) \in \widetilde{X} \ \wedge \ (x, s(N)) \in C_Y,$$

where N witnesses that $\widetilde{X}$ is a cylinder.

(2) Let N and C witness that $\widetilde{X}$ is a cylinder and assume that N is large enough for $2^{-(N+2)} < \epsilon/2$. Write C as a countable union of closed sets $\{C_i\}$ of diameter $\leq \epsilon/2$. Then $|C_i \times \mathcal{N}^{\omega \setminus N}| < \epsilon$ for each i and there must be C_i with the positive projection into $\mathcal{D}$. Let $\widetilde{Y} = C_i \times \mathcal{N}^{\omega \setminus N}$. □

3.3. Severing schemes. Now we introduce *severing schemes,* the central notion of the proof. They will be used later in the construction as finite approximations of a Luzin scheme.

We use cl to denote the topological closure in $\mathcal{D}$. If S is a tree on $\omega \times \omega$, $\{Z_{\sigma\tau}\}_{(\sigma\tau)\in S}$ is a family of sets, and $\sigma \in \omega^{<\omega}$, we write

$$Z_\sigma = \bigcup \{Z_{\sigma\tau} : \tau \in S_\sigma\}.$$

Definition. Let S be a finite tree on $\omega \times \omega$. Call a family $\{\widetilde{X}_{\sigma\tau}\}_{(\sigma\tau)\in S}$ of solid cylinders a *severing scheme* if

- for each $(\sigma, \tau) \in S$ we have $X_{\sigma\tau} \subseteq \operatorname{cl} X_{\sigma^* \tau^*}$,
- the family $\{f(X_{\sigma\tau})\}$ is relatively discrete.

Note that if $\{\widetilde{X}_{\sigma\tau}\}_{(\sigma\tau)\in S}$ is a severing scheme, then each of the families $\{\widetilde{X}_{\sigma\tau}\}$, $\{X_{\sigma\tau}\}$, and $\{f(X_{\sigma\tau})\}$, consists of pairwise disjoint sets.

The following lemma will be a crucial ingredient in the construction of the maps $p_{\mathcal{C}}$ and $p_{\mathcal{N}}$. We postpone its proof until Section 4.

Lemma 3.3. (Crucial Lemma) *Fix $\epsilon > 0$ and a severing scheme $\{\widetilde{X}_{\sigma\tau}\}_{(\sigma\tau)\in S}$. Suppose that nonempty subsets Y and Y' of $\mathcal{D}$ are such that $Y' \subseteq \operatorname{cl} Y$, and that $i \in \omega$ and $\eta \in \operatorname{pr} S$ are such that $\eta^\frown i \notin \operatorname{pr} S$. Then we can find finite sets (possibly empty) $J_{\eta\vartheta} \subseteq \omega$ for $\vartheta \in S_\eta$ such that $\bigcup_{\vartheta \in S_\eta} J_{\eta\vartheta} \neq \emptyset$ and we can associate with the tree*

$$S' = S \cup \{(\eta^\frown i, \vartheta^\frown j) : \vartheta \in S_\eta,\ j \in J_{\eta\vartheta}\}$$

a severing scheme $\{\widetilde{X}'_{\sigma\tau}\}_{(\sigma\tau)\in S'}$ so that

(1) *for all $\vartheta \in S_\eta$ and $j \in J_{\eta\vartheta}$ we have*

$$|\widetilde{X}'_{\eta^\frown i\, \vartheta^\frown j}| < \epsilon$$

(2) *for all $(\sigma, \tau) \in S$,*

$$\widetilde{X}'_{\sigma\tau} \subseteq \widetilde{X}_{\sigma\tau} \quad \text{and} \quad |X'_{\sigma\tau}, X_{\sigma\tau}| < \epsilon$$

and the sets $X'_{\sigma\tau}$ are solid analytic; in particular, for all $\sigma \in \operatorname{pr} S$,

$$|X'_\sigma, X_\sigma| < \epsilon;$$

(3) *for all $\vartheta \in S_\eta$,*

$$\forall j \in J_{\eta\vartheta} \ \ X'_{\eta^\frown i, \vartheta^\frown j} \subseteq X_{\eta\vartheta},$$

and

$$|X'_{\eta^\frown i}, Y'| < |X_\eta, Y| + \epsilon.$$

3.4. Defining embeddings. The construction of the maps $p_{\mathcal{C}}$ and $p_{\mathcal{N}}$ will proceed in several steps.

Step 1. In order to define the maps $p_{\mathcal{C}}$ and $p_{\mathcal{N}}$ we shall build a tree T on $\omega \times \omega$ such that for each $s \in \omega^\omega$

$$T_s \text{ is finitely branching and infinite,}$$

and we shall associate with T a scheme $\{\widetilde{Z}_{\sigma\tau}\}_{(\sigma\tau)\in T}$ of solid analytic sets such that

(**A**) the schemes $\{\widetilde{Z}_{\sigma\tau}\}_{(\sigma\tau)\in T}$, $\{Z_{\sigma\tau}\}_{(\sigma\tau)\in T}$, and $\{f(Z_{\sigma\tau})\}_{(\sigma\tau)\in T}$, are all Luzin and have vanishing diameters;

(**B**) the levels of the scheme $\{f(Z_{\sigma\tau})\}_{(\sigma\tau)\in T}$ are relatively discrete;

(**C**) the scheme $\{\widetilde{Z}_\sigma\}_{\sigma\in\omega^{<\omega}}$ is Luzin, has vanishing diameters and for each $\sigma \in \omega^{<\omega}$, writing $l = \operatorname{lh}\sigma$ and $m = \max\sigma$ we have

$$\rho \in \omega^l \ \wedge\ \|\rho, \sigma\| < 4^{-\max(l,m)} \quad \Rightarrow \quad |Z_\rho, Z_\sigma| < 3 \cdot 2^{-l}.$$

Step 2. Suppose we have done this. Now we define the functions $p_{\mathcal{C}}$ and $p_{\mathcal{N}}$. Let $\Phi : [T] \to \widetilde{\mathcal{D}}$ be the map associated with the scheme $\{\widetilde{Z}_{\sigma\tau}\}$. Note that indeed $\operatorname{dom} \Phi = [T]$ because we have here a Luzin scheme of closed subsets of a Polish space and the diameters are vanishing (in the space $\mathcal{D} \times \mathcal{R} \times \mathcal{N}^\omega$ with its product Polish metric).

Now, for each $s \in \omega^\omega$ the tree T_s is a finitely branching infinite tree, hence its body is nonempty. Since $|\widetilde{Z}_{s\restriction n}| \to 0$, condition (**B**) and the fact that $\operatorname{dom} \Phi = [T]$ imply that $[T_s]$ must be a singleton, say $\{\varphi(s)\}$. Let $\bar{\varphi} : \omega^\omega \to [T]$ be the bijection $s \mapsto (s, \varphi(s))$.

