TOPOLOGICAL REPRESENTATIONS

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Abstract. This paper studies the combinatorics of ideals which recently appeared in ergodicity results for analytic equivalence relations. The ideals have the following topological representation. There is a separable metrizable space X, a σ-ideal I on X and a dense countable subset D of X such that the ideal consists of those subsets of D whose closure belongs to I. It turns out that this definition is indepedent of the choice of D. We show that an ideal is of this form if and only if it is dense and countably separated. The latter is a variation of a notion introduced by Todorčević for gaps. As a corollary, we get that this class is invariant under the Rudin–Blass equivalence. This also implies that the space X can be always chosen to be compact so that I is a σ-ideal of compact sets. We compute the possible descriptive complexities of such ideals and conclude that all analytic equivalence relations induced by such ideals are $\mathbf{\Pi}^0_3$. We also prove that a coanalytic ideal is an intersection of ideals of this form if and only if it is weakly selective.

1. Introduction

The aim of this paper is to reveal a connection between the structure of ideals on countable sets and ideals of compact sets in Polish spaces. A family of subsets of a given set is an *ideal* if it is closed under taking subsets and finite unions. We always assume that an ideal of subsets of a set S contains all singletons $\{s\}$ for $s \in S$, i.e. an ideal J is an ideal of subsets of $\bigcup J$. Given an ideal J, we say that a set is *J-positive* if it does not belong to J. Sometimes, we write J^+ for the family of J-positive sets and co-J for the dual filter. Throughout this paper we often identify subsets of ω with elements of 2^ω via the characteristic functions. Thus, for example, given an ideal of subsets of ω we define its descriptive complexity as if it were a subset of 2^ω. On the other hand, given an ideal of compact subsets of a given Polish space X we refer to its descriptive complexity in the Vietoris space $K(X)$.

The study of definable ideals of compact sets has become a classical subject in descriptive set theory. A well-known result of Kechris, Louveau and Woodin [14] and Dourgherty, Kechris and Louveau (see [16]) says that an analytic ideal of compact sets is a σ-ideal if and only if it is $\mathbf{\Pi}^0_2$. The descriptive complexity of more complicated ideals of compact sets is the subject

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of a trichotomy theorem proved by Matheron, Solecki and Zelený [18]. Recently, Solecki [25] described a special class of $\mathbf{\Pi}^0_2$ σ-ideals of compact sets and proved several structure theorems representing ideals in that class via the meager ideal (see also [15]).

The structure of ideals on countable sets has been studied from a different perspective but some results reveal the similarities. An analogy to the Kechris–Louveau–Woodin theorem appears in the work of Solecki [24] on analytic P-ideals, where it is shown that if a P-ideal is analytic, then its descriptive complexity is $\mathbf{\Pi}^0_3$. Solecki also shows [23, Corollary 3.4] that if J is an analytic P-ideal then it is either $\mathbf{\Sigma}^0_2$ or $\mathbf{\Pi}^0_3$-complete. The structure of ideals on ω is often described in terms of the *Rudin–Blass order*. Given two ideals J, K on ω we say that J is *Rudin–Blass below* K and write $J \leq_{\mathrm{RB}} K$ if there is a finite-to-one function $f : \omega \to \omega$ such that $a \in J$ if and only if $f^{-1}(a) \in K$, for every $a \subseteq \omega$. The Jalali-Naini–Mathias–Talangrand theorem [2, Theorem 4.1.2] then says that every ideal with the Baire property is Rudin–Blass above the ideal Fin of finite sets.

A connection between ideals of compact sets and ideals on countable sets that appears in this paper uses the following operation. Suppose X is a separable metrizable space and I is a σ-ideal on X that contains all singletons. Given a dense countable set $D \subseteq X$, define the ideal J_I on D as the family $\{a \subseteq D : \mathrm{cl}(a) \in I\}$. Obviously, J_I depends only on the family of closed sets that belong to I. In principle, J_I also depends on the set D, which is equal to $\bigcup J_I$, but we will see (Proposition 2.1) that, up to isomorphism, this definition is independent of the choice of D. The ideals of the form J_I have been recently studied in [22] and used in canonization (see [13]) of smooth equivalence relations for σ-ideals generated by closed sets. Given an ideal J on a countable set E we say that J *has a topological representation* if there are I, D, X as above and a bijection $\rho : E \to D$ such that $J = \{a \subseteq E : \rho(a) \in J_I\}$. In such a case we say that J is *represented by* I.

Two examples of ideals with topological representations have been studied by Farah and Solecki [8], who showed that there are at least two isomorphism types of such ideals (namely that the ideals represented by the meager sets and by the meager null sets are not isomorphic).

The study of ideals on ω is closely connected and largely motivated by the study of equivalence relations on 2^ω given in the following way. Given an ideal J on ω we write E_J for the equivalence relation on 2^ω with $x \mathrel{E_J} y$ if $x \Delta y \in J$. Rosendal [21] proved that any Borel equivalence relation is Borel-reducible to one of the form E_J. A motivation for the results in this paper is a recent work of Zapletal [27], who shows that if J has a topological representation, then the equivalence relation E_J has the following ergodicity property. First, every Borel homomorphism from a turbulent equivalence relation F to E_J maps a comeager set to a single E_J-equivalence class. Second, if J is represented by a $\mathbf{\Pi}^0_2$ σ-ideal, then every homomorphism from E_J to an equivalence relation classifiable by countable structures maps a measure 1 set to a single equivalence class. This is in contrast with the turbulence dichotomy of Hjorth [10]. Note that if J is has a topological representation, then it is not a P-ideal and hence E_J is not induced by a Polish group action by the Solecki characterization of Polishable P-ideals [24, Corollary 4.1].

The main result of this paper establishes a combinatorial characterization of ideals which admit topological representations. One of the necessary conditions says that the ideal is *dense*, i.e. any infinite set contains an infinite subset that belongs to the ideal. The other condition is a variation of Todorčević's notion of countably separated gaps [26] (see also [5, 7, 1]). We say that an ideal J on a countable set D is *countably separated* if there is a countable family $\{a_n : n < \omega\}$ of subsets of D such that for any $a, b \subseteq D$ with $a \in J$ and $b \notin J$ there is $n \in \omega$ with $a \cap a_n = \emptyset$ and $b \cap a_n \notin J$. In such a case we say that the family $\{a_n : n < \omega\}$ *separates* J. We prove the following characterization.

Theorem 1.1. *For any ideal J on a countable set the following are equivalent:*

(i) *J is dense and countably separated,*
(ii) *J has a topological representation.*

As a corollary we get the following

Corollary 1.2. *The class of ideals which have topological representations is invariant under the Rudin–Blass equivalence.*

Proof. It is enough to show that if $J \leq_{\mathrm{RB}} K$ and K is countably separated, then J is countably separated, and if J is dense, then K is dense. Let $f : \omega \to \omega$ be a Rudin–Blass reduction witnessing $J \leq_{\mathrm{RB}} K$.

Suppose first that K is countably separated by $\{c_n : n < \omega\}$ and let $d_n = f''c_n$. We claim that d_n's witness that J is countably separated. Indeed, take $a \in J$ and $b \notin J$. Then $a' = f^{-1}(a) \in K$ and $b' = f^{-1}(b) \notin K$. Pick n such that $c_n \cap a' = \emptyset$ and $c_n \cap b' \notin K$. Then $d_n \cap a = \emptyset$ and $d_n \cap b \notin K$.

Suppose now that J is dense and let $b \subseteq \omega$ be infinite. Since f is finite-to-one, $b' = f''b$ is also infinite and hence there is $c' \subseteq b'$ with $c' \in J$. Let $c = f^{-1}(c') \cap b$ and note that c is an infinite subset of b which belongs to K, as a subset of $f^{-1}(c')$. □

Corollary 1.2 in particular implies that ideals with topological representations are invariant under $\equiv^{++}_{\mathrm{RB}}$ (see [12, Section 3.2]) and hence the class of equivalence relations of the form E_J, for J with a topological representation, is invariant under additive Borel reductions, by a result of Farah [6].

