EXTREME AMENABILITY OF ABELIAN L_0 GROUPS

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ABSTRACT. We show that for any abelian topological group G and arbitrary diffused submeasure μ, every continuous action of $L_0(\mu, G)$ on a compact space has a fixed point. This generalizes earlier results of Herer and Christensen, Glasner, Furstenberg and Weiss, and Farah and Solecki. This also answers a question posed by Farah and Solecki. In particular, it implies that if H is of the form $L_0(\mu, \mathbb{R})$, then H is extremely amenable if and only if H has no nontrivial characters. The latter gives an evidence for an affirmative answer to a question of Pestov. The proof is based on estimates of chromatic numbers of certain graphs on $\mathbb{Z}^n$. It uses tools from algebraic topology and builds on the work of Farah and Solecki.

1. INTRODUCTION

A topological group is called *extremely amenable* if every its continuous action on a compact space has a fixed point. This terminology comes from a characterization of amenable groups saying that a (locally compact) group is amenable if and only if every its continuous affine action on a compact convex set in a locally convex vector space has a fixed point. No nontrivial locally compact groups are extremely amenable as they all admit free actions on compact spaces. Still, however, many non locally compact groups are extremely amenable. These groups have received considerable attention recently (see [18]) and the following groups are extremely amenable: the group $\mathrm{Homeo}_+([0,1])$ of order-preserving homeomorphisms of the unit interval (Pestov, [19]), the group $U(\ell^2)$ of unitary transformations of ℓ^2 (Gromov and Milman, [5]), the group $\mathrm{Aut}([0,1],\lambda)$ of measure-preserving automorphisms of the Lebesgue space (Giordano and Pestov, [3]), the group $\mathrm{Iso}(\mathbb{U})$ of isometries of the Urysohn space (Pestov, [16]) and groups of the form

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$\mathrm{Aut}(M)$ for a large class of countable structures M (Kechris, Pestov and Todorcevic, [9]). The methods of proofs in the above cases vary from the concentration of measure phenomena to structural Ramsey theory. In fact, the earliest known examples of extremely amenable groups were of the form $L_0(\mu, G)$ for a topological group G and a submeasure μ, see [8, 4]. On the other hand, recently Melleray and Tsankov [14] proved that for many abelian groups Γ and Polish groups G, the group $\overline{\pi(\Gamma)}$ is an extremely amenable L_0 group, for a generic $\pi \in \mathrm{Hom}(\Gamma, G)$,

Let us explain the notion of a submeasure and of an L_0 group. Given an algebra $\mathcal{B}$ of subsets of a set X, we say that a function $\mu : \mathcal{B} \to [0, \infty)$ is a *submeasure* if $\mu(U) \leq \mu(V)$ whenever $U \subseteq V \subseteq X$ and $\mu(U \cup V) \leq \mu(U) + \mu(V)$ for all $U, V \subseteq X$. A submeasure μ is called a *measure* if $\mu(U \cup V) = \mu(U) + \mu(V)$ for disjoint $U, V \subseteq X$. A submeasure μ is called *diffused* if for every $\varepsilon > 0$ there is a finite cover $X = \bigcup_{i \leq n} X_i$ with $X_i \in \mathcal{B}$ and $\mu(X_i) \leq \varepsilon$.

Given a topological group G and a submeasure μ on $(X, \mathcal{B})$, $L_0(\mu, G)$ is the set of all step $\mathcal{B}$-measurable functions from X to G with finite range. It is a topological group with the pointwise multiplication and the topology of convergence in submeasure μ.[1] This means that given $f \in L_0(\mu, G)$, a neibourhood V of the identity in G and $\varepsilon > 0$, a basic neighbourhood of f is given by

$$\{h \in L_0(\mu, G) : \mu(\{x \in X : h(x) \notin V \cdot f(x)\}) < \varepsilon\}.$$

Groups of this form have been studied extensively especially in case $G = \mathbb{R}$. Following the convention in the literature, we write $L_0(\mu)$ for $L_0(\mu, \mathbb{R})$. The groups $L_0(\mu, G)$ are called the Hartman–Mycielski extensions. In general form, they have been used by Hartman and Mycielski [6] to show that any topological group can be embedded into a connnected topological group. In [6] Hartman and Mycielski proved that in case μ is the Lebesgue measure, $L_0(\mu, G)$ contains G as a closed subgroup and is path-connected and locally-path connected. The same remains true if μ is a diffused submeasure. Thus, the map $G \mapsto L_0(\mu, G)$ is a convariant functor from the category of topological groups to the category of connected topological groups. On the other hand, Bessaga and Pełczynski [1] (see also Keesling [10]) showed that if G is nontrivial, separable and completely metrizable, then the

[1]This construction appears in [6]. In [2] the authors take the metric completion in case G is locally compact. Since extreme amenability is invariant under taking dense subgroups, the choice of the definition does not affect extreme amenability of L_0 groups.

Hartman–Mycielski extension $L_0(\mu, G)$ (for μ the Lebesgue measure) is homeomorphic to ℓ^2.

A *character* of an abelian group is a homomorphism to the unit circle $\mathbb{T}$. Nikodým [15] studied the groups $L_0(\mu)$ and essentially showed [15, Pages 139–141], that $L_0(\mu)$ does not have any nontrivial continuous characters if and only if μ is diffused. The same argument shows that if G is any abelian group and μ is a diffused measure, then $L_0(\mu, G)$ does not have any nontrivial continuous characters. Note at this point that if an abelian group has a nontrivial continuous character, then it admits a fixed-point free continuous action on the circle $\mathbb{T}$ and thus cannot be extremely amenable. Therefore, if μ is not diffused and G is abelian, then $L_0(\mu, G)$ is not extremely amenable. One of the major open problems is a question of Glasner [18, Problem 3.4.14] (cf. [4]) whether there exists a monothetic topological group that has no nontrivial continuous characters and yet is not extremely amenable. A slightly more general question of Pestov [17, Question 1] asks whether for any abelian topological group H, extreme amenability of H is equivalent to the fact that H has no nontrivial continuous characters.