Let $\pi_{\mathcal{D}} : \mathcal{D} \times \mathcal{R} \times \mathcal{N}^\omega \to \mathcal{D}$ and $\pi_{\mathcal{R}} : \mathcal{D} \times \mathcal{R} \times \mathcal{N}^\omega \to \mathcal{R}$ be the projection maps. Define

$$p_{\mathcal{C}} = \pi_{\mathcal{D}} \circ \Phi \circ \bar{\varphi} \quad \text{and} \quad p_{\mathcal{N}} = \pi_{\mathcal{R}} \circ \Phi \circ \bar{\varphi}.$$

Then

$$f \circ p_{\mathcal{C}} = p_{\mathcal{N}} = p_{\mathcal{N}} \circ P$$

because P is the identity function.

Lemma 3.4. *The maps $p_{\mathcal{C}}$ and $p_{\mathcal{N}}$, considered as maps from $\mathcal{C}$ into $\mathcal{D}$ and from $\mathcal{N}$ into $\mathcal{R}$, are homeomorphic embeddings.*

Proof: For $p_{\mathcal{N}}$, note that $\pi_{\mathcal{R}} \circ \Phi$ is the map associated with the Luzin scheme $\{f(Z_{\sigma\tau})\}$ and use (**B**) and Lemma 2.4 to see that this map is a homeomorphism from $[T]$ onto $\mathcal{N}$. Then use Lemma 2.3 to identify $[T]$ and $\mathcal{N}$.

For $p_{\mathcal{C}}$, note that $p_{\mathcal{C}}$ is the map associated with the Luzin scheme $\{Z_\sigma\}$. Since $\mathcal{C}$ is compact, we only need to prove that $p_{\mathcal{C}}$ is continuous. Fix $s \in \omega^\omega$ and $l \in \omega$ large enough for $|Z_{s\restriction l}| < 2^{-l}$. Let $m = \max s\restriction l$ and consider $r \in \mathcal{C}$ such that $\|r, s\| < 4^{-\max(l,m)}$. Then, $\|r\restriction l, s\restriction l\| < 4^{-\max(l,m)}$, so, using (**C**) and Lemma 2.2(1), we get

$$\begin{aligned} |p_{\mathcal{C}}(r), p_{\mathcal{C}}(s)| &\le |Z_{r\restriction l}, Z_{s\restriction l}| + |Z_{s\restriction l}| \\ &< 3 \cdot 2^{-l} + 2^{-l}. \end{aligned}$$

This proves continuity of $p_{\mathcal{C}}$. □

3.5. Step 1 – construction. Now we describe the construction of the tree in Step 1 above. To build the tree T and the scheme $\{\widetilde{Z}_{\sigma\tau}\}_{(\sigma\tau)\in T}$ we inductively construct for each $n < \omega$ a finite tree T^n on $\omega \times \omega$, so that T^{n+1} is an end-extension of T^n and a severing scheme $\{\widetilde{Z}^n_{\sigma\tau}\}_{(\sigma\tau)\in T^n}$, and then we let

$$T = \bigcup_n T^n, \quad \text{and}$$

$$\widetilde{Z}_{\sigma\tau} = \widetilde{Z}^{\#\sigma}_{\sigma\tau}, \qquad \text{for } (\sigma, \tau) \in T.$$

We start with $T^0 = \{(\varnothing, \varnothing)\}$ and $\widetilde{Z}^0_{\varnothing\varnothing} = \widetilde{\ker \mathcal{D}}$. Assume that for $m \le n$ we have T^m and $\{\widetilde{Z}^m_{\sigma\tau}\}_{(\sigma\tau)\in T^m}$ constructed so that

$$\operatorname{pr} T^m = \{\sigma : \#\sigma \le m\}.$$

Then choose sufficiently small $\epsilon > 0$ and apply Lemma 3.3 to

- $S = T^n$,
- $i < \omega$ and $\eta \in \omega^{<\omega}$ such that $\#\eta^\frown i = n + 1$,
- the severing scheme
$$\{\widetilde{X}_{\sigma\tau}\}_{(\sigma\tau)\in S} = \{\widetilde{Z}^n_{\sigma\tau}\}_{(\sigma\tau)\in T^n},$$
- the sets
$$Y = Z^n_{\eta^\frown i^{\circ *}} \quad \text{and} \quad Y' = \begin{cases} \{y\} \text{ for some } y \in Y & \text{if } \eta^\frown i = \eta^\frown i^\circ, \\ Z^n_{\eta^\frown i^\circ} & \text{if } \eta^\frown i \ne \eta^\frown i^\circ. \end{cases}$$

(Note that the $Y' \subseteq \operatorname{cl} Y$ requirement of Lemma 3.3 is fulfilled)

Let

$$T^{n+1} = S', \quad \text{and}$$

$$\widetilde{Z}^{n+1}_{\sigma\tau} = \widetilde{X}'_{\sigma\tau}, \quad \text{for } (\sigma,\tau) \in T^{n+1}.$$

Then

$$\operatorname{pr} T^{n+1} = \operatorname{pr} T^n \cup \{\eta^\frown i\} = \{\sigma : \#\sigma \le n+1\},$$

so that the construction can continue and in the end we get

$$T^n = \{(\sigma,\tau) \in T : \#\sigma \le n\}.$$

Lemma 3.5. *For all $\sigma \in \omega^{<\omega}$, writing $l = \operatorname{lh}\sigma$,*

(1) *for all $\tau \in T_\sigma$ we have*

$$|\widetilde{Z}^{\#\sigma}_{\sigma\tau}| \le 2^{-l},$$

and

$$\widetilde{Z}^{\#\sigma-1}_{\sigma^*\tau^*} \supseteq \widetilde{Z}^{\#\sigma}_{\sigma\tau} \supseteq \widetilde{Z}^{\#\sigma+1}_{\sigma\tau} \supseteq \widetilde{Z}^{\#\sigma+2}_{\sigma\tau} \supseteq \cdots$$

(2) *if $\sigma \ne \sigma^\circ$, then*

$$|Z^n_\sigma, Z^n_{\sigma^\circ}| < \|\sigma, \sigma^\circ\|,$$

hence for $k > 0$,

$$|Z^n_\sigma, Z^n_{\sigma^{(k)}}| < \|\sigma, \sigma^{(k)}\|;$$

(3) *if $\sigma = \sigma^\circ$, then*

$$|Z^{\#\sigma}_\sigma| \le 2^{-l};$$

(4) *for all $n \in \omega$ we have*

$$|Z^{n+1}_\sigma, Z^n_\sigma| < 2^{-n},$$

hence for $n' > 0$, by $l \le \#\sigma$ we have

$$\begin{aligned} |Z^{n+n'}_\sigma, Z^n_\sigma| &< 2^{-(n+n'-1)} + \cdots + 2^{-n} \\ &< 2 \cdot 2^{-n} \le 2 \cdot 2^{-l}. \end{aligned}$$

Proof: We need to verify (2) and (3). Lemma 3.3 and the smallness of ϵ take care of (1), (4), and the remaining properties of the severing schemes. So, pick $\sigma \in \operatorname{pr} T^{n+1}$ and consider the following cases.