The proof of Theorem 1.1 shows that if J on ω has a topological representation, then it can be represented on the Cantor space via an identification of ω with the set of rationals in the Cantor space. The following shows that there is some extent of control over the sets in the σ-ideal that witness that J is represented on the Cantor space.

Theorem 1.3. *If J has a topological representation, then it is represented on the Cantor space by a σ-ideal generated by a family of compact nowhere dense sets.*

In partiular, the family representing an ideal can be chosen to consist of compact sets. For definable ideals, this implies computations of possible descriptive complexities.

Theorem 1.4. *If an ideal J has a topological representation and it is analytic, then it is $\mathbf{\Pi}^0_3$-complete. In particular, if E_J is analytic, then it is $\mathbf{\Pi}^0_3$.*

This gives an analogoue of the Kechris–Louveau–Woodin dichotomy.

Corollary 1.5. *If a coanalytic ideal J has a topological representation, then J is either $\mathbf{\Pi}^0_3$-complete, or $\mathbf{\Pi}^1_1$-complete. J is $\mathbf{\Pi}^0_3$-complete if and only if it is represented by a $\mathbf{\Pi}^0_2$ ideal of compact sets.*

Besides the descriptive complexity, there is one more combinatorial condition that determines the structure of ideals which have topological representations. An ideal J is *weakly selective* if for every $b \notin J$ and a function $f : b \to \omega$ there is a J-positive subset a of b such that $f \restriction a$ is either one-to-one or constant. Equivalently, J is weakly selective if any partition of a J-positive set into sets in J admits a J-positive selector. Weakly selective ideals have been studied by several authors (see Farah [4] or Baumgartner and Laver [3]) and [22, Proposition 4.3] shows that if J has a topological representation, then it is weakly selective. Here we prove the following characterization.

Theorem 1.6. *Let J be a coanalytic ideal. The following are equivalent:*

(i) *J is weakly selective,*
(ii) *J is an intersection of a family of ideals with topological representations.*

The paper is organized as follows. A preliminary discussion and a proof of Theorem 1.1 are given in Section 2. Theorem 1.3 is proved in Section 3. Theorem 1.4 together with Corollary 1.5 are proved in Section 4. Section 5 contains a proof of Theorem 1.6.

2. A characterization of ideals with topological representations

The definition on the ideals J_I formally depends on the choice of the dense set D. It turns out, however, that no matter what dense set D is chosen, the ideal J_I is the same, up to isomorphism.

Proposition 2.1. *Given a σ-ideal I on a separable metric space X and two dense countable sets D and E in X, if $J = \{a \subseteq D : \mathrm{cl}(a) \in I\}$ and $K = \{a \subseteq E : \mathrm{cl}(a) \in I\}$, then J and K are isomorphic.*

Proof. Using a back-and-forth argument, enumerate $D = \{d_n : n < \omega\}$ and $E = \{e_n : n < \omega\}$ so that the distance of d_n and e_n is smaller than $1/n$. Now for $a \subseteq \omega$ write $\mathrm{cl}_D(a) = \mathrm{cl}(\{d_n : n \in a\})$ and $\mathrm{cl}_E(a) = \mathrm{cl}(\{e_n : n \in a\})$, where cl stands for the closure taken in X. To see that J and K are isomorphic, it is enough to show that $\mathrm{cl}_D(a)$ belongs to I if and only if $\mathrm{cl}_E(a)$ belongs to I. Note that

(1) $\mathrm{cl}_D(a) \subseteq \mathrm{cl}_E(a) \cup \{d_n : n \in a\}$,
(2) $\mathrm{cl}_E(a) \subseteq \mathrm{cl}_D(a) \cup \{e_n : n \in a\}$.

Here, (1) follows from the fact that if x belongs to $\mathrm{cl}_D(a)$ and is not one of the d_n's (for $n \in a$), then there is an infinite subsequence of d_n's (indexed with elements of a) that converges to x. Since e_n is $(1/n)$-close to d_n, there is also an infinite subsequence of e_n's (with the same index set) converging to x. (2) follows by symmetry. Now, (1) and (2) imply that $\mathrm{cl}_D(a)$ and $\mathrm{cl}_E(a)$ can differ by an at most countable set. Since the singletons belong to I (we always assume that ideals contain all singletons), it follows that $\mathrm{cl}_D(a) \in I$ if and only if $\mathrm{cl}_E(a) \in I$. □

Now we will prove Theorem 1.1. Let us first comment on the sharpness of condition (i) in that theorem: neither being dense nor countably separated alone implies that the ideal has a topological representation.

To see that, first consider the ideal $\emptyset \times \mathrm{Fin} = \{a \subseteq \omega \times \omega : \forall n \in \omega\ a_n \in \mathrm{Fin}\}$, where $a_n = \{m \in \omega : (n, m) \in a\}$. This ideal is countably separated, by the sets $c_{n,k} = \{(n, m) \in \omega \times \omega : m > k\}$ but it is clearly not dense.

On the other hand, consider the ideal $\mathrm{Fin} \times \mathrm{Fin} = \{a \subseteq \omega \times \omega : \{n \in \omega : a_n \in \mathrm{Fin}\} \in \mathrm{Fin}\}$. The ideal $\mathrm{Fin} \times \mathrm{Fin}$ is not weakly selective, as witnessed by the projection function $(n, m) \mapsto n$. By [22, Proposition 4.3], an ideal which has a topological representation is weakly selective. Thus, $\mathrm{Fin} \times \mathrm{Fin}$ does not have a topological representation. On the other hand, it is clearly dense. Thefore, by Theorem 1.1 it cannot be countably separated.

Below, for $\varepsilon > 0$ and $A \subseteq 2^\omega$, write

$$\mathrm{ball}(\varepsilon, A) = \{x \in 2^\omega : \exists y \in A \quad d(x, y) < \varepsilon\}.$$

Proof of Theorem 1.1. (i)⇒(ii) Suppose J is represented on X and let $D \subseteq X$ be countable dense, I be a σ-ideal such that $J = J_I$. First note that J_I is dense. Indeed, take an infinite $a \notin J_I$. Then $\mathrm{cl}(a) \notin I$, so in particular $\mathrm{cl}(a)$ is uncountable. Let $x \in \mathrm{cl}(a) \setminus a$ and pick a sequence $\langle x_n : n \in \omega \rangle$ of elements of a converging to x. Then $b = \{x_n : n < \omega\}$ is an infinite subset of a, which is in J_I since $\mathrm{cl}(b) = b \cup \{x\}$ is countable and hence in I.

To see that J_I is countably separated, fix a countable basis $\{U_n : n < \omega\}$ of X and let $c_n = U_n \cap D$. We claim that $\{c_n : n < \omega\}$ witnesses that J is countably separated. Indeed, let $a, b \subseteq D$ be such that $a \notin J_I, b \in J_I$. Then $\mathrm{cl}(a) \notin I$ and $\mathrm{cl}(b) \in I$. By countable additivity of I, there exists n such that $U_n \cap \mathrm{cl}(b) = \emptyset$ and $U_n \cap \mathrm{cl}(a) \notin I$. Then clearly $c_n \cap b = \emptyset$ and $c_n \cap a$ is J-positive since $\mathrm{cl}(c_n \cap a)$ contains $U_n \cap \mathrm{cl}(a)$.

(ii)⇒(i) Suppose now that J is countably separated and dense. Assume that J is an ideal on ω. We will first show that the family $\{c_n : n \in \omega\}$ witnessing that J is countably separated can be improved a little. We say that a family of subsets of ω *separates points* if for each $n \neq m \in \omega$ there is a set a in that family such that $n \in a$ and $m \notin a$.