A submeasure μ is *pathological* if there is no nonzero measure ν on X with $\nu \leq \mu$. Recall that a submeasure μ is *exhaustive* if for every disjoint family $A_n \in \mathcal{B}$ we have $\lim_n \mu(A_n) = 0$. For a recent construction by Talagrand (answering an old question of Maharam) of an exhaustive pathological submeasure see [22]. Extreme amenability of the groups $L_0(\mu, G)$ has been studied for various groups and submeasures. The group $L_0(\mu, G)$ is extremely amenable in each of the following cases:

- if $G = \mathbb{R}$ and μ is pathological (Herer and Christensen, [8]),
- if $G = \mathbb{T}$ and μ is a diffused measure (Glasner [4] and, independently, Furstenberg and Weiss, unpublished),
- if G is amenable and μ is a diffused measure (Pestov, [16]),
- if G is compact solvable and μ is diffused (Farah and Solecki, [2])

There is a number of cases in which the answer was not known. The situation was unclear even for the groups $\mathbb{Z}$ and $\mathbb{R}$. In [2, Question 2] Farah and Solecki ask, for what submeasures are the groups $L_0(\mu, \mathbb{Z})$ and $L_0(\mu)$ extremely amenable. The same question appears also in Pestov's book [17, Question 8]. In this paper we answer this question and prove the following.

Theorem 1. *For any abelian topological group G and arbitrary diffused submeasure μ the group $L_0(\mu, G)$ is extremely amenable.*

Theorem 1 easily generalizes to solvable groups in place of abelian, using [2, Lemma 3.4]. Combined with the result of Nikodým, Theorem 1 implies the following.

Corollary 2. *Let μ be a submeasure. The following are equivalent:*

- *μ is diffused,*
- *$L_0(\mu, G)$ is extremely amenable for every abelian G,*
- *$L_0(\mu, \mathbb{Z})$ is extremely amenable.*

On the other hand, Theorem 1 provides an evidence for an affirmative answer to the question of Pestov mentioned above, as it implies the following.

Corollary 3. *Let μ be a submeasure. The following are equivalent:*

- *μ is diffused,*
- *$L_0(\mu)$ is extremely amenable,*
- *$L_0(\mu)$ has no nontrivial characters.*

The proof of Theorem 1 uses methods of algebraic topology and the Borsuk–Ulam theorem. It builds on the work of Farah and Solecki [2], who discovered that algebraic topological methods can be applied to study extremely amenable groups. Farah and Solecki obtained a new Ramsey-type theorem, and its proof was based on a construction of a family of simplicial complexes and an application of the Borsuk–Ulam theorem. In this paper, rather than proving a Ramsey-type result, we show a connection of extreme amenability of abelian L_0 groups with the existence of finite bounds on chromatic numbers of certain graphs on $\mathbb{Z}^n$. We use methods of algebraic topology to establish bounds from below on these chromatic numbers. The application of the Borsuk–Ulam theorem is based on the ideas of Farah and Solecki but it involves different (smaller) simplicial complexes.

The methods of algebraic topology have been also used by Lovász [11] for showing lower bounds on chromatic numbers of Kneser's graphs and by Matoušek [12] for the Kneser hypergraphs.

This paper is organized as follows. In Section 2 we connect extreme amenability of abelian L_0 groups with colorings of graphs. In Section 3 we construct a family of siplicial complexes used later in the proof of Theorem 1. The main result establishing the bounds on the chromatic numbers is proved in Section 4.

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2. Extreme amenability and chromatic numbers

Now we define the basic object of this paper, the graphs associated to a submeasure μ and a real $\varepsilon > 0$. We call a finite partition $\mathcal{P}$ of X a *measurable partition* if all elements of $\mathcal{P}$ belong to $\mathcal{B}$.

Definition 4. Let μ be a submeasure on a set X, let $\varepsilon > 0$ and let $\mathcal{P}$ be a finite measurable partition of X. We define the (undirected) graph $\Gamma^{\mathcal{P}}_{\varepsilon}(\mu)$ as follows. The set of nodes is $\mathbb{Z}^{\mathcal{P}}$ and two nodes $k = (k_A : A \in \mathcal{P})$ and $l = (l_A : A \in \mathcal{P})$ are connected with an edge if

$$\mu(\bigcup\{A \in \mathcal{P} : k_A \neq l_A + 1\}) < \varepsilon.$$

A *coloring* of a graph is a function defined on the set of its vertices such that no two vertices connected with an edge are assigned the same value. Given a graph Γ we write $\chi(\Gamma)$ for its *chromatic number*, i.e. the least number of colors in a coloring of the graph Γ.

Note that for a given submeasure μ and a given measurable partition $\mathcal{P}$ with $|\mathcal{P}| = n$, the graphs $\Gamma^{\mathcal{P}}_{\varepsilon}(\mu)$ are all graphs on $\mathbb{Z}^n$ and increase in edges as ε increases to 1. For $\varepsilon = 0$, the graph $\Gamma^{\mathcal{P}}_{\varepsilon}(\mu)$ has no edges at all, and for $\varepsilon > 1$, it is the full graph on $\mathbb{Z}^n$. Therefore, the chromatic numbers of $\Gamma^{\mathcal{P}}_{\varepsilon}(\mu)$ increase to infinity as ε approaches 1. The point of our analysis will be to fix $\varepsilon > 0$ and study the chromatic numbers of $\Gamma^{\mathcal{P}}_{\varepsilon}(\mu)$ as $\mathcal{P}$ runs over all finite measurable partitions.

The following lemma establishes the main connection between extreme amenability of abelian L_0 groups and chromatic numbers of the graphs $\Gamma^{\mathcal{P}}_{\varepsilon}(\mu)$.

Lemma 5. *Let μ be a submeasure on a set X. The following are equivalent:*

(i) *the group $L_0(\mu, G)$ is extremely amenable for every abelian topological group G,*

(ii) *for every $\varepsilon > 0$ there is no finite bound on the chromatic numbers*

$$\chi(\Gamma^{\mathcal{P}}_{\varepsilon}(\mu)),$$

where $\mathcal{P}$ runs over all finite measurable partitions of X.