Case 1. $\sigma \ne \eta^\frown i$: We only need to verify (2). We have

$$\begin{aligned} |Z^{n+1}_\sigma, Z^{n+1}_{\sigma^\circ}| &\le |Z^{n+1}_\sigma, Z^n_\sigma| + |Z^n_\sigma, Z^n_{\sigma^\circ}| + |Z^n_{\sigma^\circ}, Z^{n+1}_{\sigma^\circ}| \\ &< \epsilon + |Z^n_\sigma, Z^n_{\sigma^\circ}| + \epsilon \\ &< \|\sigma, \sigma^\circ\|, \end{aligned}$$

where the second inequality is by Lemma 3.3(2) and the third one by the inductive hypothesis and the smallness of ϵ.

Case 2. $\sigma = \eta^\frown i = \eta^\frown i^\circ$: We only need to verify (3). Note that if $\eta^\frown i = \eta^\frown i^\circ$, then $\eta = (\eta^\frown i^\circ)^*$, so then $Y = Z^n_\eta$. By Lemma 2.2(2), and 3.3(3) and the choice of Y and Y' we have

$$\begin{aligned} |Z^{n+1}_{\eta^\frown i}| &\le 2 \cdot |Z^{n+1}_{\eta^\frown i}, Y'| + |Y'| \\ &< 2 \cdot |Z^n_\eta, Y| + 2\epsilon + 0 = 2 \cdot |Z^n_\eta, Z^n_\eta| + 2\epsilon = 2 \cdot 0 + 2\epsilon = 2\epsilon, \end{aligned}$$

We are done by the smallness of ϵ.

Case 3. $\sigma = \eta^\frown i \neq \eta^\frown i^\circ$: We only need to verify (2). We have

$$\begin{aligned}|Z^{n+1}_{\eta^\frown i}, Z^{n+1}_{\eta^\frown i^\circ}| &\leq |Z^{n+1}_{\eta^\frown i}, Z^{n}_{\eta^\frown i^\circ}| + |Z^{n}_{\eta^\frown i^\circ}, Z^{n+1}_{\eta^\frown i^\circ}| \\ &< |Z^{n+1}_{\eta^\frown i}, Z^{n}_{\eta^\frown i^\circ}| + \epsilon = |Z^{n+1}_{\eta^\frown i}, Y'| + \epsilon \\ &< |Z^{n}_{\eta}, Y| + 2\epsilon = |Z^{n}_{\eta}, Z^{n}_{\eta^\frown i^{\circ *}}| + 2\epsilon,\end{aligned}$$

where the second inequality follows from Lemma 3.3(2) and the third one from Lemma 3.3(3) and the choice of Y and Y'.

Now, if $\eta = \eta^\frown i^{\circ *}$, then $|Z^n_\eta, Z^n_{\eta^\frown i^{\circ *}}| = 0$ and we are done by the smallness of ϵ. If $\eta \neq \eta^\frown i^{\circ *}$, then $\min\max\eta = \min\max\eta^\frown i$, so $\eta^\frown i^{\circ *} = \eta^\circ$. Thus, by the smallness of ϵ and the inductive hypothesis,

$$\begin{aligned}|Z^{n}_{\eta}, Z^{n}_{\eta^\frown i^{\circ *}}| + 2\epsilon &= |Z^{n}_{\eta}, Z^{n}_{\eta^\circ}| + 2\epsilon \\ &< \|\eta, \eta^\circ\| = \|\eta^\frown i, \eta^\frown i^\circ\|,\end{aligned}$$

where the last equality holds because of the changes occurring at the same place.

This ends the proof of the Lemma. □

To see that the assertions (**A**),(**B**) and (**C**) of Step 1 in 3.4 are satisfied, do the following.

- For (**B**), apply Lemma 2.4 to the Luzin scheme whose n-th level is $\{f(Z^n_{\sigma\tau}) : (\sigma, \tau) \in T^n\}$.
- For vanishing diameters, in (**A**) use Lemma 3.5(1), and in (**C**) use

  $$\begin{aligned}|Z_\sigma| &\leq 2 \cdot |Z^{\#\sigma}_{\sigma}, Z^{\#\sigma}_{\sigma^{(l)}}| + |Z^{\#\sigma}_{\sigma^{(l)}}| \\ &\leq 2 \cdot \|\sigma, \sigma^{(l)}\| + 2^{-l} \\ &< 2 \cdot 2 \cdot 2^{-l} + 2^{-l},\end{aligned}$$

  where the first inequality follows from Lemma 2.2(2), the second one from (2) and (3) of Lemma 3.5, using $(\sigma^{(l)})^\circ = \sigma^{(l)}$, and the last one from Lemma 2.4(3).
- For the estimation in (**C**), use Lemma 2.4(4) to get $k \leq \operatorname{lh}\sigma$ with $\rho^{(k)} = \sigma$. Then, by Lemma 3.5(2) and Lemma 3.5(4) applied to ρ, using $\#\rho \geq \#\rho^{(k)}$, we get

  $$\begin{aligned}|Z_\rho, Z_{\rho^{(k)}}| &\leq |Z^{\#\rho}_{\rho}, Z^{\#\rho}_{\rho^{(k)}}| + |Z^{\#\rho}_{\rho^{(k)}}, Z^{\#\rho^{(k)}}_{\rho^{(k)}}| \\ &< \|\rho, \rho^{(k)}\| + 2 \cdot 2^{-l} \\ &< 4^{-\max(l,m)} + 2 \cdot 2^{-l} \\ &\leq 3 \cdot 2^{-l}\end{aligned}$$