Lemma 2.2. *If J is countably separated, then there is a family witnessing that J is countably separated, which separates points and is such that all Boolean combinations of its elements are either infinite or empty.*

Proof. Let $\{c_n : n \in \omega\}$ be a family witnessing that J is countably separated. Enumerate all pairs of distinct natural numbers as $\langle (k_n, l_n) : n \in \omega \rangle$. We will construct a family $\{d_n : n \in \omega\}$ of subsets of ω such that for each n the following is true

(a) if $n = 2m$ is even, then d_n is a subset of c_m such that $c_m \setminus d_n \in J$;
(b) if $n = 2m + 1$ is odd, then $k_m \in d_n$ and $l_n \notin d_n$;
(c) all Boolean combinations of d_i for $i \leq n$ are infinite or empty.

Notice that such a family will also witness that J is countably separated by (a). It will separate points by (b) and have all Boolean combinations either empty of infinite by (c). Hence, $\{d_n : n < \omega\}$ will be the required family.

To construct the sets d_n inductively, we start with $d_0 = c_0$. Suppose that d_k for $k < n$ have been constructed. All Boolean combination of $\{d_k : k < n\}$ define a finite partition $\{a_k : k < k_n\}$ of ω into infinite subsets.

Case 1. Suppose that $n = 2m$ is even. For each $k < k_n$ we define a set $e_k \subset a_k \cap c_m$ in the following way. There are three possibilities:

- if $a_k \cap c_m \in J$, then $e_k = \emptyset$;
- if $a_k \cap c_m \notin J$ and $a_k \setminus c_m$ is infinite, then $e_k = a_k \cap c_m$;
- if $a_k \cap c_m \notin J$ and $a_k \setminus c_m$ is finite, then find an infinite subset $e'_k \in J$ of $a_k \cap c_m$ (using the fact that J is dense) and define $e_k = (a_k \cap c_m) \setminus e'_k$.

The set $d_n = \bigcup_{k<k_n} e_k$ is a subset of c_m such that $c_m \setminus d_n \in J$. Also, d_n is either empty or both infinite and coinfinite in every a_k, therefore it is as needed.

Case 2. Suppose that $n = 2m + 1$ is odd. There is $k < k_n$ such that $k_m \in a_k$. Let d_n be any infinite subset of a_k such that $k_m \in d_n$, $l_m \notin d_n$ and $a_k \setminus d_n$ is infinite. Then d_n separates the pair (k_m, l_m) and in each a_k it is either empty or infinite and coinfinite, therefore it is as needed. □

We can now assume that a family $\{c_n : n < \omega\}$ witnessing that J is countably separated is as in Lemma 2.2. Define a topology τ on ω by letting all c_n's be clopen basic sets. This is a Hausdorff, second-countable and regular topology, since it is zero-dimensional. By Urysohn's metrization theorem [17, Theorem 1.1], it is metrizable. Note that since all Boolean combinations of the elements of the basis are either empty or infinite, this space has no isolated points. Then, as a countable metrizable topological space without isolated points, is homeomorphic to the rationals, by a theorem of

Sierpiński [17, Exercise 7.12]. Embed (ω, τ) into the Cantor set 2^ω so that it is homeomorphic to $D = \mathbb{Q} \cap 2^\omega$. Thus, via this embedding, we treat now J as an ideal on D.

Define an ideal of $K(2^\omega)$ by $I = \{A \in K(2^\omega) : \exists a \in J \quad A \subset \mathrm{cl}(a)\}$. It turns out that I is a σ-ideal on $K(2^\omega)$.

Lemma 2.3. *I is a σ-ideal of compact sets.*

Proof. Suppose it is not a σ-ideal. In that case there is a sequence of compact sets $A_n \in I$ such that their union $A = \bigcup_{n<\omega} A_n$ is also compact and does not belong to I. Without loss of generality we can assume that A_n's are increasing. Fix a metric d on 2^ω of diameter ≤ 1. The metric notions below refer to the metric d.

We claim that there is $n \in \omega$ such that $A \setminus \mathrm{ball}(\varepsilon, A_n) \in I$ for every $\varepsilon > 0$. Suppose otherwise and construct an increasing sequence of natural numbers n_i and a sequence of reals $\varepsilon_i > 0$ such that:

- $A \setminus \mathrm{ball}(\varepsilon_i, A_{n_i})$ does not belong to I,
- $A_{n_{i+1}}$ is not contained in $\mathrm{ball}(\varepsilon_i, A_{n_i})$.

This is easy to do using our assumption and the fact that A_n's exhaust A. But then $A \setminus \bigcup_{i<\omega} \mathrm{ball}(\varepsilon_i, A_{n_i})$ is nonempty, by compactness of A. On the other hand, if $x \in A \setminus \bigcup_{i<\omega} \mathrm{ball}(\varepsilon_i, A_{n_i})$, then $x \in A \setminus \bigcup_{n<\omega} A_n$, which gives a contradiction.

Fix a number n as in the previous paragraph and without loss of generality assume that $n = 0$. Let $B_0 = A \setminus \mathrm{ball}(1, A_0)$ and for each $k \geq 1$ let $B_k = A \cap (\mathrm{ball}(\frac{1}{k}, A_0) \setminus \mathrm{ball}(\frac{1}{k+1}, A_0))$. Note that each B_k is in I, by our assumption. Next, for each $k \in \omega$ find a set $b_k \subseteq D$ such that $b_k \in J$, $B_k \subseteq \mathrm{cl}(b_k)$ and $b_k \subseteq \mathrm{ball}(\frac{1}{k}, B_k)$. Find also $a \in J$ such that $A_0 \subseteq \mathrm{cl}(a)$. Let $b = \bigcup_{k<\omega} b_k \cup a$. Note that $A \subseteq \mathrm{cl}(b)$, so $b \notin J$. Since J is countably separated by c_n's, there is n such that $a \cap c_n = \emptyset$ and $b \cap c_n \notin J$. Now, since c_n's are clopen on D, we get a clopen set $C \subseteq 2^\omega$ such that $a \subseteq C$ and $C \cap c_n = \emptyset$. Let $\varepsilon > 0$ be such that $\mathrm{ball}(\varepsilon, A_0) \subseteq C$. By the definition of b_k's, all but finitely many of them are contained in $\mathrm{ball}(\varepsilon, A_0)$. Hence $b \setminus C$ is covered with finitely many of the sets b_k, and so is $b \cap c_n \subseteq b \setminus C$. Since each b_k belongs to J, this contradicts the fact that $b \cap c_n \notin J$. □

Now, to finish the proof we will show that $J = J_I$. One inclusion is obvious: if $a \in J$, then $\mathrm{cl}(a) \in I$ by the definition of I and so $a \in J_I$. On the other hand, if $a \in J_I$, then $\mathrm{cl}(a) \in I$. Thus, there is $b \in J$ with $\mathrm{cl}(a) \subseteq \mathrm{cl}(b)$. We must prove that $a \in J$. However, if $a \notin J$, then for some n we have $c_n \cap b = \emptyset$ and $c_n \cap a \neq \emptyset$. Let $C \subseteq 2^\omega$ be a basic clopen set with $C \cap D = c_n$ and note that $\mathrm{cl}(b) \cap C = \emptyset$ and $\mathrm{cl}(a) \setminus C \neq \emptyset$, which contradicts $\mathrm{cl}(a) \subseteq \mathrm{cl}(b)$. Thus, it must be the case that $a \in J$, which concludes the argument that $J = J_I$ and ends the entire proof.

□

3. Representation via compact nowhere dense sets

It is fairly easy to see that if J has a topological representation, then it is also represented on a compact metric space. Indeed, if J is represented on X via I, then let $\hat{X}$ be a metric compactification of X and let $\hat{I}$ be the σ-ideal on $\hat{X}$ generated by the sets $\mathrm{cl}(K)$ for $K \in I$ (the closure is taken in $\hat{X}$) and the singletons $\{x\}$ for $x \in \hat{X} \setminus X$. Then J_I is represented on $\hat{X}$ via $\hat{I}$ as witnessed by the same dense set $D \subseteq \hat{X}$. The proof of Theorem 1.1 shows something more: if J has a topological representation, then we can actually find a σ-ideal of I closed subsets of the Cantor space 2^ω such that J is represented by I. In the proof of Theorem 1.3 we will use similar arguments and we will make sure that all closed sets in I are nowhere dense.