Proof. (i)$\Rightarrow$(ii) Suppose that for some ε there is a finite bound d on the chromatic numbers $\chi(\Gamma^{\mathcal{P}}_{\varepsilon}(\mu))$. It is enough to show that the group $L_0(\mu, \mathbb{Z})$ is not extremely amenable. For each measurable partition $\mathcal{P}$ of X pick a coloring $c_{\mathcal{P}} : \mathbb{Z}^{\mathcal{P}} \to \{1, \ldots, d\}$ of $\Gamma^{\mathcal{P}}_{\varepsilon}(\mu)$ into d many colors. Fix a nonprincipal ultrafilter $\mathcal{U}$ on the set of all finite measurable partitions of X. For every step $\mathcal{B}$-measurable function $f : X \to \mathbb{Z}$ with finite range, let $\mathcal{P}_f$ be the partition of X induced by f, i.e the one

$\{f^{-1}(\{k\}) : k \in \mathbb{Z}\}$. For every partition $\mathcal{P}$ refining $\mathcal{P}_f$ let $f_{\mathcal{P}}$ be the element of $\mathbb{Z}^{\mathcal{P}}$ such that $f_{\mathcal{P}}(i) = k$ if and only if the i-th element of $\mathcal{P}$ is contained in $f^{-1}(\{k\})$. Now let c be a coloring of $L_0(\mu, \mathbb{Z})$ defined as the limit over the ultrafilter:

$$c(f) = \lim_{\mathcal{U}} c_{\mathcal{P}}(f_{\mathcal{P}}).$$

Let $X_i = \{f \in L_0(\mu, \mathbb{Z}) : c(f) = i\}$ for each $i \in \{1, \ldots, d\}$ and note that the sets X_i cover the group $L_0(\mu, \mathbb{Z})$. Let $\bar{1} \in L_0(\mu, \mathbb{Z})$ be the constant function with value 1.

For each partition $\mathcal{P}$ refining $\mathcal{P}_f$, no two nodes in $\mathbb{Z}^{\mathcal{P}}$ connected with an edge can have the same color $c_{\mathcal{P}}$. Therefore, for no two functions $f, g \in L_0(\mu, Z)$ which are constant on the elements of $\mathcal{P}$ and are such that $f - g \in \bar{1} + V_\varepsilon$ it can be the case that $c_{\mathcal{P}}(f) = c_{\mathcal{P}}(g)$. By passing to the limit over $\mathcal{U}$ we get that for each $i \in \{1, \ldots, d\}$

$$(\bar{1} + V_\varepsilon) \cap (X_i - X_i) = \emptyset,$$

where $V_\varepsilon = \{f \in L_0(\mu, \mathbb{Z}) : \mu\{x \in X : f(x) \neq 0\} < \varepsilon\}$ is the basic neighbourhood of the identity in $L_0(\mu, \mathbb{Z})$. This implies that $\bar{1} \notin \overline{X_i - X_i}$ for each $i \leq d$, so $L_0(\mu, \mathbb{Z})$ is not extremely amenable by Pestov's characterization of extreme amenability [18, Theorem 3.4.9].

(ii)$\Rightarrow$(i) Let G be an abelian topological group and suppose that $L_0(\mu, G)$ is not extremely amenable. By [18, Theorem 3.4.9], there is a syndetic set $S \subseteq L_0(\mu, G)$ such that $G \neq \overline{S - S}$. Write e for the neutral element of G. Assume that there are d many translates $S_1, \ldots, S_d$ of S which cover $L_0(\mu, G)$ and an element f of $L_0(\mu, G)$ such that $f \notin \overline{S - S}$. This means that there is a neighborhood W of e in G and an $\varepsilon > 0$ such that

$$(f + W_\varepsilon) \ \cap \ (S - S) = \emptyset$$

where $W_\varepsilon = \{h \in L_0(\mu, G) : \mu(\{x \in [0, 1] : h(x) \notin W\}) < \varepsilon\}$. Let $V_\varepsilon = \{h \in L_0(\mu, G) : \mu(\{x \in [0, 1] : h(x) \neq e\}) < \varepsilon\}$ and note that $V_\varepsilon \subseteq W_\varepsilon$, so $(f + V_\varepsilon) \ \cap \ (S - S) = \emptyset$.

Let $\mathcal{P}_f$ be the partition generated by f, i.e. $\mathcal{P}_f = \{f^{-1}(\{g\}) : g \in G\}$. Now we will construct d-colorings of the graphs $\Gamma^{\mathcal{P}}_\varepsilon(\mu)$ for all $\mathcal{P}$ refining $\mathcal{P}_f$. Since we have $\chi(\Gamma^{\mathcal{P}_0}_\varepsilon(\mu)) \leq \chi(\Gamma^{\mathcal{P}_1}_\varepsilon(\mu))$ whenever $\mathcal{P}_1$ refines $\mathcal{P}_0$, this will end the proof.

Let $\mathcal{P}$ be a measurable partition of X refining $\mathcal{P}_f$ and let $k \in \mathbb{Z}^{\mathcal{P}}$. Define $g_k \in L_0(\mu, G)$ so that

$$g_k(x) = k(A) f(x) \quad \text{if } x \in A.$$

Put

$$c(k) = i \quad \text{if and only if} \quad g_k \in S_i.$$

We claim that c defines a coloring of the graph $\Gamma^{\mathcal{P}}_{\varepsilon}(\mu)$. Indeed, suppose $k, l \in \mathbb{Z}^{\mathcal{P}}$ have the same color i and are connected with an edge. Let $B = \bigcup\{A \in \mathcal{P} : k(A) \neq l(A)+1\}$ and note that $\mu(B) < \varepsilon$, since k and l are connected with an edge. For $x \in X \setminus B$ we have $g_k(x) - g_l(x) = (l(A)+1)f(x) - l(A)f(x) = f(x)$, so

$$\mu(\{x \in X : g_k(x) - g_l(x) \neq f(x)\}) < \varepsilon.$$

As $g_k, g_l \in S_i$, this implies that $g_k - g_l \in (f + V_\varepsilon) \cap S_i - S_i$ and since $S_i - S_i = S - S$, we get

$$\emptyset \neq (f + V_\varepsilon) \cap (S - S),$$

which contradicts the choice of ε and f.

□

3. Simplicial complexes

Here we gather the definitions and constructions of simplicial complexes used in the proof o Theorem 18 in the next section.

Given a finite set V, a *symplicial complex* with vertex set V is a finite family of subsets of V, closed under taking subsets. Elements of this family are called *simplices* of the simplicial complex. Given a simplicial complex K with vertex set V and an action of a group H on V, we say that K is an *H-complex* if for every $F \in K$ and $h \in H$ we have $\{h(v) : v \in F\} \in K$. Given two simplicial complexes K and L with disjoint sets of vertices, we define their *join* and denote it by $K * L$ as the simplicial complex on the union of vertex sets of K and L with

$$K * L = \{F \cup G : F \in K, G \in L\}.$$

Given a simplicial complex X write $\mathrm{Cone}(X)$ for the join of X with a single point. Note that $\mathrm{Cone}(X)$ is always contractible.

