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ON IDEALIZED FORCING

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Introduction

The concept of a σ-ideal seems to have been present in forcing from its very beginnings. The classical forcing notions, like the Cohen, random, Miller or Sacks forcing can be described in terms of σ-ideals on Polish spaces. In fact, most of the forcing notions used to prove results in classical descriptive set theory and analysis fit into this general scheme. In [**35**] Zapletal has taken on general study of forcing notions which can be defined in terms of σ-ideals on Polish spaces and developed a far-reaching and deep theory. It connects the approach of proper forcing, introduced by Shelah (see [**29**]), with the study of certain classes of σ-ideals. The theory of idealized forcings also sheds new light on the countable-support iteration of the Sacks or Miller forcing, particularly on the structure of forcing conditions and the axiomatic approach to the iteration in terms of the Covering Property Axiom (see [**7**] and [**14**]).

This dissertation is motivated by the program of *Idealized Forcing* [**35**]. Our approach is, however, reverse to the one from [**35**]. Starting with σ-ideals on Polish spaces we give combinatorial description of forcing notions associated to them. This gives new insight into the structure of idealized forcings and enables us to answer some open questions and sharpen some existing results in this theory. The central point of this thesis is Theorem 3.4, where we establish a combinatorial description of an arbitrary forcing associated to a σ-ideal generated by closed sets. We also focus on the aspects of the theory coming from applications in classical descriptive set theory and the theory of real functions.

Motivation and background

An *idealized forcing* is a forcing with the positive elements of the Boolean algebra $\mathbf{Bor}(X)/\mathcal{I}$ (which is usually denoted by $\mathbb{P}_\mathcal{I}$), where X is a Polish space and $\mathcal{I}$ is a σ-ideal on X. Equivalently, $\mathbb{P}_\mathcal{I}$ can be described as the family of $\mathcal{I}$-positive Borel sets, ordered by inclusion. Most of the classical forcing notions can be obtained in this way. The most important examples are: the Cohen forcing associated to the σ-ideal of meager sets, the random forcing (σ-ideal of null sets), the Sacks forcing (σ-ideal of countable sets) or the Miller forcing (σ-ideal generated by compact sets in the Baire space).

Classical forcings of this form have been applied for decades in descriptive set theory, with applications to measure theory, topology and analysis. The results obtained by forcing are not only independece results but also pure ZFC theorems.

Classical applications of forcing in descriptive set theory range from the study of Borel functions, structure of the Borel sets in products of Polish spaces, to uniformization results (see [**9**]). The approach of [**35**] gives new and important results in all these areas.

What lies in the heart of applications of forcing, is the structure of names for reals in the generic extension. It is known (see [**9**] or [**6**]) that in most classical examples the names for reals can be realized as Borel functions from the ground model (see Chapter 2). In [**35**] Zapletal proved that this holds under the assumption that the forcing is proper. This has opened a wide range of possibilities for applying proper forcing notions in descriptive set theory.

In all classical examples, however, the names for reals can be realized by continuous functions from the ground model. This property is called *continuous reading of names* (see Chapters 2 and 4). It is important for the uniformization results and for the study of Borel sets in products of Polish spaces (see [**9**] and [**34**]).

In [**35**, Theorem 4.1.2] Zapletal has shown that both properness of the forcing and continuous reading of names hold for the forcing $\mathbb{P}_{\mathcal{I}}$ if the σ-ideal $\mathcal{I}$ is generated by closed sets. σ-ideals generated by closed sets have also been investigated in descriptive set theory. Note that if a σ-ideal $\mathcal{I}$ on a Polish space X is generated by closed sets, then $\mathcal{I}$ is determined by $\mathcal{I} \cap \mathbf{\Pi}^0_1(X)$ (the family of closed sets which are in $\mathcal{I}$), which is a σ-ideal of closed sets. Efforts have been made to analyze possible complexities of σ-ideals of closed sets in the hyperspace of X. Kechris, Louveau and Woodin have established stringent constraints. They proved (see [**15**, Theorem 33.3]) that if a σ-ideal of closed sets is coanalytic in the hyperspace of X, then it is either $\mathbf{\Pi}^1_1$-complete, or else a $\mathbf{G}_\delta$ set. Moreover, Kechris, Louveau and Woodin [**16**, Theorem 7] proved that if a σ-ideal of closed sets on X is analytic, then it is actually a $\mathbf{G}_\delta$ set.