Proof of Theorem 1.3. Suppose J is represented on X via a σ-ideal I. By the remarks above, we can assume X is the Cantor space and J is a family of subsets of $D = \mathbb{Q} \subseteq 2^\omega$. Note that in this case a family witnessing that J is countably separated can be chosen to consist of those basic clopen subsets of 2^ω which are J-positive. Enumerate the J-positive basic clopen subsets of D as $\{c_n : n < \omega\}$. Below, the notions of open and clopen will refer to the topology on D.

Lemma 3.1. *For each open J-positive $a \subseteq D$ and distinct $k, l \in a$ there are disjoint J-positive clopen subsets $b, c \subseteq a$ such that $k \in b$ and $l \in c$.*

Proof. Let W_k, W_l be disjoint clopen neighborhoods of k and l in a such that $a \setminus (W_k \cup W_l)$ is J-positive. Note that such neighborhoods must exist, since otherwise $\mathrm{cl}(a)$ would be covered by

$$\bigcup\{\mathrm{cl}(a \setminus (W_k \cup W_l) : W_k, W_l \text{ clopen neighborhoods of } k, l\} \cup \{k, l\}.$$

Let $a' = a \setminus (W_k \cup W_l)$. It is enough to show that there are two disjoint J-positive clopen subsets of a'. This is to say that J is not a maximal ideal below a'. Write $A = \mathrm{cl}(a')$ and let $A' = A \setminus \bigcup\{U : U \text{ basic open and } U \cap A \in I\}$. Obviously, $A' \notin I$ and pick $x \in A'$. Again, note that here must be a basic clopen neighboorhood V of x such that $A \setminus V \notin I$ since otherwise A would be covered by $\bigcup\{A \setminus V : V \text{ basic clopen neighboorhood of } x\} \cup \{x\}$ and belong to I. Pick such V and let $b = a' \cap V$ and $c = a' \setminus V$. Now b and c are disjoint J-positive subsets of a'. □

Lemma 3.2. *Suppose $c \subseteq D$ is a J-positive clopen set and $b \subseteq c$ is open such that $c \setminus b \in J$. If $b_n \subseteq b$ are J-positive clopen sets with $b = \bigcup_n b_n$, then for every J-positive set $d \subseteq b$ there is n with $d \cap b_n \notin J$.*

Proof. Suppose that $d \subseteq b$ is J-positive. We need to show that $b_n \cap d \notin J$ for some n. Suppose otherwise. Since J is weakly selective [22, Proposition 4.3], there is J-positive $e \subseteq d$ such that $|e \cap b_n| \leq 1$ for each n. This means that $e \cap b$ is discrete and hence $\mathrm{cl}(e) \subseteq e \cup \mathrm{cl}(c \setminus b)$ since c was a clopen set. This implies that $\mathrm{cl}(e) \in I$ and contradicts the fact that $e \notin J$. □

We will construct a Hausdorff, zero-dimensional topology τ on D which has no isolated points and such that:

(i) all sets in J are nowhere dense in τ,
(ii) for each $b \notin J$ and $a \in J$ there is a τ-clopen set $U \subseteq D$ such that $U \cap a = \emptyset$ and $U \cap b \notin J$.

Given that, as in the proof of Theorem 1.1, find a homeomorphism of (ω, τ) and $\mathbb{Q}$ and embed it into the Cantor set. Let I be the σ-ideal generated by the sets $\mathrm{cl}_\tau(a)$ for $a \in J$ (here cl_τ stands for the closure taken in the Cantor set in which (ω, τ) is embedded).

The condition (i) implies that the elements of I are nowhere dense in the Cantor set and we need to show that $J = J_I$. Indeed, if $a \in J$, then $\mathrm{cl}_\tau(a) \in I$ and so $a \in J_I$. What is left to prove is that if $b \notin J$ and $a_n \in J$, then $\mathrm{cl}_\tau(b) \not\subseteq \bigcup_n \mathrm{cl}_\tau(a_n)$. By induction construct a decreasing sequence of τ-clopen sets U_n with $a_n \cap U_n = \emptyset$ and $U_n \cap b \notin J$ and $\mathrm{diam}(U_n) < 1/n$ (the diameter is computed with respect to the usual metric on the Cantor set in which (ω, τ) is embedded). Having U_n constructed, let $b_n = U_n \cap b$. Using (ii), find a τ-clopen set U_{n+1} such that $U_{n+1} \cap a_n = \emptyset$ and $U_{n+1} \cap b_n \notin J$. If needed, shrink it so that $\mathrm{diam}(U_{n+1}) < 1/(n+1)$ and still $U_{n+1} \cap b_n \notin J$ (this is possible as U_{n+1} is covered with finitely many relatively clopen sets of diameter less than $1/(n+1)$). At the end, let $x \in 2^\omega$ belong to $\bigcap_n U_n$. Then $x \in \mathrm{cl}_\tau(b) \setminus \bigcup \mathrm{cl}_\tau(a_n)$. This shows that if $b \notin J$, then $\mathrm{cl}_\tau(b) \notin I$ and thus proves that $J = J_I$.

To construct the topology τ we will construct sets a_n such that $\{a_n : n < \omega\}$ separates points in D and (ii), (iii) and (iv) hold, where

(iii) each a_n is clopen and all elements in the algebra generated by $\{a_i : i < n\}$ are either empty or J-positive,
(iv) for each n, m and J-positive set $b \subseteq a_n \cap c_m$ there is $k > n$ such that $a_k \subseteq a_n \cap c_m$ and $a_k \cap b \notin J$.

Having the sets a_n constructed, take them as a clopen basis of the topology τ, which is then Hausdorff, zero-dimensional and has no isolated points by (iii). To see (i) note that if a_n is τ-clopen and $b \in J$, then there is m such that $a_n \cap c_m$ is disjoint from b since c_m's separate J. Then, by (iv) applied to $b = a_n \cap c_m$ there is k with $a_k \subseteq a_n \cap c_m$ and in particular a_k is disjoint from b. This shows that b is nowhere dense in τ.

The construction of the sets a_n will be by induction with $a_0 = D$. In the construction we will make sure that (iii) and (v) hold where

(v) for each n, m there is a sequence k_i such that $\bigcup_i a_{k_i} \subseteq a_n \cap c_m$ and $(a_n \cap c_m) \setminus \bigcup_i a_{k_i} \in J$ and for each $b \subseteq a_n \cap c_m$ with $b \notin J$ there is i such that $b \cap a_{k_i} \notin J$.

Note that (v) implies that (ii) and (iv) hold. Indeed, (iv) follows from (v) immmediately. To see (ii), take $b \notin J$ and $a \in J$ and let c_m be such that $c_m \cap a = \emptyset$ and $c_m \cap b \notin J$. Apply (v) to $a_0 = D$ and c_m and find k_i such that $a_{k_i} \subseteq c_m$ and $b \cap a_{k_i} \notin J$. This proves (ii).

Now we are ready for the induction that takes care of (iii) and (v). Start with $a_0 = D$. Given k write A_k for the algebra generated by $\{a_i : i < k\}$. At each step of the induction we will make a sequence of promises. A *promise* is a pair (j, c), where $j \in D$ and $c \subseteq D$ is clopen and J-positive. The meaning of a promise is as follows. Given a set a in the algebra generated by the sets constructed so far and its subset $c \subseteq a$ which is clopen and such that $a \setminus c \in J$ we will make sure that c is τ-open and promise to construct a sequence of τ-clopen and clopen sets a_{k_i} in the future so that $c = \bigcup_i a_{k_i}$. To do so, we must construct one $a_{k_i} \subseteq c$ for each $j \in c$ with $j \in a_{k_i}$. Thus, the set of promises made in such case is the set $\{(j, c) : j \in c\}$. The inductive construction will use bookkeeping in order to fulfill all promises made during its steps.