If K and L are simplicial complexes with vertex sets V and W with $V \cap W = \{x\}$ (here x is the *basepoint* of both K and L) the *wedge* of K and L, denoted by $K \vee L$ is the simplicial complex on the vertex set $V \cup W$ with $K \vee L = K \cup L$. The wedge of complexes with disjoint vertex sets is obtained as above after identification of the basepoints of these complexes. Note that if $K = K_1 \vee \ldots \vee K_n$ and $L = L_1 \vee \ldots \vee L_m$ are simplicial complexes with basepoints x and y, respectively, then $K * L$ is homotopy equivalent to the wedge $\bigvee_{1 \leq i \leq n, 1 \leq j \leq m} K_i * L_j$, after collapsing the cones $\{x\} * L$ and $K * \{y\}$ to a single point, which becomes the basepoint of $K * L$.

If K is a simplicial complex, then its *barycentric subdivision*, denoted by $\mathrm{sd}(K)$, is the simplicial complex with the vertex set K defined as

$$\mathrm{sd}(K) = \{C \subseteq K \setminus \{\emptyset\} : C \text{ is a chain}\}.$$

If K is an H-complex, then $\mathrm{sd}(K)$ becomes an H-complex too, in the natural way. Given a simplicial complex K, we write $||K||$ for its *geometric realization* (see [7, Page 537], [13, Page 14] or [21, Chapter 3, Section 2]). Note that if a group H acts on K, then this action can be naturally extended to an action on $||K||$.

Given a natural number p, write $\mathbb{Z}/p$ for the cyclic group of rank p and $+_p$ for the addition modulo p.

Definition 6. Given $n, p \in \mathbb{N}$ and a partial function $\tau : \{0, \ldots, n\} \to \{0, \ldots, p-1\}$ a *component interval* of τ is any maximal interval $I \subseteq \{0, \ldots, n\}$ such that τ is constant on $I \cap \mathrm{dom}(f)$.

The component intervals of τ cover $\mathrm{dom}(\tau)$ and their number is equal to the number of "steps" of τ.

Definition 7. Let l, n be natural numbers and let p be a prime number. Let $V_p^{n,l}$ be the set of nonempty partial functions τ from $\{0, \ldots, n\}$ to $\{0, \ldots, p-1\}$ which have the following properties:

(i) $n - l \leq |\mathrm{dom}(\tau)|$,
(ii) the number of component intervals of τ is at most $l + 1$.

Note that for $n \leq l$ the set $V_p^{n,l}$ consists of all nonempty partial functions from $\{0, \ldots, n\}$ to $\mathbb{Z}/p$.

Definition 8. Let $K_p^{n,l}$ be the simplicial complex whose vertices set is $V_p^{n,l}$ and whose simplices are those subsets of $V_p^{n,l}$ which form a chain with respect to inclusion.

There is a natural action of $\mathbb{Z}/p$ on $K_p^{n,l}$ defined as follows. If $\tau \in V_p^{n,l}$ and $k \in \mathbb{Z}/p$, then $k + \tau \in V_p^{k,l}$ is such that $\mathrm{dom}(k + \tau) = \mathrm{dom}(\tau)$ and $(k + \tau)(i) = k +_p \tau(i)$ for each $i \in \mathrm{dom}(\tau)$. .

Let S_p^n be the complex whose vectices are partial functions τ from $\{0, \ldots, n\}$ to $\{0, \ldots, p-1\}$ with $|\mathrm{dom}(\tau)| = 1$ and the simplices of S_p^n are those subsets of its vertices whose union forms a partial function. In other words, S_p^n is the join $(\mathbb{Z}/p)^{*(n+1)}$. Note that if $p = 2$, then S_p^n is topologically equivalent to the n-dimensional sphere S^n. For each $a \in (\mathbb{Z}/p \setminus \{0\})^{n+1}$ write S_a^n for $\{0, a(0)\} * \{0, a(1)\} * \ldots * \{0, a(n)\}$ and note that S_a^n is also topologically equivalent to the n-dimensional sphere S^n. Now, S_p^n is homotopy equivalent to the wedge of $n+1$-many copies of the complex $\mathbb{Z}_p$, which is the wedge of $p-1$-many zero-dimensional spheres, with the basepoint 0. Hence S^n_p is homotopy equivalent to the wedge $\bigvee_{a\in(\mathbb{Z}/p\setminus\{0\})^{n+1}} S^n_a$.

Recall that $V^{n,n}_p$ consists of all nonempty partial functions from the set $\{0,\dots n\}$ to $\{0,\dots,p-1\}$. It is equal to the set of vertices of $\mathrm{sd}(S^n_p)$ and the complexes $K^{n,n}_p$ and $\mathrm{sd}(S^n_p)$ are isomorphic as $\mathbb{Z}/p$-complexes. Since the barycentric subdivision preserves the homotopy type, $K^{n,n}_p$ is homotopy equivalent to the wedge of $(p-1)^{n+1}$-many n-dimensional spheres.

To push this analysis even further, write D^n for the set of all nonempty partial functions from $\{0,\dots,n\}$ to $\{0\}$ and note that $D^n \subseteq K^{n,n}_p = \mathrm{sd}(S^n_p)$ is the barycentric subdivision of the simplex $\{0\}^{*(n+1)}$. Let $\bar{0}$ be the constant total function equal to 0. For each $a \in (\mathbb{Z}/p \setminus \{0\})^{n+1}$ write

$$K^a_p = \{\tau \in K^{n,n}_p : \forall m \in \mathrm{dom}(\tau) \quad \tau(m) \in \{0, a(m)\}$$

and note that $K^a_p = \mathrm{sd}(S^n_a)$. Note also that D^n is contained in each K^a_p and hence $\bar{0}$ belongs to each of these sets. By the abstract considerations above, we know that $K^{n,n}_p$ is homotopy equivalent to the wedge $\bigvee_{a\in(\mathbb{Z}/p\setminus\{0\})^{n+1}} K^a_p$ with the basepoint $\bar{0}$. In this case, the homotopy equivalence can be realized in the following way. Note that $\bigvee_{a\in(\mathbb{Z}/p\setminus\{0\})^{n+1}} K^a_p$ contains $(p-1)^{n+1}$-many copies of D^n with the point $\bar{0}$ glued together, whereas $K^{n,n}_p$ contains just one copy of D^n. Since D^n is contractible, both $\bigvee_{a\in(\mathbb{Z}/p\setminus\{0\})^{n+1}} K^a_p$ and $K^{n,n}_p$ are homotopy equivalent to the space obtained by collapsing the copies of D^n to $\bar{0}$.