Although usually we apply forcing to descriptive set theory, sometimes descriptive set theory is used to solve problems arising in forcing. Recall a theorem of Solecki [**30**, Theorem 1] which says that if $\mathcal{I}$ is a σ-ideal generated by closed sets, then any analytic set either belongs to $\mathcal{I}$, or else it contains an $\mathcal{I}$-positive $\mathbf{G}_\delta$ set. This theorem has important consequences in forcing, for example it implies that the forcing of $\mathcal{I}$-positive Borel sets is equivalent to the forcing of $\mathcal{I}$-positive analytic sets. In this thesis we will find many more applications of the Solecki theorem in Idealized Forcing.

Overview of the results

In the first part of the dissertation we study idealized forcing, for σ-ideals generated by closed sets. The main motivation here comes from the examples of the Sacks and the Miller forcing.

Zapletal proved [**35**, Theorem 4.1.2] that if $\mathcal{I}$ is generated by closed sets, then the forcing $\mathbb{P}_{\mathcal{I}}$ is proper. For the Miller and the Sacks forcing, however, much more can said since the two are forcing with trees which have fusion (satisfy Axiom A). For σ-ideals generated by closed sets we introduce *generalized Banach-Mazur games*, which describe the σ-ideals in terms of existence of a winning strategy for one of the players. Using these games we show that if $\mathcal{I}$ is generated by closed sets, then the forcing $\mathbb{P}_{\mathcal{I}}$ is equivalent to a forcing with trees satisfying Axiom A. This strenghtens a result of Zapletal [**35**, Theorem 4.1.2]. The meaning of our results is that in the forcing $\mathbb{P}_{\mathcal{I}}$, there is always a rich structure, which implies properness and can be used for better understanding generic extensions.

We study σ-ideals generated by closed sets which are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ (for definition see Chapter 2). We introduce a Borel structure on the family of $\mathbf{G}_\delta$ subsets of a Polish space and examine the projective complexity of $\mathcal{I}$-positive $\mathbf{G}_\delta$ sets. Using this, we give a new proof of the theorem saying that if a σ-ideal $\mathcal{I}$ generated by a hereditary coanalytic family of closed sets, then it is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Next, we study the Fubini powers of $\mathcal{I}$. We prove that if $A \subseteq X^\omega$ is analytic, then either $A \in \mathcal{I}^\omega$ (for definition see Section 3.6), or else A contains a $\mathbf{G}_\delta$ "tree" of $\mathcal{I}$-positive $\mathbf{G}_\delta$ sets (for details see Section 3.6). This result can be treated as an amalgamation of the results of Solecki [**30**, Theorem 1] and Kanovei and Zapletal [**35**, Theorem 5.1.9].

We illustrate the theory of idealized forcing for σ-ideals generated by closed sets with several examples motivated by analysis and measure theory.

We study a class of σ-ideals arising from Baire class 1, not piecewise continuous functions (for definitions see Section 3.7). We give a complete description of the forcing in this case. Namely, we prove that for any Baire class 1 function which is not piecewise continuous, the associated forcing notion is equivalent to the Miller forcing. We also prove that in this case the σ-ideal is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ and we provide a Banach-Mazur game characterization of the σ-ideal.

We also study the σ-ideal $\mathcal{E}$ generated by closed null sets. We prove that the forcing $\mathbb{P}_{\mathcal{E}}$ does not add Cohen reals, which implies that the forcing extensions are minimal. We provide a Banach-Mazur game characterization for the σ-ideal $\mathcal{E}$ too.

In the second part of the dissertation we study the property of continuous reading of names and the "canonical" example without this property — the

Steprāns forcing. We introduce the notion of a *wide tree* and show that the Steprāns forcing is equivalent to the forcing with wide trees. We use wide trees to show that the Steprāns forcing has continuous reading of names in an extended topology (in the so-called Baire topology on $(\omega+1)^\omega$). The latter was announced by Hrušak and Zapletal [**10**, Example 2.7] but the proof they provided is incorrect.

We also show that the σ-ideal associated to the Pawlikowski function is not generated by closed sets in the Baire topology on $(\omega+1)^\omega$. This disproves a claim of Hrušak and Zapletal [**10**, Example 2.7] and answers a question of Steprāns [**32**, §4, p. 1273].

In [**11**, Question 5.5] Hrušak and Zapletal asked whether there exists an idealized forcing which does not have continuous reading of names in any presentation. We answer this question positively, constructing a class of examples of such forcing notions.