Enumerate also all pairs of distinct elements of D as (k_n, l_n). At each step k we will consider one of the three possibilities:

(a) either we construct a_k to separate k_n and l_n,
(b) or we consider a pair a_i, c_m for some $i < n$ and make sure (possibly making promises) that there will be a sequence a_{k_n} such that that $\bigcup_n a_{k_n} \subseteq c_m \cap a_i$ and for each J-positive $b \subseteq a_i \cap c_m$ there is j with $b \cap a_{k_j} \notin J$,
(c) or we consider a promise (j, c) made so far and construct a_k so that $j \in a_k$ and $a_k \subseteq c$.

We start with $a_0 = D$. Suppose everything is constructed so far and we are at step k. There are three cases.

Case (a). We need to separate k_n and l_n. If k_n and l_n are already separated by the algebra A_k, then put $a_k = \emptyset$. Otherwise, find an atom d of this algebra with $k_n, l_n \in d$. Use Lemma 3.1 to find two disjoint clopen J-positive subsets $U, V \subseteq d$ with $k_n \in U$ and $l_n \in V$. Put $a_k = U$.

Case (b). Suppose we consider the pair a_i, c_m. Enumerate all the atoms of the algebra A_k below a_i as $\{d_l : l < L\}$. Let c' be obtained by removing from c_m all the intersections $c_m \cap d_l$ which are in J. Note that c' is still clopen and $c \setminus c' \in J$. We will make sure to construct a sequence a_{k_j} as in (iii) so that $c' = \bigcup_j a_{k_j}$ and for each $b \subseteq c' \cap a_i$ there is j with $a_{k_j} \cap b \notin J$. Define P, Q to be subsets of the set of atoms of A_k with

- $d \in P$ if $c' \cap d \notin J$ and $d \setminus c' \notin J$,
- $d \in Q$ if $c' \cap d$ is co-J in d.

For each $d \in P$ let $a'_i = d \cap c'$ and put $a_k = \bigcup_{i<|P|} a'_i$. This defines a_k and if Q is empty, then there is nothing more to do. However, if Q is nonempty, then for each $d \in Q$ we make a sequence of promises to construct for each $d \in Q$ a sequence $\langle a_{k_j} : j < \omega \rangle$ of sets which are clopen and such that $d \cap c' = \bigcup_j a_{k_j}$. The fact that a_{k_j} are clopen together with Lemma 3.2 will guarantee that (v) is satisfied. Thus, we add to our list of promises all pairs $(j, c' \cap d)$ with $d \in Q$ and $j \in c' \cap d$.

Case (c). Suppose we are in a position to fulfill a promise (j, c). Find an atom d of A_k with $j \in d$. Since c was co-J in an atom d' of some A_i with

$i < k$ with $j \in d'$, it must be the case that $c \cap d$ is co-J in d. Use Lemma 3.1 to find two disjoint J-positive clopen subsets U, V of $c \cap d$ with $j \in U$. Put $a_k = U$.

This ends the inductive construction and the proof. □

4. Descriptive complexity of ideals with topological representations

Proof of Theorem 1.4. Suppose J is analytic and has a topological representation. By Theorem 1.3 there is a σ-ideal I of compact subsets of a Polish space X with a countable dense set D such that J_I is isomorphic to J. Consider the function $a \mapsto \mathrm{cl}(a)$ from $P(D)$ to $K(X)$ and note that it is Baire class 1. Since J_I is analytic, $I = \{A \in K(X) : \exists b \in J_I\ A \subseteq \mathrm{cl}(b)\}$ is also analytic, and hence $\mathbf{\Pi}^0_2$ by the theorem of Kechris–Louveau–Woodin [14, Theorem 11]. Therefore, J must be $\mathbf{\Pi}^0_3$ as a preimage of a $\mathbf{\Pi}^0_2$ set by a Baire class 1 function.

To check that J is in fact $\mathbf{\Pi}^0_3$-complete, we need the following standard fact.

Lemma 4.1. *All analytic ideals are $\mathbf{\Sigma}^0_2$-hard.*

Proof. This follows directly from the Jalali-Naini–Mathias–Talagrand theorem [2, Theorem 4.1.2]. Indeed, if J is analytic, then it has the Baire property and hence $\mathrm{Fin} \leq_{\mathrm{RB}} J$. From this we easily get a continuous reduction from $\mathrm{Fin} \subseteq 2^\omega$ (which is $\mathbf{\Sigma}^0_2$-complete) to J. □

We will now show that J_I is $\mathbf{\Pi}^0_3$-hard. The argument is based on ideas from [18]. Fix a point $x \in X$. Fix also a compatible metric on X and let $V_n = \mathrm{ball}(2^{-n}, x)$ for each $n \in \omega$. For each n define an ideal $J^n = \{a \cap V_n : a \in J_I\}$ on $D \cap V_n$. Note that each J^n is analytic, and hence $\mathbf{\Sigma}^0_2$-hard by Lemma 4.1. Therefore for each n there is $\phi_n : 2^\omega \to P(D \cap V_n)$ such that $\phi_n^{-1}(J^n) = \mathrm{Fin}$. Define $\phi : (P(\omega))^\omega \to P(D)$ by $\phi(\langle a_n : n \in \omega\rangle) = \bigcup_{n\in\omega} \phi_n(a_n)$. Let W be the set

$$\{\langle a_n : n \in \omega\rangle : \forall n \in \omega\ a_n \in \mathrm{Fin}\} = \{\langle a_n : n \in \omega\rangle : \forall n \in \omega\ \phi_n(a_n) \in J^n\}.$$

and note that W is $\mathbf{\Pi}^0_3$-complete. To finish the proof it suffices to show that $\phi^{-1}(J_I) = W$.

Suppose first that $\phi(\langle a_n : n \in \omega\rangle) \in J_I$, i.e $\bigcup_{n\in\omega} \phi_n(a_n) \in J_I$. Then for each n we have $\phi_n(a_n) \in J_I$ and $\phi_n(a_n) \subseteq V_n$. Hence, for each n the set $\phi_n(a_n)$ is in J^n and $\langle a_n : n \in \omega\rangle \in W$.

On the other hand, if $\langle a_n : n \in \omega\rangle \in W$, i.e. for each n we have $\phi_n(a_n) \in J^n \subseteq J_I$, then $\mathrm{cl}(\phi_n(a_n)) \in I$. Since I is a σ-ideal, $\bigcup_{n\in\omega} \mathrm{cl}(\phi_n(a_n)) \cup \{x\}$ is in I. To prove that $\bigcup_{n\in\omega} \phi_n(a_n) \in J_I$ it suffices to show that $\mathrm{cl}(\bigcup_{n\in\omega} \phi_n(a_n)) \subseteq \bigcup_{n\in\omega} \mathrm{cl}(\phi_n(a_n)) \cup \{x\}$. Indeed, if $y \in \mathrm{cl}(\bigcup_{n\in\omega} \phi_n(a_n))$, then there is a sequence x_n of elements of $\bigcup_{n\in\omega} \phi_n(a_n)$ convergent to y. There are two cases:

- either there is m such that there are infinitely many x_n's in $\phi_m(a_m)$. In this case y is an element of $\mathrm{cl}(\phi_m(a_m))$;

- or in each of $\phi_m(a_m)$'s there are only finitely many x_n's. But then x_n's must converge to x and hence $y = x$.

In both cases we have $y \in \bigcup_{n\in\omega} \mathrm{cl}(\phi_n(a_n)) \cup \{x\}$, which ends the proof. □

Corollary 1.5 is now an adaptation of the proof of the Kechris–Louveau–Woodin theorem.