- To see that $\{\widetilde{Z}_{\sigma\tau}\}_{(\sigma\tau)\in T}$ is Luzin, for the inclusion condition use

  $$\widetilde{Z}^{\#\sigma}_{\sigma\tau} \subseteq \widetilde{Z}^{\#\sigma-1}_{\sigma^*\tau^*} \subseteq \widetilde{Z}^{\#\sigma^*}_{\sigma^*\tau^*},$$

  where the first inclusion follows from Lemma 3.5(1) and the second from $\#\sigma^* \leq \#\sigma$ and Lemma 3.5(1) applied to σ^*. To see the intersection condition suppose that $(\sigma_0, \tau_0) \neq (\sigma_1, \tau_1)$ and $(\sigma_0^*, \tau_0^*) = (\sigma_1^*, \tau_1^*)$. If $\sigma_0 = \sigma_1$ then $\widetilde{Z}^{\#\sigma_0}_{\sigma_0\tau_0}$ and $\widetilde{Z}^{\#\sigma_1}_{\sigma_1\tau_1}$ are distinct sets from the same severing scheme and thus are disjoint. If $\sigma_0 \neq \sigma_1$, say $\#\sigma_1 < \#\sigma_0$, then

  $$\widetilde{Z}^{\#\sigma_0}_{\sigma_0\tau_0} \subseteq \widetilde{Z}^{\#\sigma_0-1}_{\sigma_0^*\tau_0^*} \subseteq \widetilde{Z}^{\#\sigma_1}_{\sigma_0^*\tau_0^*} = \widetilde{Z}^{\#\sigma_1}_{\sigma_1^*\tau_1^*},$$

  where the first inclusion is by Lemma 3.5(1) applied to σ_0, the second by

  $$\#\sigma_0^* = \#\sigma_1^* \leq \#\sigma_1 < \#\sigma_0$$

and Lemma 3.5(1) applied to σ_0^*, and the equality holds by $(\sigma_0^*, \tau_0^*) = (\sigma_1^*, \tau_1^*)$. Since $\widetilde{Z}^{\#\sigma_1}_{\sigma_1^*\tau_1^*}$ and $\widetilde{Z}^{\#\sigma_1}_{\sigma_1\tau_1}$ are disjoint, as distinct sets from the same severing scheme, it follows that $\widetilde{Z}^{\#\sigma_0}_{\sigma_0\tau_0}$ and $\widetilde{Z}^{\#\sigma_1}_{\sigma_1\tau_1}$ are also disjoint.

This finishes the proof of Theorem 1.1 modulo the Crucial Lemma.

□

4. Proof of the Crucial Lemma

In this section we prove the Crucial Lemma. We first need some auxuliary lemmas.

Lemma 4.1. *Suppose that X is solid analytic. Then there are solid analytic sets $X_0, X_1 \subseteq X$ such that*

- $X_0 \subseteq \mathrm{cl} X_1$,
- *and the family $\{f(X_0), f(X_1)\}$ is relatively discrete.*

Proof: There exists a basic open set $U \subseteq \mathcal{D}$ such that $f^{-1}(U) \cap X$ is not relatively open in X modulo a small set. Indeed, otherwise, removing from X countably many offending sets of the form $f^{-1}(V) \cap X$, for basic open sets $V \subseteq \mathcal{D}$, we get $X' \subseteq X$ such that $f^{-1}(U) \cap X'$ is relatively open in X' for each basic open $U \subseteq \mathcal{D}$. This makes f continuous on X', and X becomes the union of the set X', which is small, and countably many small offending sets. Hence X itself is countably continuous, which contradicts our assumption that it is nonempty and solid.

Fix $U \subseteq \mathcal{D}$ as above and put $Z = X \smallsetminus f^{-1}(U)$. Note that both Z and $X \cap f^{-1}(U) \cap \mathrm{cl}\, Z$ are positive. Let $U = \bigcup_{n<\omega} V_n$ with $V_n \subseteq \mathcal{D}$ open such that $\mathrm{cl}\, V_n \subseteq V_{n+1}$. Note that for some $n \in \omega$ the set $X \cap f^{-1}(V_n) \cap \mathrm{cl}\, Z$ is positive.

Fix $n \in \omega$ as above and put $X_1 = X \smallsetminus f^{-1}(\mathrm{cl}\, V_{n+1})$. Then X_1 is an analytic solid set, as a simple subset of the analytic solid set X, by Lemma 3.1(3). Since $f^{-1}(V_n)$ is disjoint from X_1, the set

$$X \cap f^{-1}(V_n) \cap \big(\mathrm{cl}\, X_1 \smallsetminus X_1\big) = X \cap f^{-1}(V_n) \cap \mathrm{cl}\, X_1$$

contains $X \cap f^{-1}(V_n) \cap \mathrm{cl}\, Z$ and thus is positive.

Since $X \cap f^{-1}(V_n) \cap \big(\mathrm{cl}\, X_1 \smallsetminus X_1\big)$ is positive, by Lemma 3.1 there is a solid analytic set

$$X_0 \subseteq X \cap f^{-1}(V_n) \cap \big(\mathrm{cl}\, X_1 \smallsetminus X_1\big).$$

Since $X_0 \subseteq \mathrm{cl}\, X_1$, the sets X_0 and X_1 are as needed. This ends the proof. □

Below we use int_Y to denote the topological interior relatively to a subspace Y; cl denotes the topological closure in the surrounding topological space, which should be clear from the context.

Lemma 4.2. *Let $\mathcal{X}$ be a topological space and let X_0, $X_1 \subseteq X \subseteq \mathcal{X}$ be such that*

$$X_0 \cup X_1 \ \textit{ is dense in } \ X.$$

Then for any $Y \subseteq \mathrm{cl}\, X$,

$$\mathrm{int}_Y(Y \cap \mathrm{cl}\, X_0) \cup \mathrm{int}_Y(Y \cap \mathrm{cl}\, X_1) \ \textit{ is dense in } \ Y.$$

Proof: Note that since $X = \mathrm{cl}\, X_0 \cup \mathrm{cl}\, X_1$, we have $Y = (Y \cap \mathrm{cl}\, X_0) \cup (Y \cap \mathrm{cl}\, X_1)$. It is enough to show that if $Y = Y_0 \cup Y_1$ with Y_0 and Y_1 closed in Y, then $\mathrm{int}_Y\, Y_0 \cup \mathrm{int}_1\, Y_1$ is dense in Y. But if $U \subseteq Y$ is open in Y and disjoint from $\mathrm{int}_Y\, Y_0 \cup \mathrm{int}_1\, Y_1$, then $Y_0 \cap U$ and $Y_1 \cap U$ are relatively closed nowhere dense on U, which contradicts the fact that $U = (U \cap Y_0) \cup (U \cap Y_1)$. □

The following lemma is a topological version of the pigeonhole principle. Given a set X in a metric space and $\epsilon > 0$, we say that a set $E \subseteq X$ is an *ϵ-net for X* if for all $x \in X$ we have $|x, E| \leq \epsilon$.