In the third part of the dissertation we study the structure of Borel functions. In [**31**, Theorem 4.1] Solecki proved the following dichotomy. A Baire class 1 function is either σ-continuous, or else "contains" the Pawlikowski function P.

The Pawlikowski function P is a function from $(\omega+1)^\omega$ into the Baire space. Although $(\omega+1)^\omega$ is homeomorphic to the Cantor space, there appears to be some additional, "geometric" structure on that space. We introduce a family of projections on $(\omega+1)^\omega$ and use them to study the geometry of $(\omega+1)^\omega$.

We use the structure of projections on $(\omega+1)^\omega$ to give a new proof of the Solecki dichotomy. Our proof works for arbitrary Borel functions $f:\omega^\omega\to\omega^\omega$.

Navigation

Let us state very briefly how this dissertation is organized. In Chapter 3 we study idealized forcing with σ-ideals generated by closed sets (these results are also included in [**26**]). In Chapter 4 we study continuous reading of names and the Steprāns forcing (these results are also included in [**27**]). In Chapter 5 we study σ-continuity of Borel functions (these results are also included in [**25**]).

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2
Preliminaries

2.1 Descriptive set theory

A *Polish space* is a separable completely metrizable space. We are going to consider several Polish spaces in this dissertation, with two the most important cases: the Baire space ω^ω and the Cantor space 2^ω. We will make use of the basic theory of Polish spaces as exposed in [**15**]. A fundamental fact about Polish spaces is that $\mathbf{G}_\delta$ subsets of Polish spaces are also Polish and that each Polish space is homeomorphic to a $\mathbf{G}_\delta$ set in the Hilbert cube $[0,1]^\omega$.

Basic operations on Polish spaces include products of two and of countably many Polish spaces. For a Polish space X we denote by $K(X)$ the hyperspace of X, consisting of all compact subsets of X, with the Vietoris topology. $K(X)$ is also Polish and if X is compact then $K(X)$ is compact too. By $F(X)$ we denote the Effros Borel space, i.e. the family of closed subsets of X with the σ-algebra generated by the sets $\{F \in F(X) : F \cap U \neq \varnothing\}$, where U vaires over open subsets of X.

We will also used some effective descriptive set theory, as exposed in [**20**]. We will always tacitly assume that a Polish space X is given with a fixed *recursive presentation* (cf. [**20**, Section 3.B]). In particular, a Polish space is always given with a fixed basis of open sets.

We will use the following model-theoretic notation. We denote first order formulas by $\varphi(v)$ or $\varphi(v,w)$ (v, w are variables). Suppose $\varphi(v)$ is a formula with a single variable v, M is a (class) model and $A \in M$. By $\varphi^M(A)$ we denote the set $\{a \in A \cap M : M \models \varphi(a)\}$. If $M = V$, then we write only $\varphi(A)$ for this set.

We say that a set $T \subseteq Y^{<\omega}$ is a *tree on* Y if for each $\sigma \subseteq \tau \in Y^{<\omega}$ if $\tau \in T$, then $\sigma \in T$ as well. Elements of trees are called *nodes*. A tree is *pruned* if it has no terminal nodes. By $\mathrm{Lev}_n(T)$ we denote the set $\{\tau \in T : |\tau| = n\}$ ($|\tau|$ stands for the length τ). We call $\mathrm{Lev}_n(T)$ the *n-th level of T*. For a tree

$T \subseteq Y^{<\omega}$ we denote by $\lim T$ the set $\{x \in Y^\omega : \forall n < \omega\ x{\restriction}n \in T\}$. For a node $\tau \in T$ the *end-extension of* τ *in* T, denoted by $T(\tau)$, stands for the subtree $\{\sigma \in T : \tau \subseteq \sigma \vee \sigma \subseteq \tau\}$. For a set $T_0 \subseteq T$ the *end-extension of* T_0 *in* T is the union of end-extensions of elements of T_0 in T. For $\tau \in T$ we write $[\tau]_T$ for the set $\{x \in \lim T : \tau \subseteq x\} = \lim T(\tau)$. If the tree $T = Y^{<\omega}$ and Y is clear from the context, then we drop the subscript T and write only $[\tau]$ for the above set. We say that a subset $F \subseteq T$ is a *front of* T if

- for each $t \in \lim T$ there is $n < \omega$ and $\tau \in F$ such that $\tau \subseteq t{\restriction}n$,
- for each $\sigma, \tau \in F$, if $\sigma \neq \tau$, then neither $\sigma \subseteq \tau$, nor $\tau \subseteq \sigma$.