Proof of Corollary 1.5. Suppose $J = J_I$ for a σ-ideal I on a compact space X with a dense subset $D \subseteq X$ on which J lives. Consider the family $F = \{C \in K(X) : C \cap D \text{ is dense in } C\}$ and note that F is Borel and the map $C \mapsto C \cap D$ is a Borel function from F to $P(D)$. Let $I' = I \cap F$ and note that I' is coanalytic as it is the preimage of J by the above function.

If I' is $\mathbf{\Pi}^0_2$, then J is $\mathbf{\Pi}^0_3$-complete by Theorem 1.4. On the other hand, if I' is not $\mathbf{\Pi}^0_2$, then by the Hurewicz separation theorem [17, Theorem 21.18], there is a Cantor set $C \subseteq K(X)$ such that $C \cap I' = \mathbb{Q}$. Now consider the function $K \mapsto (\bigcup K) \cap D$ from $K(C)$ to $P(D)$. Since $\bigcup K$ is compact for a compact $K \subseteq K(C)$, we have that if $K \subseteq \mathbb{Q}$, then $\bigcup K \in I'$ and hence $(\bigcup K) \cap D \in J_I$. On the other hand, if $K \not\subseteq \mathbb{Q}$, then $\bigcup K$ does not belong to I but still belongs to F, and hence $(\bigcup K) \cap D \notin J$. This proves that the above function is a reduction from $K(\mathbb{Q})$ (which is a $\mathbf{\Pi}^1_1$-complete subset of $K(2^\omega)$) to J and shows $\mathbf{\Pi}^1_1$-completeness of J. □

5. Weakly selective ideals

We follow standard set-theoretic notation concerning trees. In particular, a *branch* through a tree $T \subseteq \omega^{<\omega}$ is a sequence $t \in \omega^\omega$ such that $t \restriction n \in T$ for every $n \in \omega$. The set of all branches through a tree T is denoted by $[T]$. Given a tree $T \subseteq \omega^{<\omega}$, we say that branch $(n_1, n_2, \ldots) \in [T]$ is J-*positive* if $\{n_1, n_2, \ldots\} \notin J$. Given a tree $T \subseteq \omega^{<\omega}$ and $t \in T$ we write $\mathrm{split}_T(t) = \{n \in \omega : t^\frown n \in T\}$. Given a family A of subsets of ω, we say that a tree $T \subseteq \omega^{<\omega}$ is A-*splitting* if for each $t \in T$ we have $\mathrm{split}_T(t) \in A$. Given a tree T, we call the sets $\mathrm{split}_T(t)$ for $t \in T$ the *splitting sets of* T.

A subclass of weakly selective ideals are the selective ideals (see Mathias [19], Farah [4] and Grigorieff [9]). An ideal J is *selective* [9, Definition 1.7] if every J-partition of ω admits a J-positive selector. Here, a J-*partition* of ω is a partition $\omega = \bigcup_n a_n$ with $\bigcup_{m>n} a_m \notin J$ for every m. Equivalently [4, Definition 1.1], J is selective if any sequence of J-positive sets a_n has a J-positive *diagonalization*, i.e. a set $a \notin J$ such that $a \setminus n \subseteq a_n$ for each n. Selective ideals have been studied by Grigorieff, who proved [9, Corollary 1.15] a characterization of selectivity in terms of branches of trees: an ideal J is selective if and only if every tree T with the property that any finite intersection of splitting sets of T is in J^+, has a J-positive branch. The following lemma provides a similar characterization of weak selectivity. The proof is similar to that of Grigorieff and one implication is implicit in [20].

Lemma 5.1. *Let J be an ideal on ω. The following are equivalent:*

(a) *J is weakly selective,*

(b) *for each J-positive b, every co-$(J \restriction b)$-splitting tree has a J-positive branch.*

Proof. (b)⇒(a). Suppose J is not weakly selective, i.e. there is a J-positive b and a function $f : b \to \omega$ which is neither constant nor one-to-one on any J-positive subset of b. This means that preimages of single points belong to J. We will produce a co-$(J \restriction b)$-splitting tree whose all branches are in J. Define the tree $T \subseteq b^{<\omega}$ inductively as follows. If $t = (n_0, \dots, n_k) \in b^{<\omega}$ belongs to T, then add to T all $t^\frown n$ with $n \notin \bigcup_{i \leq k} f^{-1}(n_i)$. Note that this is a co-$(J \restriction b)$-splitting tree and if $x = (n_0, n_1, \dots)$ is a branch through T, then f is one-to-one on $\{n_0, n_1, \dots\}$. Thus, all branches of T are in J.

(a)⇒(b). Suppose now J is weakly selective, b is J-positive and T is a co-$(J \restriction b)$-splitting tree. We need to produce a J-positive branch through T. First, note that if J is weakly selective, then any sequence a_n of sets in co-$(J \restriction b)$ has a J-positive pseudointersection. Indeed, if $\bigcap_n a_n \notin J$, then this intersection is in particular a pseudointersection. Otherwise the sets $\bigcap_{m<n} a_m \setminus \bigcup_{m \geq n} a_m$ and $\bigcap_n a_n$ define a partition of b into sets in J and any selector of that partition is a pseudointersection of a_n's.

Now, let c be a J-positive pseudointersection of the splitting sets of T. Define a sequence k_n of elements of c by induction in the following way. Let k_0 be the minimum of c. Let k_{n+1} be the minimal element of c bigger than k_n such that $c \setminus k_{n+1}$ is contained in $\mathrm{split}_T(t)$ for each $t \in T$ whose length is not greater than k_n and whose all elements are not greater than k_n. The intervals $[k_n, k_{n+1})$ define a partition of c into finite sets. Let $d \subseteq c$ be a J-positive selector of that partition. Write $d_0 = d \cap \bigcup_n [k_{2n}, k_{2n+1})$ and $d_1 = d \cap \bigcup_n [k_{2n+1}, k_{2n+2})$. Note that by the definition of k_n's, both sets d_0 and d_1 are branches through T. Since $d = d_0 \cup d_1$, one of these sets must be J-positive and hence T has a J-positive branch, as needed. ☐

The proof of Theorem 1.6 will be based on the above lemma as well as on some ideas of Hrušák from his Category Dichotomy [11, Theorem 5.20]. In particular, we will use a game $H(J)$, invented by Hrušák in [11].

Proof of Theorem 1.6. (ii)⇒(i) Suppose $J = \bigcap_{l \in \Lambda} J_l$, where each J_l has a topological representation. Let $f : b \to \omega$ be a function with $b \notin J$. Suppose f is not constant on any J-positive subset of b. Pick $l \in \Lambda$ such that $b \notin J_l$. We will find a J_l-positive subset of b on which f is 1-1. Let X be a separable metric space, D its dense countable subset and I a σ-ideal of subsets of X such that J_l is isomorphic to J_I. Without loss of generality assume $b \subseteq D$. Write $B = \mathrm{cl}(b)$ and let $B' = B \setminus \{U : U$ is basic open and $B \cap U \in I\}$. Note that B' is still I-positive. Enumerate all basic open sets in X which intersect B' into a sequence $\langle V_n : n < \omega \rangle$ and by induction on i construct a sequence of points $n_i \in b$ such that $f(n_i) \notin \{f(n_j) : j < i\}$ and $n_i \in V_i$. Once this is done put $b' = \{n_i : i < \omega\}$ and note that $b' \notin J_l$ since $\mathrm{cl}(b')$ contains B'. Obviously, then f is 1-1 on b'.

To perform the construction, suppose that points n_i are chosen for $i<k$ and let $a=b\cap V_k$. Note that $a\notin J_l$. By our assumption, f is not constant on any J_l-positive set, so it cannot assume finitely many values on a, which implies that there is $n_k\in a$ such that $f(n_k)\neq f(n_i)$ for all $i<k$. This ends the construction.