Definition 9. Write $L^{n,l}_p$ for the subcomplex of $K^{n,n}_p$ whose vertices are those partial functions which have at most $l+1$ many component intervals and write $J^{n,l}$ for the subcomplex of $K^{n,l}_p$ whose vertices are the partial functions τ with $n-l \le |\mathrm{dom}(\tau)|$.

Note that $K^{n,l}_p = L^{n,l}_p \cap J^{n,l}_p$. The complex $J^{n,l}_p$ was used by Solecki and Farah in [2] and its isomorphic copy appears in Matoušek [13, Page 2513], where it is shown that the $\mathbb{Z}/p$-index of $J^{n,l}_p$ is greater or equal to l. Matoušek's computation of the $\mathbb{Z}/p$-index is based on the Sarkaria inequality, which is a combinatorial version of the Alexander duality (see [21, Chapter 6, Section 2] or [7, Theorem 3.44]). We will use Alexander's duality explicitly to compute the cohomology of $J^{n,l}_p$ over $\mathbb{Z}/p$. In fact, Alexander's duality can be used in the same way for the complex $L^{n,l}_p$ but we will show that the latter is even homotopy equivalent to S^l_p.

We will show that cohomology groups of $K_p^{n,l}$ over $\mathbb{Z}/p$, up to $l-1$, are trivial. It seems that it can be even shown that $K_p^{n,l}$ is homotopy equivalent to S_p^l but a proof of that should be more subtle.

Given a simplicial complex X together with a finite sequence of its subsimplices $\langle A_0, \ldots, A_{p-1}\rangle$, write $\mathrm{Cone}(X, \langle A_i : i < p\rangle)$ for the space $X \cup \mathrm{Cone}(A_0) \cup \ldots \cup \mathrm{Cone}(A_{p-1})$, where $\mathrm{Cone}(A_i)$ are cones to distinct cone-points.

Lemma 10. *If X is a simplicial complex and $A_1, \ldots, A_p$ are its subcomplexes, each of which is either contractible or empty, then the complex $\mathrm{Cone}(X, \langle A_1, \ldots, A_p\rangle)$ is homotopy equivalent to X.*

Proof. This follows from the fact that collapsing a contractible subcomplex is a homotopy equivalence. Details can be found in [7, Example 0.13]. □

We use the following notation. Given a natural number m write p^{*m} for the set of all nonempty partial functions from $\{0, \ldots, m-1\}$ to $\{0, \ldots, p-1\}$. Now we define the complexes which are obtained by an iterated operation of coning-off a sequence of subcomplexes.

Definition 11. Suppose X is a simplicial complex, m, p are natural numbers and a family of subcomplexes A_σ of X is given for $\sigma \in p^{*m}$. Assume that the complexes A_σ are such that $A_\sigma \subseteq A_\tau$ if $\sigma \supseteq \tau$. We define the complex $\mathrm{Cone}(X, \langle A_\sigma : \sigma \in p^{*m}\rangle)$ by an inductive construction. Let $X^0 = X$ and $A^0_\sigma = A_\sigma$ for each $b \in p^{*m}$. By induction on $k \leq m$ define X^k and A^k_σ for $\sigma \in p^{*\{k,\ldots,m-1\}}$ as follows

$$X^{k+1} = \mathrm{Cone}(X^k, \langle A^k_{\{(k+1,i)\}} : i < p\rangle)$$
$$A^{k+1}_\sigma = \mathrm{Cone}(A^k_\sigma, \langle A^k_{\{(k,i))\}^\frown\sigma} : i < p\rangle)$$

(here $\{(j,i)\}$ is the partial function with domain $\{j\}$ and the value i). Finally, let $\mathrm{Cone}(X, \langle A_\sigma : \sigma \in p^{*m}\rangle) = X^{m-1}$.

Given a nonempty finite set b we also write p^{*b} for the set of all nonempty partial functions from b to $\{0, \ldots, p-1\}$. Slightly abusing the above notation, if b is a set of natural numbers, naturally ordered, we define $\mathrm{Cone}(X, \langle A_\sigma : \sigma \in p^{*b}\rangle)$ in analogous way as above.

Lemma 12. *If X is a simplicial complex and A_σ, for $\sigma \in p^{*m}$ are its subcomplexes, each of which is either empty or contractible, then $\mathrm{Cone}(X, \langle A_\sigma : \sigma \in p^{*m}\rangle)$ is homotopy equivalent to X.*

Proof. By induction on $k \leq m$ we prove that X^k is homotopy equivalent to X and each A^k_σ is either empty or contractible for $\sigma \in p^{*\{k,\ldots,m\}}$. The induction step follows directly from the definition and Lemma 10. □

To show that $L_p^{n,l}$ is homotopy equivalent to S_p^l, define for each $\sigma \in p^{*\{l+1,\ldots,n\}}$ the set

$$A_\sigma = \{\tau \in V_p^{l,l} : \tau^\frown\sigma \text{ has at most } l \text{ component intervals}\}.$$

Now $L_p^{n,l}$ is naturally isomorphic to $\mathrm{Cone}(K_p^{l,l}, \langle A_\sigma : \sigma \in p^{*\{l+1,\ldots,n\}}\rangle)$. The isomorphism takes the functions $\{(k,i)\}$ to the cone points of the cones over $A^k_{\{(k,i)\}}$.

Lemma 13. *For each $\sigma \in p^{*\{l+1,\ldots,n\}}$ the set A_σ is either empty or contractible.*

Proof. We prove by induction on $i \leq l$ that given $\rho \in p^{\{i,\ldots,l\}}$ with $l-i$ component intervals (i.e. with $\rho(j) \neq \rho(j+1)$ for $j < l-i$) and $\rho(l) \neq \sigma(l+1)$, the set

$$A_{\sigma,\rho} = \{\tau \in V_p^{i,l} : \tau^\frown\rho^\frown\sigma \text{ has at most } l \text{ component intervals}\}$$

is either empty or contractible.

First, for $i = 1$ note that $\rho^\frown\sigma$ already has at least l component intervals by the assumption on ρ. If $\rho^\frown\sigma$ has more than l many component intervals, then $A_{\sigma,\rho}$ is empty. If $\rho^\frown\sigma$ has exactly l many component intervals, then $A_{\sigma,\rho}$ consists of the single point $\{\rho(0)\}$.