Lemma 4.3. *Let $\epsilon > 0$, $M < \omega$, and let $\mathcal{X}$ be a compact metric space with its subspace X and sets $\{X_m \subseteq X : m < M\}$. There is $L < \omega$ such that for any M-many families*

$$\{X_m^l \subseteq X_m : l < L\}, \quad m < M,$$

if for all $m < M$ and all $l \neq l' < L$,

$$X_m^l \cup X_m^{l'} \text{ is dense in } X_m,$$

then there exists $\ell < L$ such that for each m the set X_m^ℓ is nonempty and is an ϵ-net for X_m.

Proof: Using compactness of $\mathcal{X}$, find in each X_m a finite ϵ-net E_m. Let $L < \omega$ be any number greater than the sum of all cardinalities of these nets.

Suppose that the sets X_m^l are as postulated. It is enough to find ℓ such that for each m we have $E_m \subseteq \operatorname{cl} X_m^\ell$. If every ℓ fails at some point of some E_m, then there are $l \neq l'$ that fail at the same point of the same E_m violating $X_m \subseteq \operatorname{cl} X_m^l \cup \operatorname{cl} X_m^{l'}$. $\square$

Finally, we prove the Crucial Lemma 3.3.

Proof of the Crucial Lemma: The proof proceeds in three steps.
Step 1. Choose large enough $L < \omega$. For $\vartheta \in S_\eta$ let

$$S/(\eta, \vartheta) = \{(\sigma, \tau) \in S : \eta \subseteq \sigma, \vartheta \subseteq \tau\}.$$

Claim 1. There is $J < \omega$ such that in each $X_{\eta\vartheta}$, with $\vartheta \in S_\eta$, there are $\epsilon/3$-nets

$$\{e_{\eta\vartheta}^{lj} : j < J\} \subseteq X_{\eta\vartheta}, \quad l < L,$$

such that all points

$$f(e_{\eta\vartheta}^{lj}), \quad l < L, \, j < J, \, \vartheta \in S_\eta$$

are distinct.

Proof: Since $X_{\eta\vartheta}$ is solid, its nonempty open balls are positive and have uncountable f-images. The claim follows from compactness of $\operatorname{cl} X_{\eta\vartheta}$. $\square_{\textit{Claim 1}}$.

Let $J \in \omega$ be as in Claim 1 and for each $\vartheta \in S_\eta$ let $I_{\eta\vartheta}$ be a set of cardinality J such that $I_{\eta\vartheta} \cap I_{\eta\vartheta'} = \emptyset$ if $\vartheta, \vartheta' \in S_\eta$ and $\vartheta \neq \vartheta'$. Let $\{e_{\eta\vartheta}^{lj} : j \in I_{\eta,\vartheta}\} \subseteq X_{\eta\vartheta}, l < L$ be as in Claim 1. Let $\delta > 0$ be the least distance between the points $f(e_{\eta\vartheta}^{lj})$. For $l < L$, $\vartheta \in S_\eta$, and $(\sigma, \tau) \in S/(\eta, \vartheta)$, inductively on the rank in $S/(\eta, \vartheta)$, define solid analytic sets $X_{\sigma\tau}^l \subseteq X_{\sigma\tau}$ as follows.

- If (σ, τ) is minimal, i.e., $(\sigma, \tau) = (\eta, \vartheta)$, note that for $j \in I_{\eta\vartheta}$ the set

  $$Z_{\eta\vartheta}^{lj} = \{x \in X_{\eta\vartheta} : |x, e_{\eta\vartheta}^{lj}| < \epsilon/3 \ \wedge \ |f(x), f(e_{\eta\vartheta}^{lj})| < \delta/3\}$$

  is solid analytic by Lemma 3.1(5). By Lemma 4.1, it has solid analytic subsets $X_{\eta\vartheta0}^{lj}$ and $X_{\eta\vartheta1}^{lj}$ such that

  $$X_{\eta\vartheta0}^{lj} \subseteq \operatorname{cl} X_{\eta\vartheta1}^{lj} \quad \text{and} \quad \{f(X_{\eta\vartheta0}^{lj}), f(X_{\eta\vartheta1}^{lj})\} \text{ is relatively discrete.} \tag{4.1}$$

  Let

  $$E_{\eta\vartheta}^l = \{x \in X_{\eta\vartheta} : \forall j \in I_{\eta,\vartheta} \ |f(x), f(e_{\eta\vartheta}^{lj})| > \delta/3\}.$$

  Note that $E_{\eta\vartheta}^l$ is an analytic solid set by Lemma 3.1(5), as it is the preimage in $X_{\eta,\vartheta}$ of the open set $V = \{v \in \mathcal{R} : \forall j \in I_{\eta\vartheta} \ |v, f(e_{\eta\vartheta}^{lj})| > \delta/3\}$. Put

  $$X_{\eta\vartheta}^l = E_{\eta\vartheta}^l \cup \bigcup_{j \in I_{\eta\vartheta}} X_{\eta\vartheta1}^{lj},$$

  and note that as a union of finitely many analytic solid sets, $X_{\eta\vartheta}^l$ is an analytic solid set by Lemma 3.1(4).

- If $(\sigma,\tau) \in S/(\eta,\vartheta)$ is not minimal, put

$$X^l_{\sigma\tau} = \operatorname{int}_{X_{\sigma\tau}} \big(X_{\sigma\tau} \cap \operatorname{cl} X^l_{\sigma^*\,\tau^*}\big). \tag{4.2}$$

Note that if $X^l_{\sigma,\tau} \neq \emptyset$, then $X^l_{\sigma,\tau}$ is an analytic solid set by Lemma 3.1(6).

Claim 2. For $\vartheta \in S_\eta$ and $(\sigma,\tau) \in S/(\eta,\vartheta)$, if $l \neq l'$ then

$$X^l_{\sigma\tau} \cup X^{l'}_{\sigma\tau} \text{ is dense in } X_{\sigma\tau}.$$

Proof: The proof is by induction on the rank in $S/(\eta,\vartheta)$. By the way δ was chosen, $X_{\eta\vartheta} = E^l_{\eta\vartheta} \cup E^{l'}_{\eta\vartheta}$, which takes care of the minimal case of $(\sigma,\tau) = (\eta,\vartheta)$. If (σ,τ) is not minimal, use $X_{\sigma\tau} \subseteq \operatorname{cl} X_{\sigma^*\,\tau^*}$, the inductive hypothesis, and Lemma 4.2. $\square_{\textit{Claim 2}}$

Step 2. Now we use the pigeonhole principle. Since L is large, by Claim 2 and Lemma 4.3, find $\ell < L$ such that for each $\vartheta \in S_\eta$ and $(\sigma,\tau) \in S/(\eta,\vartheta)$ we have $X^\ell_{\sigma\tau} \neq \emptyset$ and

$$|X^\ell_{\sigma\tau}, X_{\sigma\tau}| < \epsilon. \tag{4.3}$$

Note that in particular each $X^\ell_{\sigma\tau}$ is an analytic solid set by the remark after (4.2).