(i)⇒(ii) Suppose now that J is coanalytic ideal of subsets of ω. Let $D\subseteq[\omega]^\omega\times\omega^\omega$ be closed set such that $[\omega]^\omega\setminus J$ is the projection of D.

Consider the following game $H'(J)$. In his n-th turn Player II picks a set $a_n\in J$. Player I responds with a pair (k_n,m_n) with $k_n\in\omega\setminus a_n$ and $m_n\in n\cup\{\text{pass}\}$. Player I wins if at the end he has chosen infinitely many m_n's different than 'pass' and $(\bar{k},\bar{m})$ belongs to D, where $\bar{k}=\{k_n:n<\omega\}$ and $\bar{m}$ is the sequence of those m_n's which are not equal to 'pass'.

The game $H'(J)$ is an unfolded version of the game $H(J)$ in which Player II picks $a_n\in J$ and Player I responds just with a number $k_n\notin a_n$. Player I wins in $H(J)$ if $\{k_n:n<\omega\}$ does not belong to J.

Claim 5.2. *If Player II has a winning strategy in $H'(J)$, then he also has a winning strategy in $H(J)$.*

Proof. Let σ be a winning strategy for Player II in $H'(J)$. We describe a strategy σ' for Player II in the game $H(J)$. Suppose Player II is about to make his n-th move after Player I has played $k_0,\ldots,k_{n-1}$. Let F be the finite set of all sequences $m_0,\ldots,m_n$ such that $m_i\in i\cup\{\text{pass}\}$ and for each $(m_0,\ldots,m_{n-1})\in F$ let a_f be the n-th move in the game $H'(J)$ according to the strategy σ after Player I has played $(k_0,m_0),\ldots,(k_{n-1},m_{n-1})$. Let the move of Player II in $H(J)$ be $\bigcup_{f\in F}a_f$.

We claim that this is a winning strategy for Player II. Suppose it is not and there is a counterplay of Player I. The counterplay is a sequence $(k_n:n<\omega)$ such that $\{k_n:n<\omega\}\notin J$. We will find a counterplay to the strategy σ in $H'(J)$. Since $\{k_n:n<\omega\}\notin J$, there is a sequence $(m_n:n<\omega)\in\omega^\omega$ such that $(\{k_n:n<\omega\},(m_n:n<\omega))\in D$. Let m'_n be the sequence such that $m_n\in i\cup\{\text{pass}\}$ and the elements of m'_n different from 'pass' enumerate $(m_n:n<\omega)$. Consider now the play in which Player I plays (k_n,m'_n) and Player II plays according to σ. Note that this is a legal play in $H'(J)$ by the definition of σ'. It is also a counterplay to σ in which Player I wins. □

Claim 5.3. *If J is weakly selective, then Player II cannot have a winning strategy in $H(J)$.*

Proof. Suppose there is such strategy and let T be the tree of all counterplays of Player I, i.e. $T=\{(k_0,\ldots,k_n):n<\omega\text{ and }k_i\notin\sigma(k_0,\ldots,k_{i-1})\text{ for each }i\leq k\}$. Note that T is a co-J-splitting tree whose all branches belong to J. But by Lemma 5.1, any co-J-splitting tree must have a J-positive branch. □

Claim 5.4. *If J is weakly selective, then for every $b\notin J$ there is a countable family Y of J-positive subsets of b such that for every $a\in J$ there is $x\in Y$ with $x\cap a=\emptyset$.*

Proof. Given a set $b \notin J$ consider the ideal $J \restriction b = \{a \subseteq b : a \in J\}$ of subsets of b. Note that it is still weakly selective and coanalytic. Hence, by Claims 5.2 and 5.3 and the fact that the game $H'(J \restriction b)$ is a closed game, there is a winning strategy σ for Player I in $H'(J \restriction b)$. Let T be the tree of all partial plays in $H'(J \restriction b)$, i.e. all sequences $(a_0, n_0, m_0, a_1, n_1, m_1, \ldots, a_k, n_k, m_k)$ such that $(n_i, m_i) = \sigma((a_0, n_0, m_0, \ldots, a_i))$. Now, inductively, for each k find a subset T'_k of T consisting of sequences $(a_0, n_0, m_0, \ldots, a_k, n_k, m_k)$ such that:

- $(a_0, n_0, m_0, \ldots, a_{k-1}, n_{k-1}, m_{k-1})$ belongs to T'_{k-1},
- if $(a_0, n_0, m_0, \ldots, a_k, n_k, m_k)$ and $(a'_0, n_0, m_0, \ldots, a'_k, n_k, m_k)$ belong to T'_k, then $a_i = a'_i$ for each $i \leq k$.

Let $T' = \bigcup_k T'_k$ and put

$$S = \{(n_0, \ldots, n_k) \in \omega^{<\omega} : \exists (a_0, \ldots, a_k), (m_0, \ldots, m_k) \\ (a_0, n_0, m_0, \ldots, a_k, n_k, m_k) \in T'\}.$$

The tree S is a subtree of $b^{<\omega}$, whose all branches are $(J \restriction b)$-positive, as if $(n_0, n_1, \ldots) \in [S]$, then there are $(a_0, a_1, \ldots), (m_0, m_1, \ldots)$ such that $(a_0, n_0, m_0, a_1, n_1, m_1, \ldots) \in [T']$ and all branches through T' follow the strategy σ. The tree S is also $(J \restriction b)^+$-splitting. Indeed, if $a = \mathrm{split}_S(t) \in J$ for some $t = (n_0, \ldots, n_k) \in S$, then pick $(a_0, \ldots, a_k)$ and $(m_0, \ldots, m_k)$ such that $(a_0, n_0, m_0, \ldots, a_k, n_k, m_k) \in T'_k$. Let then

$$(n, m) = \sigma((a_0, n_0, m_0 \ldots, a_k, n_k, m_k, a)).$$

Note that $n \notin a$ and by the construction, there is a' such that

$$(a_0, n_0, m_0 \ldots, a_k, n_k, m_k, a', n, m) \in T'_{k+1}.$$

So $(n_0, \ldots, n_k, n) \in S$ and this contradicts the fact that $n \notin a$.

Consider now the family of all splitting sets of S. We claim that this is the desired family. Indeed, if a intersects all these sets, then a contains a branch through S and therefore, it is J-positive. □

For each $b \notin J$ let Y_b be a countable family of subsets of b as in Claim 5.4. We say that a family X of J-positive sets is *almost separating* if

- for every $x \in X$ and $n \in x$ there is $y \in X$ with $y \subseteq x$ and $n \notin y$,
- for every $x \in X$ there are $x_0, x_1 \in X$ with $x_0, x_1 \subseteq x$ such that $x_0 \cap x_1 = \emptyset$ and $x \setminus (x_0 \cup x_1) \in J$.

We also say that a family X of J-positive sets is *almost closed under finite intersections* if

- every finite intersection of elements of X is either empty or J-positive,
- for every $x_0, \ldots, x_n \in X$ if $\bigcap_{i \leq n} x_i \notin J$, then there is $y \in X$ such that $y \subseteq \bigcap_{i \leq n} x_i$ and $(\bigcap_{i \leq n} x_i) \setminus y \in J$.

Claim 5.5. *For each $b \notin J$ there is a countable family X of J-positive subsets of b which is almost closed under finite intersections, almost separating* *and such that for each $x \in X$ and $a \in J$ there is $y \in X$ with $y \subseteq x$ and $y \cap a = \emptyset$.*

Proof. First note that since $J \restriction x$ is not a maximal ideal (as it has the Baire property) for every J-positive set $x \subseteq b$, there are two disjoint complementary J-positive subsets of x, say $x(0), x(1)$. Moreover, there is a countable family S_x of J-positive subsets of x which separates points in x. Let Z be a countable family of J-positive subsets of b such that $b \in Z$ and

- for every $c \in Z$ we have $Y_c \subseteq Z$,
- for every $c \in Z$ we have $c(0), c(1) \in Z$ and $S_c \subseteq Z$,
- for every $c_0, \ldots, c_n \in Z$ if $\bigcap_{i \leq n} c_i \notin J$, then $\bigcap_{i \leq n} c_i \in Z$.