Now let us argue for the induction step from $i-1$ to i. Let $\rho(i) = j_0$. Let

$$B = \{\tau \in V_p^{l,i-1} : \tau^\frown\rho^\frown\sigma \text{ has at most } l \text{ component intervals}\}$$

and for each $j < p$ let

$$B_j = \{\tau \in V_p^{l,i-1} : \tau^\frown j^\frown\rho^\frown\sigma \text{ has at most } l \text{ component intervals}\}.$$

Note that $B = B_{j_0}$ and for each $j \neq j_0$ the set B_j is either empty or contractible, by the inductive assumption. If $\rho^\frown\sigma$ has more than l many component intervals, then $A_{\rho,\sigma}$ is empty. Otherwise, $A_{\rho,\sigma}$ is equal to $\mathrm{Cone}(B, \langle B_j, j : j < p\rangle)$ and by Lemma 10 it is is homotopy equivalent to $\mathrm{Cone}(B, B_{j_0}, j_0)$, which is equal to $\mathrm{Cone}(B)$ since $B = B_{j_0}$, and hence it is contractible. $\square$

Together, Lemmas 12 and 13 give the following.

Corollary 14. *For each natural numbers p and $n \geq l$ the complex $L_p^{n,l}$ is homotopy equivalent to S_p^l.*

Below $\tilde{H}^i(X,\mathbb{Z}/p)$ stands for the reduced cohomology and $\tilde{H}_i(X,\mathbb{Z}/p)$ for the reduced homology of X over $\mathbb{Z}/p$. The following lemma is a topological version of Sarkaria's trick used in Matoušek [12, Page 2513].

Lemma 15. *If p and $n \geq l$ are natural numbers, then for each $0 \leq i \leq l$ we have*
$$\tilde{H}_i(J_p^{n,l}, \mathbb{Z}/p) = 0.$$

Proof. Write $C = \{\tau \in K_p^{n,n} : |\mathrm{dom}(\tau)| < n - l\}$ and note that the dimension of C is equal to $n - l$ (see [12, Page 2513]). Write E^n for $D^n \setminus C$ and $J_p^a = K_p^a \setminus C$ for each $a \in (\mathbb{Z}/p \setminus \{0\})^{n+1}$. Recall that $K_p^{n,n}$ is homotopy equivalent to the wedge $\bigvee_{a \in (\mathbb{Z}/p \setminus \{0\})^{n+1}} K_p^a$. Note that $\bar{0} \notin C$ and E^n is still contractible to $\bar{0}$. As in the case of the wedge of K_p^a's, $\bigvee_{a \in (\mathbb{Z}/p \setminus \{0\})^{n+1}} J_p^a$ contains $(p-1)^{n+1}$ many copies of E^n with the point $\bar{0}$ glued together, whereas $J_p^{n,l}$ contains just one copy of E^n. Therefore, both $\bigvee_{a \in (\mathbb{Z}/p \setminus \{0\})^{n+1}} J_p^a$ and $J_p^{n,l}$ are homotopy equivalent to the space obtained by collapsing the copies of E^n to $\bar{0}$.

Now, each J_p^a is obtained by removing a subset of dimension $n - l$ from K_p^a, which is the n-dimensional sphere. By Alexander's duality [21, Chapter 6, Section 2], $\tilde{H}_i(J_p^a, \mathbb{Z}/p) = 0$ for $0 \leq i \leq l$. Since the reduced homology of a wedge of siplicial compexes is the direct sum of reduced homologies of these complexes, we get $\tilde{H}_i(J_p^{n,l}, \mathbb{Z}/p) = 0$ for $0 \leq i \leq l$, as needed. □

Lemma 16. *If p is a prime number and $n \geq l$, then for each $0 \leq i \leq l - 1$ we have*
$$\tilde{H}^i(K_p^{n,l}, \mathbb{Z}/p) = 0.$$

Proof. First we prove that $K_p^{n,l}$ is connected. Note that every $\tau \in K_p^{n,l}$ is connected by a path to a total function in $K_p^{n,l}$. This follows from the fact that if $i \notin \mathrm{dom}(\tau)$ and $\tau' \in K_p^{n,l}$ is such that $\mathrm{dom}(\tau') = \mathrm{dom}(\tau) \cup \{i\}$, $\tau(j) = \tau'(j)$ for $j \in \mathrm{dom}(\tau)$, then τ and τ' are connected with an edge in $K_p^{n,l}$. Next, notice that every total function is connected to $\bar{0}$. The latter follows from the fact that if $\tau \in K_p^{n,l}$ is a total function and $i \leq n+1$ is such that $\tau(i-1) = \tau(i) \neq \tau(i+1)$ and $\tau' \in K_p^{n,l}$ is such that $\tau' \upharpoonright [0, i) = \tau \upharpoonright [0, i)$, $\tau' \upharpoonright (i, n+1] = \tau \upharpoonright (i, n+1]$ and $\tau'(i) = \tau(i+1)$, then τ and τ' are connected by a path via $\tau'' \in K_p^{n,l}$ where $\mathrm{dom}(\tau'') = \{0, \ldots, n+1\} \setminus \{i\}$ and $\tau'' \upharpoonright \mathrm{dom}(\tau'') = \tau \upharpoonright \mathrm{dom}(\tau'') = \tau' \upharpoonright \mathrm{dom}(\tau'')$.

Now we compute the cohomology. By Lemma 15, for $i \leq l$ we have $\tilde{H}^i(J_p^{n,l}, \mathbb{Z}/p) = 0$. ($J_p^{n,l} \cup L_p^{n,l}$ is even bigger, so it also has small codimension in $K_p^{n,n}$ and the same argument using Alexander's duality gives that for $i \leq l$ we have $\tilde{H}_i(J_p^{n,l} \cup L_p^{n,l}, \mathbb{Z}/p) = 0$ as well. On the other hand, Corollary 14 implies that $\tilde{H}_i(L_p^{n,l}, \mathbb{Z}/p) = 0$ for $i \leq l$. Write the Mayer–Vietoris sequence [21, Chapter 4, Section 6] of reduced homology groups:

$$\begin{aligned} \ldots \to \tilde{H}_i(K_p^{n,l}, \mathbb{Z}/p) \to \tilde{H}_i(J_p^{n,l}, \mathbb{Z}/p) \oplus \tilde{H}_i(L_p^{n,l}, \mathbb{Z}/p) \to \\ \to \tilde{H}_i(J_p^{n,l} \cup L_p^{n,l}, \mathbb{Z}/p) \to \tilde{H}_{i-1}(K_p^{n,l}, \mathbb{Z}/p) \to \ldots \to 0 \end{aligned}$$

to see that $H_i(K_p^{n,l}, \mathbb{Z}/p) = 0$ for $1 \le i \le l-1$. Since $\mathbb{Z}/p$ is a field, by the universal coefficients theorem (see [21, Chapter 5, Section 6]) we get that $\tilde{H}^i(K_p^{n,l}, \mathbb{Z}/p) = 0$ as well, for $1 \le i \le l-1$. □

Although Lemma 16 is by no means optimal, it is just enough to apply Volovikov's version [23] of the Borsuk–Ulam theorem for the complex $K_p^{n,l}$ to get the following.