Write $I = \bigcup_{\vartheta\in S_\eta} I_{\eta\vartheta}$ and for each $j \in I$ let $\vartheta(j)$ be the unique $\vartheta \in S_\eta$ such that $j \in I_{\eta\vartheta}$.

Claim 3. There is a nonempty set $I' \subseteq I$ such that

$$|\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I'}, Y'| < |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| + \epsilon/3.$$

Proof: Pick ϵ' such that $0 < \epsilon' < \epsilon$. We have $Y' \subseteq \operatorname{cl} Y$ and

$$|\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| = |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, \operatorname{cl} Y|.$$

Put

$$I' = \{j \in I : |e^{\ell j}_{\eta\vartheta(j)}, Y'| < |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| + \epsilon'/3\}.$$

We claim that I' is as needed.

To see that I' is nonempty, pick any $y_0 \in Y'$ and let $j_0 \in I$, be such that $|e^{lj_0}_{\eta\vartheta(j_0)}, y_0| = |\{e^{lj}_{\eta\vartheta(j)}\}_{j\in I}, y_0|$ Then

$$\begin{aligned} |e^{\ell j_0}_{\eta\vartheta(j_0)}, Y'| &\le |e^{\ell j_0}_{\eta\vartheta(j_0)}, y_0| = |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, y_0| \le \sup_{y\in Y'} |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, y| \le \sup_{y\in \operatorname{cl} Y} |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, y| \\ &\le |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, \operatorname{cl} Y| = |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| < |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| + \epsilon'/3, \end{aligned}$$

which shows that $j_0 \in I'$. To prove the claim we will show that

- if $j \in I'$, then $|e^{\ell j}_{\eta\vartheta(j)}, Y'| < |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| + \epsilon'/3$,
- if $y' \in Y'$, then $|\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I'}, y'| < |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| + \epsilon'/3$.

The first item follows directly from the definition of the set I'. To see the second one, note that if $y' \in Y'$ and $j \in I \setminus I'$, then

$$|y', e^{\ell j}_{\eta\vartheta(j)}| \ge |Y', e^{\ell j}_{\eta\vartheta(j)}| \ge |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j<J}, Y| + \epsilon'/3,$$

where the second inequality follows from the definition of the set I'. On the other hand,

$$\begin{aligned} |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, y'| &\le \sup_{y\in Y'} |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, y| \le \sup_{y\in \operatorname{cl} Y} |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, y| = \sup_{y\in Y} |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, y| \\ &\le |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| < |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| + \epsilon'/3 \end{aligned}$$

and since $|\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, y'| = \min(|\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I'}, y'|, |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I\setminus I'}, y'|)$, it must be the case that

$$|\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I'}, y'| < |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| + \epsilon'/3,$$

as needed. This ends the proof of the Claim. $\square_{\textit{Claim 3}}$

Step 3. Now we define the sets S', $X'_{\sigma\tau}$ and $\widetilde{X}'_{\sigma\tau}$. Let $I' \subseteq I$ be as in Claim 3. For each $\vartheta \in S_\eta$ let $J_{\eta\vartheta} = I_{\eta\vartheta} \cap I'$ and let

$$S' = S \cup \{(\eta, \vartheta)^\frown(i, j) : \vartheta \in S_\eta,\ j \in J_{\eta\vartheta}\}.$$

Note that some of the sets $J_{\eta\vartheta}$ may be empty but $S'_{\eta^\frown i} \neq \emptyset$ as I' is nonempty.

Now we define the sets $X'_{\sigma\tau}$:

- if $(\sigma, \tau) = (\eta^\frown i, \vartheta^\frown j)$, use 3.2 to get a solid cylinder $\widetilde{X}'_{\eta^\frown i\,\vartheta^\frown j} \subseteq \widetilde{X}_{\eta\vartheta}$ with diameter less than ϵ and base inside $X^{\ell j}_{\eta\vartheta 0}$;
- if $(\sigma, \tau) \in S/(\eta, \vartheta)$, $\vartheta \in S_\eta$, let $\widetilde{X}'_{\sigma\tau}$ be a cylinder that unfolds $X^{\ell}_{\sigma\tau}$ inside $\widetilde{X}_{\sigma\tau}$;
- otherwise put $\widetilde{X}'_{\sigma\tau} = \widetilde{X}_{\sigma\tau}$.

Claim 4. $\{\widetilde{X}'_{\sigma\tau} : (\sigma, \tau) \in S'\}$ is a severing scheme with the desired properties.

Proof: First, $\{f(X'_{\sigma\tau})\}_{(\sigma\tau)\in S'}$ is relatively discrete by (4.1), the choice of δ, the definition of $Z^{lj}_{\eta\vartheta}$ and the assumption that $\{f(X_{\sigma\tau})\}_{(\sigma\tau)\in S}$ is relatively discrete. To see that $X'_{\sigma\tau} \subseteq \operatorname{cl} X'_{\sigma^*\,\tau^*}$ use

- (4.1) if $(\sigma, \tau) = (\eta, \vartheta)^\frown(i, j)$, $j \in J_{\eta\vartheta}$,
- (4.2) if $(\sigma, \tau) \in S/(\eta, \vartheta)$ is not minimal,
- the inductive hypothesis if $(\sigma, \tau) = (\eta, \vartheta)$ or $(\sigma, \tau) \in S \setminus S/(\eta, \vartheta)$.

Now, in the Crucial Lemma 3.3, (1) follows immediately from the way $\widetilde{X}'_{\sigma\tau}$ were defined and (2) follows from (4.3). To see (3) note that since $X'_{\eta^\frown i\,\vartheta^\frown j} \subseteq X^{\ell j}_{\eta\vartheta 0} \subseteq Z^{\ell j}_{\eta\vartheta}$

- $X'_{\eta^\frown i}$ is contained in $\epsilon/3$-neighborhood of $\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I'}$
- and $\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I'}$ is contained in $\epsilon/3$-neighborhood of $X'_{\eta^\frown i}$,

which means that $|X'_{\eta^\frown i}, \{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I'}| \leq \epsilon/3$. Also, since $\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I_{\eta\theta}}$ is an $\epsilon/3$-net in $X_{\eta\theta}$, the set $\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}$ is an $\epsilon/3$-net in X_η, so $|\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, X_\eta| \leq \epsilon/3$. Using Lemma 2.2(3) and Claim 3, we get

$$|X'_{\eta^\frown i}, Y'| \leq |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I'}, Y'| + \epsilon/3 < |\{e^{\ell j}_{\eta\vartheta(j)}\}_{j\in I}, Y| + 2\epsilon/3 \leq |X_\eta, Y| + \epsilon,$$

which proves (3). $\square_{\textit{Claim 4}}$

We have proved the Crucial Lemma and thus completed the proof of Theorem 1.1. $\square$

5. Estimationss of Borel complexity of witnessing sets

In this section we give estimates on the possible complexity of witnessing sets for countably continuous functions.