Enumerate Z with infinite repetitions as $\{z_0, z_1, \ldots\}$. Now, by induction construct sets x_i as follows. Let $x_0 = z_0$ and $x_{i+1} = z_i \setminus \bigcup\{z_i \cap x_j : j < i \text{ and } z_i \cap x_j \in J\}$. Now, the family $X = \{x_i : i < \omega\}$ is as needed. Indeed, note that for each $i < \omega$ we have $x_i \subseteq z_i$ and $z_i \setminus x_i \in J$, so the properties of the family X follow immediately from the construction of the family Z. $\square$

Given a set $b \notin J$ let X_b be a countable family of J-positive subsets of b as in Claim 5.5. Let

$$J_b = \{a \subseteq \omega : \forall x \in X_b \ \exists y \in X_b \quad y \subseteq x \ \wedge \ a \cap y = \emptyset\}.$$

Note that J_b is an ideal of subsets of ω.

Claim 5.6. *For each $b \notin J$ the ideal J is contained in J_b.*

Proof. This follows directly from the properties of X_b. $\square$

Claim 5.7. *For each $b \notin J$ the ideal J_b has a topological representation.*

Proof. We will check that J_b is countably separated and dense.

J_b is countably separated by X_b. Indeed, Let $a \in J_b$ and $c \notin J_b$. Since $c \notin J_b$, there is $x \in X_b$ such that for no $y \in X_b$ with $y \subseteq x$ it is the case that $y \cap c = \emptyset$. Note that actually for each such y we have $y \cap c \notin J_b$. Now, since $a \in J_b$, there is $y \in X_b$ with $y \subseteq x$ and $y \cap a = \emptyset$.

To see that J_b is dense, let $c \subseteq \omega$ be infinite. We need to find an infinite $a \subseteq c$ such that $a \in J_b$. We can assume that $c \notin J_b$. This means that there is $x \in X_b$ such that for each y with $y \subseteq x$ we have $y \cap c \notin J_b$. Enumerate X_b as $\{x_i : i < \omega\}$. By induction on i, construct a strictly increasing sequence n_i and J-positive sets $y_i \in X_b$ such that

(a) $y_{i+1} \subseteq y_i$, $y_i \subseteq x$ (so $y_i \cap c \notin J_b$) and $n_i \in y_i \cap c$
(b) $x_i \setminus y_i$ contains an element of X_b.

We start with $y_{-1} = x$. To perform the induction step, use the fact that X_b is almost separating and find two J_b-positive sets $x_i(0), x_i(1) \in X_b$ which are subsets of x_i and such that $x_i \setminus (x_i(0) \cup x_i(1)) \in J$. If $y_{i-1} \cap x_i$ is empty, then put $y_i = y_{i-1}$ and pick any $n_i \in y_i \cap c$ bigger than n_{i-1}. If $y_{i-1} \cap x_i$ is nonempty, then it is J-positive. Note that at least one of $y_{i-1} \cap x_i(0)$ or $y_{i-1} \cap x_i(1)$ must be nonempty. Since X_b is almost closed under finite intersections, one of these sets contains an element of X_b, say y_i. Pick any $n_i \in y_i \cap c$ bigger than n_{i-1}. This ends the construction.

Put $a = \{n_i : i < \omega\}$. We claim that a belongs to J_b. Indeed, pick $x_i \in X_b$. By (b) $x_i \setminus y_i$ contains an element of X_b, say y. Hence, by (a), $a \cap y$ is finite. Since X_b almost separates points, we can further shrink y to $z \in X_b$ such that $z \cap a$ is empty. This shows that $a \in J_b$. □

Now, for each $b \notin J$ we have the ideal J_b such that $J \subseteq J_b$, $b \notin J_b$ and J_b has a topological representation. This implies that $J = \bigcap_{b \notin J} J_b$ is an intersection of ideals which have topological representations and ends the proof. □

References

[1] Antonio Avilés and Stevo Todorčević. Multiple gaps. *Fund. Math.*, 213(1):15–42, 2011.

[2] Tomek Bartoszyński and Haim Judah. *Set theory*. A K Peters Ltd., Wellesley, MA, 1995. On the structure of the real line.

[3] James E. Baumgartner and Richard Laver. Iterated perfect-set forcing. *Ann. Math. Logic*, 17(3):271–288, 1979.

[4] Ilijas Farah. Semiselective coideals. *Mathematika*, 45(1):79–103, 1998.

[5] Ilijas Farah. Analytic quotients: theory of liftings for quotients over analytic ideals on the integers. *Mem. Amer. Math. Soc.*, 148(702), 2000.

[6] Ilijas Farah. Basis problem for turbulent actions. II. c_0-equalities. *Proc. London Math. Soc. (3)*, 82(1):1–30, 2001.

[7] Ilijas Farah. Analytic Hausdorff gaps. II. The density zero ideal. *Israel J. Math.*, 154:235–246, 2006.

[8] Ilijas Farah and Sławomir Solecki. Two $F_{\sigma\delta}$ ideals. *Proc. Amer. Math. Soc.*, 131(6):1971–1975 (electronic), 2003.

[9] Serge Grigorieff. Combinatorics on ideals and forcing. *Ann. Math. Logic*, 3(4):363–394, 1971.

[10] Greg Hjorth. A dichotomy theorem for turbulence. *J. Symbolic Logic*, 67(4):1520–1540, 2002.

[11] Michael Hrušák. Combinatorics of filters and ideals. In *Set theory and its applications*, volume 533 of *Contemp. Math.*, pages 29–69. Amer. Math. Soc., Providence, RI, 2011.

[12] Vladimir Kanovei. *Borel equivalence relations*, volume 44 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008. Structure and classification.

[13] Vladimir Kanovei, Marcin Sabok, and Zapletal Jindřich. *Canonical Ramsey Theory on Polish Spaces*, volume 202 of *Cambridge Tracts in Mathematics*. Cambridge University Press, 2013.

[14] A. S. Kechris, A. Louveau, and W. H. Woodin. The structure of σ-ideals of compact sets. *Trans. Amer. Math. Soc.*, 301(1):263–288, 1987.

[15] A. S. Kechris and S. Solecki. Approximation of analytic by Borel sets and definable countable chain conditions. *Israel J. Math.*, 89(1-3):343–356, 1995.

[16] Alexander S. Kechris. Hereditary properties of the class of closed sets of uniqueness for trigonometric series. *Israel J. Math.*, 73(2):189–198, 1991.

[17] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[18] É. Matheron, S. Solecki, and M. Zelený. Trichotomies for ideals of compact sets. *J. Symbolic Logic*, 71(2):586–598, 2006.

[19] A. R. D. Mathias. Happy families. *Ann. Math. Logic*, 12(1):59–111, 1977.

[20] David Meza Alcántara. Ideals and filters on countable sets. 2009. Ph.D. thesis at the Universidad National Autónoma Mexico.

[21] Christian Rosendal. Cofinal families of Borel equivalence relations and quasiorders. *J. Symbolic Logic*, 70(4):1325–1340, 2005.
[22] Marcin Sabok and Jindřich Zapletal. Forcing properties of ideals of closed sets. *J. Symbolic Logic*, 76(3):1075–1095, 2011.
[23] Sławomir Solecki. Analytic ideals. *Bull. Symbolic Logic*, 2(3):339–348, 1996.
[24] Sławomir Solecki. Analytic ideals and their applications. *Ann. Pure Appl. Logic*, 99(1-3):51–72, 1999.
[25] Sławomir Solecki. G_δ ideals of compact sets. *J. Eur. Math. Soc. (JEMS)*, 13(4):853–882, 2011.
[26] Stevo Todorčević. Analytic gaps. *Fund. Math.*, 150(1):55–66, 1996.
[27] J Zapletal. preprint in preparation.

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