Corollary 17. *If p is a prime number, $n \ge l$ and $d(p-1) \le l-1$, then for every continuous map $f : ||K_p^{n,l}|| \to \mathbb{R}^d$ there is a point in $||K_p^{n,l}||$ whose $\mathbb{Z}/p$-orbit is mapped by f to a single point in $\mathbb{R}^d$.*

4. Bounds on chromatic numbers

Given a diffused submeasure μ, for each $\delta > 0$ there is a finite covering of X with sets in $\mathcal{B}$ with submeasure less than δ. Let $k_\mu(\delta)$ be the minimal number of elements in such a covering. Given $\varepsilon > 0$ consider the function $k_\varepsilon^\mu : \mathbb{N} \to \mathbb{N}$ defined as $k_\varepsilon^\mu(d) = k_\mu(\varepsilon/4d)$ and for each $d \in \mathbb{N}$ let

$$\mathcal{Q}_d^\varepsilon = \{I_1^\varepsilon, \ldots, I_{k_\varepsilon^\mu(d)}^\varepsilon\}$$

be a measurable partition of X into $k_\varepsilon^\mu(d)$ many sets of submeasure less than $\varepsilon/4d$. Note that k_ε^μ is increasing and

$$k_\varepsilon^\mu(d) \ge d\mu(X)(4/\varepsilon). \tag{1}$$

Let $F_\varepsilon^\mu : \mathbb{N} \to \mathbb{N}$ be any increasing function such that

$$k_\varepsilon^\mu \circ F_\varepsilon^\mu(m) = m \tag{2}$$

for each $m \in \mathbb{N}$. Such a function exists since k_ε^μ is increasing and unbounded by (1). Moreover,

$$\lim_{m\to\infty} F_\varepsilon^\mu(m) = \infty. \tag{3}$$

The rate of growth of F_ε^μ may be very low if μ is not a measure (for example, for the Hausdorff submeasures constructed in [2] it is sublogarithmic). On the other hand, if μ is a measure, then F_ε^μ is linear. However, for proving Theorem 1 we only need the fact that F_μ^ε diverges to infinity.

For $\varepsilon > 0$ write $C_\varepsilon^\mu = \sqrt[3]{\mu(X)^2/16\varepsilon}$. For $n \in \mathbb{N}$ and $\varepsilon > 0$ let $\mathcal{P}_n^\varepsilon(\mu)$ be a finite measurable partition of X with, say $k_n^\varepsilon(\mu)$, many sets in $\mathcal{B}$ of submeasure less than $1/n$ which refines all $\mathcal{Q}^\varepsilon_d(\mu)$ for $d \leq F^\mu_\varepsilon(C^\mu_\varepsilon \sqrt[3]{n})$. Write $\Gamma^n_\varepsilon(\mu)$ for the graph $\Gamma_\varepsilon^{\mathcal{P}^\varepsilon_n(\mu)}(\mu)$.

Here is the main result which, together with Lemma 5 and (3), implies Theorem 1.

Theorem 18. *Let μ be a diffused submeasure. Given $\varepsilon > 0$ we have*

$$\chi(\Gamma^n_\varepsilon(\mu)) \geq F^\mu_\varepsilon(C^\mu_\varepsilon \sqrt[3]{n}).$$

Proof. Fix $n \in \mathbb{N}$. Write k_n for $k^\varepsilon_n(\mu) - 1$ and μ_n for the submeasure on $\{0, \ldots, k_n\}$ induced by μ, i.e $\mu_n(B) = \mu(\bigcup\{A_i : i \in B\})$, where $\{A_0, \ldots, A_{k_n}\}$ is an enumeration of $\mathcal{P}_n(\mu)$.

Suppose that $d < K^\varepsilon_\mu(C^\mu_\varepsilon \sqrt[3]{n})$ and there exists a coloring $c : \mathbb{Z}^{k_n+1} \to \{1, \ldots, d\}$ of the graph $\Gamma^n_\varepsilon(\mu)$. Let $k = k(\varepsilon/4d)$. By the assumption on $\mathcal{P}^\varepsilon_n(\mu)$, we can assume (possibly rearranging the numbers $0, \ldots, k_n$) that there are consequtive intervals $I_1, \ldots, I_k$ covering $\{0, \ldots, k_n\}$ each of submeasure μ_n less than $1/4d$. Using the Bertrand postulate (see [20, Chapter 10]), pick a prime number p with

$$k(d+1) < p < 2k(d+1) \tag{4}$$

Since $p > k(d+1)$, we have that

$$\begin{aligned} \frac{k}{p} &< \frac{1}{d+1} = 1 - \frac{d}{d+1}, \\ 1 - \frac{k}{p} &> \frac{d}{d+1}, \\ (1 - \frac{k}{p})(d+1) &> d, \end{aligned}$$

which implies that

$$(p-k)(d+1) > dp. \tag{5}$$

Put $l = dp$.

Claim.

$$\frac{l+1}{n} < \frac{\varepsilon}{8}$$

Proof. Otherwise, we have $dp = l \geq n\,\varepsilon/8$ and since $k = k(\varepsilon/4d)$, we have that

$$\frac{4d}{\varepsilon}\mu(X) \leq k \tag{6}$$

and

$$d + 1 \leq 4d = \frac{\varepsilon}{\mu(X)}(\frac{4d}{\varepsilon}\mu(X)) \leq \frac{\varepsilon}{\mu(X)}k. \tag{7}$$

By (4) and the assumption that the Claim does not hold, we have

$$\frac{n\varepsilon}{8} \leq dp \leq 2k(d+1)d < 2k(d+1)^2,$$

and by (7) we get

$$\frac{n\varepsilon}{16} < k(d+1)^2 \leq \frac{\varepsilon^2}{\mu(X)^2}k^3, \tag{8}$$

which gives that $k = k^{\varepsilon}_{\mu}(d) > C^{\mu}_{\varepsilon}\sqrt[3]{n}$. Since F^{ε}_{μ} is increasing, (2) implies that $d \geq F^{\varepsilon}_{\mu}(C\sqrt[3]{n})$ contrary to the assumption. This ends the proof of the Claim. □

For each $f \in V^{k_n,l}_p$ write $\bar{f} : \{0,\ldots,k_n\} \to \{0,\ldots,p-1\}$ for the function which is equal to f on its domain and 0 elsewhere. Let $\bar{c} : V^{k_n,l}_p \to \{1,\ldots,d\}$ be defined by $\bar{c}(f) = c(\bar{f})$. Note that $\bar{c}$ induces a map $c : V^{k_n,l}_p \to \mathbb{R}^d$ defined by $c(f) = \mathrm{e}_{\bar{c}(f)}$, where $\{\mathrm{e}_1,\ldots,\mathrm{e}_d\}$ are the standard basis vectors in $\mathbb{R}^d$. This in turn, extends to a continuous map

$$||c|| : ||K^{k_n,l}_p|| \to \mathbb{R}^d,$$

which is the affine extension of c.