Lemma 5.1. *Suppose that $\mathcal{D}$ is an analytic space and $\mathcal{R}$ is a separable metrizable space. If $f : \mathcal{D} \to \mathcal{R}$ is countably continuous, then witnessing sets can be chosen of the form $G \cap A$, where G is $\mathbf{\Pi}^0_2$ and A is in* $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_1$.

Proof: Suppose that $\mathcal{D} = \bigcup_{n<\omega} X_n$ and f is continuous on each X_i. Let $\overline{\mathcal{R}}$ be a metric compactification of $\mathcal{R}$. By the Kuratowski extension theorem [10, Theorem 3.8] there are $\mathbf{\Pi}^0_2$ sets $G_n \subseteq \mathcal{D}$ and continuous functions $g_n : G_n \to \overline{\mathcal{R}}$ such that $X_n \subseteq G_n$ and g_n extends the restriction of f to X_n.

Let $E_n = \{x \in G_n : g_n(x) = f(x)\}$. We claim that E_n is the intersection of G_n with a set in $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_1$. Indeed, the graph of g_n is relatively closed in $G_n \times \mathcal{R}$, say $g_n = (G_n \times \mathcal{R}) \cap F$ for a closed set $F \subseteq \mathcal{D} \times \mathcal{R}$. Then $E_n = G_n \cap (\mathrm{id}, f)^{-1}(F)$.

Now $X_n \subseteq E_n$, so $\mathcal{D} = \bigcup_{n<\omega} E_n$ and f is continuous on each E_n. $\square$

The complexity of the sets in $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_1$ seems to be crucial in estimating the possible complexity of witnessing sets.

Lemma 5.2. *Suppose that $\mathcal{D}$ is an analytic space, $\mathcal{R}$ is a separable metrizable space and $f : \mathcal{D} \to \mathcal{R}$ is a function. Let $\alpha > 1$ be a countable ordinal. Then $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_1 \subseteq \mathbf{\Pi}^0_\alpha$ if and only if $f^{-1}\mathbf{\Pi}^0_1 \subseteq \mathbf{\Pi}^0_\alpha$.*

Proof: For the nontrivial implication, note that if $U = \bigcup_{n\in\omega} V_n \times W_n$ with $V_n \subseteq \mathcal{D}$ and $W_n \subseteq \mathcal{R}$ open, then $(\mathrm{id}, f)^{-1}(U) = \bigcup_{n<\omega}(V_n \cap f^{-1}(W_n))$ belongs to $\mathbf{\Sigma}^0_\alpha$ since every $f^{-1}(W_n)$ is $\mathbf{\Sigma}^0_\alpha$. □

6. Extensions of the Jayne and Rogers theorem

In this section we prove Theorem 1.2 and Corollary 1.3. We will use the following lemma, which appears in [13, Proposition 6.6.]. We provide a proof for the sake of completeness.

Lemma 6.1. *$P^{-1}\mathbf{\Pi}^0_n \not\subseteq \mathbf{\Pi}^0_n$ for each $n > 0$.*

Proof: For each $n \in \omega$ we construct a $\mathbf{\Sigma}^0_n$ set $A_n \subseteq \mathcal{N}$ such that $P^{-1}(A_n)$ is $\mathbf{\Sigma}^0_{n+1}$-complete (see [10, Definition 22.9]). The construction is by induction on n and uses the following fact (see [16], [5] or [10, Exercise 23.3]): given a sequence of Polish spaces X_k, a countable ordinal $\alpha > 0$ and $\mathbf{\Sigma}^0_\alpha$-complete sets $B_k \subseteq X_k$, the set $\{x \in \prod_{k<\omega} X_k : \exists i \in \omega \quad x(i) \notin B_i\}$ is $\mathbf{\Sigma}^0_{\alpha+1}$-complete.

To begin with, take $A_0 = \{y \in \mathcal{N} : y(0) \neq 0\}$. Note that $P^{-1}(A_0) = \{x \in \mathcal{C} : x(0) \neq 0\}$ is open but not closed and hence $\mathbf{\Sigma}^0_1$-complete. To see this use determinacy of open games and [10, Theorem 22.10]. For a proof avoiding games, define $c_k \in \mathcal{C}$ by $c_k(0) = k$ and $c_k(i) = 0$ for $i > 0$ and given an open set $G \subseteq \mathcal{N}$, send $\mathcal{N} \setminus G$ to c_0, write G as a disjoint union of countably many clopen sets and send them to distinct c_k for $k \neq 0$. This defines a reduction from G to $P^{-1}(A_0)$.

Now suppose that we have found a $\mathbf{\Sigma}^0_n$ set $A_n \subseteq \mathcal{N}$ such that $P^{-1}(A)$ is $\mathbf{\Sigma}^0_{n+1}$-complete. To construct the set A_{n+1} we use the identification of the space $\mathcal{C}$ with its countable power $\mathcal{C}^\omega$, the space $\mathcal{N}$ with its countable power $\mathcal{N}^\omega$ and the function P with its countable power P^ω (this identification is done via a fixed bijection between ω and $\omega \times \omega$ using the fact that $P : \omega^\omega \to \omega^\omega$ is the identity function). We find a $\mathbf{\Sigma}^0_{n+1}$ set $A_{n+1} \subseteq \mathcal{N}^\omega$ such that $(P^\omega)^{-1}(A_{n+1})$ is $\mathbf{\Sigma}^0_{n+2}$-complete. Put

$$A_{n+1} = \{(y_i : i < \omega) \in \mathcal{N}^\omega : \exists i \in \omega \quad y_i \notin A_n\}.$$

Then A_{n+1} is $\mathbf{\Sigma}^0_{n+1}$ and $(P^\omega)^{-1}(A_{n+1}) = \{(x_i : i < \omega) \in \mathcal{C}^\omega : \exists i \in \omega \quad x_i \notin P^{-1}(A_n)\}$. The latter is $\mathbf{\Sigma}^0_{n+2}$-complete by the remark preceeding this paragraph. □

Lemma 6.2. *Suppose that $\mathcal{D}$ is an analytic space and $\mathcal{R}$ is a separable metrizable space. Let $n > 0$ be a natural number. If a function $f : \mathcal{D} \to \mathcal{R}$ is such that $f^{-1}\mathbf{\Pi}^0_{n+1} \subseteq \mathbf{\Pi}^0_{n+1}$ and $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_1 \subseteq \mathbf{\Sigma}^0_{n+1}$, then f is countably continuous with $\mathbf{\Pi}^0_n$ witnessing sets.*

Proof: If $n = 1$, then $f^{-1}\mathbf{\Pi}^0_2 \subseteq \mathbf{\Pi}^0_2$ and we are done by the Jayne and Rogers theorem.