We say that a node $\tau \in T$ is *a stem of* T if for every $\sigma \in T$ either $\sigma \subseteq \tau$, or $\tau \subseteq \sigma$. For a finite sequence $\tau \in Y^{<\omega}$ and a tree T on Y we denote by $\tau^\frown T$ the tree $\{\tau^\frown\sigma : \sigma \in T\}$.

If Y is an arbitrary set and T is a tree on $\omega \times Y$, then we write $\mathrm{proj}[T]$ for the set $\{x \in \omega^\omega : \exists y \in Y^\omega\ \forall n < \omega\ \ (x{\restriction}n, y{\restriction}n) \in T\}$.

Given a countable set Y we are going to consider Y^ω as a topological space. Having a topology $\mathcal{O}$ on Y we equip Y^ω with the product topology of ω copies of $\mathcal{O}$. In case Y is a discrete space, the sets $[\tau]$, for $\tau \in Y^{<\omega}$, form a base of the topology on Y^ω.

In examining closed sets in the spaces Y^ω we will use the fact that for each closed set $C \subseteq Y^\omega$ there is a tree $T \subseteq Y^{<\omega}$ such that $C = \lim T$. Moreover, if Y is a discrete space, then for each tree $T \subseteq Y^{<\omega}$ the set $\lim T$ is closed in Y^ω.

In a metric space (X, d) for $A, B \subseteq X$ we write $\mathrm{dist}(A, B)$ for $\inf\{d(a, b) : a \in A, b \in B\}$. The Hausdorff distance between A and B is denoted by $h(A, B)$.

If X is a Polish space, then we denote by $\mathbf{Bor}(X)$ the family of Borel subsets of X. We are going to use the fact that the Borel sets can coded as a Δ^1_1 set over a Π^1_1 set, i.e. there exists a Π^1_1 set $P \subseteq X$ and two sets $A, C \subseteq X \times X$ $A \in \Sigma^1_1$, $C \in \Pi^1_1$ such that for each $x \in P$ the sets A_x and C_x are equal and for each Borel set $B \subseteq X$ there is $x \in P$ such that $B = A_x = C_x$. The set P is referred as to the set of *codes for Borel sets*. For any model M of a sufficient part of ZFC we write A^M, C^M and P^M for the above sets defined in the model M. If $x \in P^M$ and B is the Borel set such that $B = A_x = C_x$, then we say that the set $A^M_x = C^M_x$ *decodes* x in M and denote it by B^M.

A *projective pointclass* on X is one of the families $\mathbf{Bor}(X)$, $\Sigma^1_n(X)$, $\Pi^1_n(X)$, $\mathbf{\Sigma}^1_n(X)$ or $\mathbf{\Pi}^1_n(X)$ for some $n < \omega$. A *Borel pointclass* is one of the families $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ for some $\alpha < \omega_1$, or Σ^0_α, Π^0_α for some $\alpha < \omega_1^{CK}$.

If ν is a probability Borel measure on a Polish space X, then we refer to (X, ν) as to a *measure Polish space*. We say that a Borel measure ν on a Polish space X is *strictly positive* if for each nonempty open set $U \subseteq X$ we have $\nu(U) > 0$. We say that ν is *continuous* if $\nu(\{x\}) = 0$ for every $x \in X$.

For a Borel measure ν on a Polish space X and a Borel set $A \subseteq X$ of positive measure we write ν_A for the *relative measure on* A, defined by $\nu_A(B) = \nu(B \cap A)/\nu(A)$.

By a *σ-ideal* on a Polish space X we mean a family $\mathcal{I} \subseteq \mathcal{P}(X)$ such that if $A, B \subseteq X$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$ and for each sequence $A_n \subseteq X$ if $A_n \in \mathcal{I}$ for each $n < \omega$, then also $\bigcup_{n<\omega} A_n \in \mathcal{I}$. If X is a Polish space, then we denote by $\mathcal{M}(X)$ the σ-ideal of meager sets in X. If (X, ν) is a measure space, then we denote by $\mathcal{N}(X, \nu)$ the σ-ideal of null sets. In case X is understood from the context, we will drop the reference to X and write simply $\mathcal{M}$ and $\mathcal{N}$.