Now, $l > d(p-1)$, so by Corollary 17, there is a point $x_0 \in ||K^{k_n,l}_p||$ such that $||c||$ is constant on the $\mathbb{Z}/p$-orbit of x_0. The point x_0 lies in a maximal simplex in $K^{k_n,l}_p$, which is of the form

$$h_l \subseteq \ldots \subseteq h_0.$$

Pick $i_0 < d$ such that the i_0-th coordinate of x_0 is nonzero. As the function $||c||$ is the affine extension of c, there is $0 \leq m \leq l$ such that $\bar{c}(h_m) = i_0$. Moreover, if $q \in \mathbb{Z}/p$ is any number, then $x_0 + q$ lies in the maximal simplex

$$h_l + q \subseteq \ldots \subseteq h_0 + q$$

and since still the i_0-th coordinate of $x_0 + q$ is nonzero, there is $0 \leq m_q \leq l$ such that $\bar{c}(h_{m_q} + q) = i_0$.

Put $h = h_{m_0}$. Consider the set $A_h = \{j \in \mathbb{Z}/p :$ there is a component interval J of h such that h is equal to j on J and J intersects at least two of the intervals $I_1,\ldots,I_k\}$. Note that since the intervals $I_1,\ldots,I_k$ are disjoint and consequtive, we have $|A_h| \leq k-1$. Let $B_h = \{0,\ldots,p-1\} \setminus A_h$ and note that $|B_h| \geq p-k+1$. For each $j \in B_h$ the preimage of j by h is a disjoint union of component intervals, each of which is contained in at most one of the intervals $I_1,\ldots,I_k$. Note that (5) implies that

$$(p-k+1)(d+1) > l+1.$$

Since the number of the component intervals of h is at most $l+1$, by the pigeonhole principle we get that one of the elements of B_h, say q_0, must have at most d many component intervals in its preimage. Since each $I_1 \ldots, I_k$ has submeasure at most $\varepsilon/4d$, we get that

$$\mu_n(h^{-1}(\{q_0\})) \leq d \cdot \frac{\varepsilon}{4d} = \frac{\varepsilon}{4}. \tag{9}$$

Let $f = h_{m_{p-1-q_0}} +_p (p-1-q_0)$ and $g = h_{m_{p-q_0}} +_p (p-q_0)$. Note that by the way we have chosen the numbers m_q, we have $c(\bar{f}) = c(\bar{g}) = i_0$. We will show that

$$\mu_n(\{i \leq k_n : \bar{f}(i) + 1 \neq \bar{g}(i)\}) < \varepsilon, \tag{10}$$

which implies that $\bar{f}$ and $\bar{g}$ are connected with an edge in $\Gamma^n_\varepsilon(\mu)$. This will give a contradiction and end the proof.

Note that h and $h_{m_{p-1-q_0}}$ as well as h and $h_{m_{p-q_0}}$ differ at at most $l+1$ many points in $\{0, \ldots, k_n\}$. Since each point has submeasure μ_n at most $1/n$, by the Claim we get that

$$\mu_n(\{i \leq k_n : h_{m_{p-1-q_0}}(i) \neq h(i)\}) \leq \frac{\varepsilon}{8} \tag{11}$$

$$\mu_n(\{i \leq k_n : h_{m_{p-q_0}}(i) \neq h(i)\}) \leq \frac{\varepsilon}{8} \tag{12}$$

By (9), (11) and the Claim we have that

$$\begin{aligned} \mu_n(\{i \leq k_n : \bar{f}(i) +_p 1 \neq \bar{f}(i) + 1\}) &= \mu_n(\bar{f}^{-1}(\{p-1\})) \\ \leq \mu_n(f^{-1}(\{p-1\})) + \varepsilon/8 &\leq \mu_n({h_{m_{p-1-q_0}}}^{-1}(\{q_0\})) + \varepsilon/8 \\ \leq \mu_n(h^{-1}(\{q_0\})) + \varepsilon/8 + \varepsilon/8 &\leq \varepsilon/8 + \varepsilon/4. \end{aligned} \tag{13}$$

On the other hand, by (11), (12) and the Claim

$$\begin{aligned} &\mu_n(\{i \leq k_n : \bar{f}(i) +_p 1 \neq h(i) +_p (p-q_0)\}) \\ &= \mu_n(\{i \leq k_n : \bar{f}(i) \neq h(i) +_p (p-1-q_0)\}) \\ &\leq \mu_n(\{i \leq k_n : f(i) \neq h(i) +_p (p-1-q_0)\}) + \varepsilon/8 \\ &= \mu_n(\{i \leq k_n : h_{m_{p-1-q_0}}(i) +_p (p-1-q_0) \neq h(i) +_p (p-1-q_0)\}) \\ &+ \varepsilon/8 = \mu_n(\{i \leq k_n : h_{m_{p-1-q_0}}(i) \neq h(i)\}) + \varepsilon/8 \leq \varepsilon/4 \end{aligned}$$

and

$$\begin{aligned} &\mu_n(\{i \leq k_n : \bar{g}(i) \neq h(i) +_p (p-q_0)\}) \\ &\leq \mu_n(\{i \leq k_n : g(i) \neq h(i) +_p (p-q_0)\}) + \varepsilon/8 \\ &= \mu_n(\{i \leq k_n : h_{m_{p-q_0}}(i) +_p (p-q_0) \neq h(i) +_p (p-q_0)\}) + \varepsilon/8 \\ &= \mu_n(\{i \leq k_n : h_{m_{p-q_0}}(i) \neq h(i)\}) + \varepsilon/8 \leq \varepsilon/4. \end{aligned}$$

So, together with (13), this gives (10), as needed. This ends the proof. □

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