Assume that $n > 1$. Theorem 1.1 and Lemma 6.1 imply that f is countably continuous. Indeed, otherwise there would be topological embeddings $p_\mathcal{C}$ and $p_\mathcal{N}$ as in Theorem 1.1. Since $f^{-1}\mathbf{\Pi}^0_{n+1} \subseteq \mathbf{\Pi}^0_{n+1}$ and $p_\mathcal{C}$ and $p_\mathcal{N}$ are topological embeddings, this would imply that $P^{-1}\mathbf{\Pi}^0_{n+1} \subseteq \mathbf{\Pi}^0_{n+1}$, contradicting Lemma 6.1.

By Lemma 5.1, as witnessing sets we can choose intersections of the form $G \cap A$, where G is $\mathbf{\Pi}^0_2 \subseteq \mathbf{\Sigma}^0_{n+1}$ and A is from $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_1 \subseteq \mathbf{\Sigma}^0_{n+1}$, i.e. witnessing sets can be chosen from $\mathbf{\Sigma}^0_{n+1}$. Decomposing further into countable unions of $\mathbf{\Pi}^0_n$ sets, we get $\mathbf{\Pi}^0_n$ witnessing sets, as needed. □

Proof of Theorem 1.2: (1)⇒(2). Suppose that f is countably continuous with $\mathbf{\Pi}^0_n$ witnessing sets $\{X_i : i < \omega\}$. It is enough to show that $(\mathrm{id}, f)^{-1}\mathbf{\Sigma}^0_{n+1} \subseteq \mathbf{\Sigma}^0_{n+1}$. If $A \subseteq \mathcal{D} \times \mathcal{R}$ is $\mathbf{\Sigma}^0_{n+1}$, then $(\mathrm{id}, f)^{-1}(A)$ is the union of the sets $(\mathrm{id}, f)^{-1}(A) \cap X_i$, which are $\mathbf{\Sigma}^0_{n+1}$, as (id, f) is continuous on X_i and $X_i \in \mathbf{\Pi}^0_n$.

(2)⇒(1). Suppose now that $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_{n+1} = \mathbf{\Pi}^0_{n+1}$. Then $f^{-1}\mathbf{\Pi}^0_{n+1} \subseteq \mathbf{\Pi}^0_{n+1}$. Also, $\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Pi}^0_{n+1}$ yields $(\mathrm{id}, f)^{-1}\mathbf{\Sigma}^0_1 \subseteq \mathbf{\Pi}^0_{n+1}$, so $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_1 \subseteq \mathbf{\Sigma}^0_{n+1}$ and Lemma 6.2 ends the proof.

□

Proof of Corollary 1.3: If f is 1-1 and open, then $(\mathrm{id}, f)^{-1}\mathbf{\Sigma}^0_1 = f^{-1}\mathbf{\Sigma}^0_1$. Indeed, if $U = \bigcup_{k<\omega} V_k \times W_k$, where $V_k \subseteq \mathcal{D}$ and $W_k \subseteq \mathcal{R}$ are open, then $(\mathrm{id}, f)^{-1}(U)$ is the f-preimage of the open set $\bigcup_{k<\omega} f(V_k) \cap W_k$. This implies that $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_n = f^{-1}\mathbf{\Pi}^0_n$ for each $n > 0$. Thus, in this case the corollary follows from Theorem 1.2.

Suppose f is Baire class $n-1$ and $n > 1$ (for $n = 1$ we are done by the Jayne and Rogers theorem). The implication from (1) to (2) follows from Theorem 1.2 since $f^{-1}\mathbf{\Pi}^0_{n+1} \subseteq (\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_{n+1}$. For the other implication, use Lemma 6.2, noting that $(\mathrm{id}, f)^{-1}\mathbf{\Pi}^0_1 \subseteq \mathbf{\Pi}^0_n$. The latter is true by Lemma 5.2 and the fact that $f^{-1}\mathbf{\Pi}^0_1 \subseteq \mathbf{\Pi}^0_n$, which follows from the assumption that f is of Baire class $n-1$. □

7. Conjectures and questions

Semmes [15, Theorem 1.0.3] proved that a function $f : \mathcal{N} \to \mathcal{N}$ is countably continuous with $\mathbf{\Pi}^0_2$ witnessing sets if and only if $f^{-1}\mathbf{\Pi}^0_3 \subseteq \mathbf{\Pi}^0_3$. Corollary 1.3 extends this result for arbitrary $n < \omega$ under the assumption that f is Baire class $n-1$. The following seems to be feasable.

Conjecture 7.1. *For each analytic space $\mathcal{D}$, separable metrizable space $\mathcal{R}$, function $f : \mathcal{D} \to \mathcal{R}$ and natural number $n > 0$ the following are equivalent:*

(1) *f is countably continuous with $\mathbf{\Pi}^0_n$ witnessing sets,*
(2) $f^{-1}\mathbf{\Pi}^0_{n+1} \subseteq \mathbf{\Pi}^0_{n+1}$.

In [15, Theorem 1.0.2] Semmes also proved that given $f : \mathcal{N} \to \mathcal{N}$, f can be decomposed into countably many Baire class 1 functions with $\mathbf{\Pi}^0_2$ domains if and only if $f^{-1}\mathbf{\Pi}^0_2 \subseteq \mathbf{\Pi}^0_3$. This suggests the following conjecture, which is even more general than Conjecture 7.1 (under the convention that Baire class 0 functions are the continuous functions).

Conjecture 7.2. *For each analytic space $\mathcal{D}$, separable metrizable space $\mathcal{R}$, function $f : \mathcal{D} \to \mathcal{R}$ and natural numbers k and $n > 0$ the following are equivalent:*

(1) *f is a union of countably many Baire class k functions with $\mathbf{\Pi}^0_n$ domains,*
(2) $f^{-1}\mathbf{\Pi}^0_{n+1-k} \subseteq \mathbf{\Pi}^0_{n+1}$.

It should be noted that Conjecture 7.2 would provide a complete classification of functions of finite Baire class, as every such function satisfies (2) of Conjecture 7.2 for some n and k.

Finally, it would be very interesting to find a generalization of the theorem of Jayne and Rogers to infinite Borel classes.

Question 7.3. *Is there an analogue of the theorem of Jayne and Rogers characterizing functions which are countably continuous with $\mathbf{\Pi}^0_\alpha$ (or $\mathbf{\Sigma}^0_\alpha$) witnessing sets for any fixed $\alpha < \omega_1$?*

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