By a *σ-ideal of closed sets* on a Polish space X we mean a family $\mathcal{K}$ of closed sets such that if $A, B \subseteq X$ are closed and $B \in \mathcal{K}$, then $A \in \mathcal{K}$ and for each sequence $A_n \subseteq X$ of closed sets if $A_n \in \mathcal{K}$ for each $n < \omega$ and $\bigcup_{n<\omega} A_n$ is closed, then also $\bigcup_{n<\omega} A_n \in \mathcal{K}$. Analogously we define *σ-ideals of analytic sets* and *σ-ideals of Borel sets*.

If $\mathcal{I}$ is a σ-ideal (or σ-ideal of analytic sets) on a Polish space X and $B \subseteq X$ is an $\mathcal{I}$-positive Borel set, then we write $\mathcal{I}{\restriction}B$ for $\{A \cap B : A \in \mathcal{I}, A \subseteq X\}$.

We say that a family $\mathcal{A}$ of analytic sets on a Polish space X is $\mathbf{\Pi}^1_1$ *on* $\mathbf{\Sigma}^1_1$ (resp. Π^1_1 *on* Σ^1_1) if for any $\mathbf{\Sigma}^1_1$ (resp. Σ^1_1) set $A \subseteq X \times X$ the set $\{x \in X : A_x \in \mathcal{A}\}$ is $\mathbf{\Pi}^1_1$ (resp. Π^1_1). A σ-ideal $\mathcal{I}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if $\mathbf{\Sigma}^1_1 \cap \mathcal{I}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. Recall that the σ-ideals $\mathcal{M}$ and $\mathcal{N}$ (on any Polish measure space), as well as the σ-ideals of countable or $\mathbf{K}_\sigma$ sets (on ω^ω) are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. We say that a σ-ideal $\mathcal{I}$ is *generated by closed sets* if for each set $A \in \mathcal{I}$ there is a sequence of closed sets $F_n \in \mathcal{I}$ such that $A \subseteq \bigcup_{n<\omega} F_n$. The σ-ideals $\mathcal{M}(X)$, $\mathbf{K}_\sigma$ and of countable sets are generated by closed sets. On the other hand, if (X, ν) is a measure Polish space with a strictly positive measure, then $\mathcal{N}(X, \nu)$ is not generated by closed sets, since there are null and comeager $\mathbf{G}_\delta$ sets in X.

If $\mathcal{I}$ is a σ-ideal on X and $B \subseteq X$ is a Borel set, then we say that B is *$\mathcal{I}$-positive* if $B \notin \mathcal{I}$. We say that B is *$\mathcal{I}$-full* if $X \setminus B \in \mathcal{I}$.

If $\mathcal{I}$ is a σ-ideal on a Polish space X and $A, B \subseteq X$, then we write $\forall^{\mathcal{I}} x \in A\ x \in B$ to denote the fact that $\{x \in A :\ x \notin B\} \in \mathcal{I}$. If A is a Polish space, then $\forall^{\mathcal{M}} x \in A\ x \in B$ means that $\{x \in A :\ x \notin B\}$ is meager in A. We will need the following version of the Kuratowski-Ulam theorem.

THEOREM 2.1 (Kuratowski, Ulam). *Let X and Y be Polish spaces and let $f : Y \to X$ be a continuous open surjection. Suppose $B \subseteq Y$ has the Baire property and*

$$\forall^{\mathcal{M}} x \in X \quad B \cap f^{-1}[\{x\}] \text{ is meager in } f^{-1}[\{x\}].$$

Then B is meager in Y.

PROOF. The proof is analogical to the proof of the "product" version of the Kuratowski-Ulam theorem [**15**, Theorem 8.41]. The only difference is that instead of [**15**, Lemma 8.42], we need to prove that if $U \subseteq Y$ is open dense, then

$$(*) \qquad \forall^{\mathcal{M}(X)} x \in X \quad U \cap f^{-1}[\{x\}] \text{ is open dense in } f^{-1}[\{x\}].$$

To show this, we take the open basis $\langle U_n : n < \omega\rangle$ of Y and we show that for each $n < \omega$ the set

$$V_n = \{x \in X : \ f^{-1}[\{x\}] \cap U_n = \varnothing \ \vee \ f^{-1}[\{x\}] \cap U_n \cap U \neq \varnothing\}$$

contains an open dense set. Indeed, let $W_n = X \setminus \overline{f[U_n]}$ and notice that the set $f[U_n] \cup W_n$ is open dense in X. Moreover, $W_n \subseteq V_n$ and $V_n \cap f[U_n]$ is dense open in $f[U_n]$. Now, notice that if $x \in \bigcap_{n<\omega} V_n$, then $U \cap f^{-1}[\{x\}]$ is open dense in $f^{-1}[\{x\}]$. This proves $(*)$. □

2.2 Infinite games

Infinite games used in descriptive set theory are two-player games in which each of the two players picks a natural number in each of the ω turns. We stick to the tradition of naming the two players Adam and Eve (Adam begins). A set $P \subseteq \omega^\omega$, called the payoff set, is given in advance. After the game has ended, a real $x \in \omega^\omega$ is constructed from the natural numbers picked by the two players (in a prescribed way). We say that Eve wins the game if $x \in P$, otherwise we say that Adam wins. A winning strategy for one of the players is understood as a set of "rules" that tells him or her how to play to win the game.

A good reference covering most of the standard applications of infinite games in descriptive set theory is [**15**]. Classical applications of games in the study of σ-ideals on Polish spaces include the Banach-Mazur games. Recall that a Banach-Mazur game $G(B)$ is defined for each set $B \subseteq \omega^\omega$ in such a way that Eve has a winning strategy in $G(B)$ if and only if B is meager.

In this dissertation we will use more general terminology than is commonly used in the context of infinite games.

By a game scheme we mean a set of rules for a two-player game. Formally, a game scheme is a pruned tree $G \subseteq Y^{<\omega}$ for a countable set Y, where the last elements of sequences at even and odd levels are understood as possible moves of Eve and Adam, respectively. In particular, in any game scheme the first move is made by Adam. Nodes of the tree G of even length are called *partial plays* and elements of $\lim G$ are called *plays*. Note that partial plays always end with a move of Eve.

If τ is a partial play in a game scheme G, then by the *relativized game scheme* G_τ we mean the tree $G(\tau)$. The game scheme G_τ consists of the games which "continue" the partial play τ.

A *payoff set* p in a game scheme G is to be understood as a subset of $\lim G$. By a game we mean a pair (G,p) where G is a game scheme and p is a payoff set in G (we say that the game (G,p) is *in the game scheme* G). For a game (G,p) we say that Eve *wins* a play $g \in \lim G$ if $g \in P$. Otherwise we say that Adam *wins* g.

A *strategy for Adam* in a game scheme G is a subtree $S \subseteq G$ such that

- for each odd $n \in \omega$ and $\tau \in S$ such that $|\tau| = n$ the set of immediate successors of τ in S contains precisely one point,

- each even $n \in \omega$ and $\tau \in S$ such that $|\tau| = n$ the sets of immediate successors of τ in S and G are equal.

Strategy for Eve is defined analogously. If (G, p) is a game in the game scheme G and S is a strategy for Adam in G, then we say that S is a *winning strategy for Eve* in the game (G, p) if $\lim S \subseteq p$. *Winning strategy for Adam* is defined analogously.

Recall the theorem of Martin [**15**, Theorem 20.5] which says that if $p \subseteq \lim G$ is a Borel set (here the topology on $\lim G$ is induced from Y^ω with the product of discrete topologies on Y), then the game (G, p) is determined, i.e. there is a winning strategy for one of the players.

2.3 Forcing

Basic forcing terminology can be found in standard reference books like [**13**] or [**18**]. If $\mathbb{P} = (P, \leq)$ is a partially ordered set, then we use the notation $p \leq q$ to denote that p is a stronger condition than q. We say that p and q are *compatible* if there exists $r \in \mathbb{P}$ such that $r \leq p$ and $r \leq q$. We write $p \perp q$ to denote that p and q are not compatible. A poset $\mathbb{P}$ is *separative* if for each $p, q \in P$ either $p \leq q$ or there is $r \leq p$ such that $r \perp q$. For every poset $\mathbb{P}$ the equivalence relation $E_{\mathbb{P}}$ is defined as follows: $pE_{\mathbb{P}}q$ if every extension of p is compatible with q and vice versa. The $E_{\mathbb{P}}$-classes of P form a partially ordered set, called the *separative quotient* of $\mathbb{P}$. The separative quotient of $\mathbb{P}$ is separative and is equivalent (in the forcing sense) to $\mathbb{P}$. Any separative partial ordering $\mathbb{P}$ can be densely embedded into a unique complete Boolean algebra, denoted by $ro(\mathbb{P})$.

If $\mathbb{P} = (P, \leq)$ is a poset and $p \in \mathbb{P}$, then by $\mathbb{P} \restriction p$ we denote the poset $\{q \in \mathbb{P} : q \leq p\}$ with the restricted ordering $\leq$.

In this dissertation we will be mainly concerned with *idealized forcings*, that is forcing notions of the form $\mathbb{P}_{\mathcal{I}} = (\mathbf{Bor}(X) \setminus \mathcal{I}, \subseteq)$, where $\mathcal{I}$ is a σ-ideal on a Polish space X. As exposed, $\mathbb{P}_{\mathcal{I}}$ is usually not separative and its separative quotient is the quotient Boolean algebra of $\mathbf{Bor}(X)$ mod $\mathcal{I}$. By $\mathbb{Q}_I$ we will always denote the forcing $(\mathbf{\Sigma}^1_1(X) \setminus \mathcal{I}, \subseteq)$.

Forcing extensions by an idealized forcing $\mathbb{P}_{\mathcal{I}}$ are of special form. Namely, the forcing $\mathbb{P}_{\mathcal{I}}$ adds one point of the space X, called the *generic point.* The generic point is defined as follows: it is the single element of

$$\bigcap \{B^{V[G]} : B \in G\},$$

where G is the generic filter over V. The generic point $g \in X$ determines the generic filter G by $G = \{B \in \mathbb{P}_{\mathcal{I}} : g \in B^{V[G]}\}$. There is a canonical name $\dot{g}$ for the generic point such that for each $B \in \mathbb{P}_{\mathcal{I}}$ we have

$$[\![\dot{g} \in \dot{B}]\!] = [\![\dot{B} \in \dot{G}]\!],$$

where $\dot{G}$ is the canonical name for the generic filter G and $\dot{B}$ is a name for the set $B^{V[G]}$. In case X is the Baire space or the Cantor space we also refer to the generic point as to the *generic real.*

One can show [**35**, the first paragraph of the proof of Proposition 2.2.2] that if $M \prec H_\kappa$ is a countable elementary submodel (for a big enough κ) such that M contains $\mathbb{P}_\mathcal{I}$, then the set of $\mathbb{P}_\mathcal{I}$-generic points over M is a Borel set.

If $\mathcal{I}$ is a σ-ideal on a Polish space X, then by a *presentation* of the poset $\mathbb{P}_\mathcal{I}$ we mean a poset $\mathbb{P}_\mathcal{J}$ where $\mathcal{J}$ is a σ-ideal on a Polish space Y such that there is a Borel bijection $f : X \to Y$ for which we have $A \in \mathcal{J}$ iff $f^{-1}[A] \in \mathcal{I}$ for any $A \in \mathbf{Bor}(Y)$. Moreover, if $g : X \to Y$ is a Borel function and $\mathcal{I}$ is a σ-ideal on X, then the σ-ideal $\{A \subseteq Y : g^{-1}[A] \in \mathcal{I}\}$ is called the *transported* (via g) σ-ideal.

The standard definition of properness of a forcing notion can be found in [**13**] or [**29**]. In the case of idealized forcing it can be stated in a more convenient form. Zapletal proved [**35**, Proposition 2.2.2] that a forcing $\mathbb{P}_\mathcal{I}$ is proper if and only if for each countable elementary submodel $M \prec H_\kappa$ (such that $\mathbb{P}_\mathcal{I} \in M$), for every $B \in M \cap \mathbb{P}_\mathcal{I}$ we have

$$\{x \in B : x \text{ is a } \mathbb{P}_\mathcal{I}\text{-generic point over } M\} \notin \mathcal{I}.$$

For proper forcings of the form $\mathbb{P}_\mathcal{I}$ there is a nice reprezentation of names for reals in the extension. Zapletal proved [**35**, Proposition 2.3.1] that if $\mathbb{P}_\mathcal{I}$ is proper, then each real in a $\mathbb{P}_\mathcal{I}$-extension is equal to the value of a ground-model Borel function at the generic point. In other words, for any $\dot{x}$ which is a name for a real, for any $B \in \mathbb{P}_\mathcal{I}$ there is $C \in \mathbb{P}_\mathcal{I}$, $C \leq B$ and a Borel function $f : C \to \omega^\omega$ such that

$$C \Vdash \dot{x} = f(\dot{g}).$$

This property is called *Borel reading of names.*

A forcing $\mathbb{P}_\mathcal{I}$ is said to have *continuous reading of names* if the above ground-model function can be chosen continuous. If $\mathbb{P}_\mathcal{I}$ is proper, then continuous reading of names can be translated to the following purely topological property: for each Borel $\mathcal{I}$-positive set $B \subseteq X$ and every Borel function $f : B \to \omega^\omega$ there is an $\mathcal{I}$-positive Borel $C \subseteq B$ such that f is continous on C.

Recall that a forcing notion $\mathbb{P}$ satisfies Baumgartner's *Axiom A* if there is a sequence $\langle \leq_n : n < \omega\rangle$ of partial orders on $\mathbb{P}$ such that $\leq_0 = \leq$, $\leq_{n+1} \subseteq \leq_n$ and

- if $\langle p_n \in \mathbb{P}, n < \omega\rangle$ is such that $p_{n+1} \leq_n p_n$, then there is $q \in \mathbb{P}$ such that $q \leq_n p_n$ for all n,
- for every $p \in \mathbb{P}$, for every n and for every name $\dot{\alpha}$ for an ordinal there exist $q \in \mathbb{P}$ and a countable set of ordinals A such that $q \leq_n p_n$ for each $n < \omega$, and $q \Vdash \dot{\alpha} \in A$.

Axiom A implies properness but the property of satisfying Axiom A is not invariant under forcing equivalence. However, if a forcing notion $\mathbb{P}$ is such that that there is a forcing notion $\mathbb{Q}$ satisfying Axiom A and $\mathbb{P}$ is forcing equivalent to $\mathbb{Q}$, then $\mathbb{P}$ is proper. In fact, this is a common way of showing that a forcing notion is proper. It is worth mentioning that Koszmider [**19**, Theorem 21] proved that Axiom A is preserved under countable support iteration.

We say that a forcing notion $\mathbb{P}$ is *ω^ω-bounding* if for any name $\dot{x}$ for an element of ω^ω, for any $p \in \mathbb{P}$ there is $q \in \mathbb{P}$ and $y \in \omega^\omega$ such that $q \leq p$ and $q \Vdash \forall n < \omega \quad \dot{x}(n) < \check{y}(n)$.

We say that a real s in a generic extension $V[G]$ is a *splitting real over* V if $V[G] \vDash \forall x \in V \cap [\omega]^\omega \ (\neg(x \subseteq s) \ \wedge \ \neg(x \cap s = \varnothing))$.

3

Forcing with closed sets

3.1 Introduction

Many idealized forcings arise from σ-ideals generated by closed sets. The main motivating examples are the Sacks forcing and the Miller forcing. The former is associated to the σ-ideal of countable subsets of the Cantor space (or any other Polish space). The latter is associated to the σ-ideal of $\mathbf{K}_\sigma$ subsets of the Baire space.

Zapletal [**35**] investigated the forcing arising from σ-ideals generated by closed sets and proved the following.

Theorem 3.1 (Zapletal, [**35**, Theorem 4.1.2]). *If $\mathcal{I}$ is a σ-ideal on a Polish space X generated by closed sets, then the forcing $\mathbb{P}_\mathcal{I}$ is proper and has continuous reading of names in the topology of X.*

Properness is a rather abstract property of a forcing notion, which was invented to generalize properties of the Sacks forcing (and its iterations). Both the Sacks and the Miller forcing are equivalent to forcings with trees. In these forcings with trees we have fusion (Axiom A), which implies properness and is used to prove continuous reading of names. The main theorem of this chapter, Theorem 3.4, generalizes this. Namely, we prove that if $\mathcal{I}$ is generated by closed sets, then the forcing $\mathbb{P}_\mathcal{I}$ is equivalent to a forcing with trees satisfying Axiom A.

In Section 3.3 we introduce the generalized Banach-Mazur games (cf. [**15**, Section 8.H]), which we use in Section 3.4 in the proof of Theorem 3.4.

In the next two sections we investigate $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals generated by closed sets and their Fubini powers. By the results of Mazurkiewicz [**15**, Theorem 29.19] and Arsenin-Kungui [**15**, Theorem 18.18], both the σ-ideal of countable subsets of 2^ω and the σ-ideal of $\mathbf{K}_\sigma$ subsets of ω^ω are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

In Section 3.5 we give a new proof of Corollary 3.14 which says that a coanalytic hereditary family of closed sets generates a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal. This is a known fact but all its known proofs use sophisticated methods. We give a proof using idealized forcing. We also prove Proposition 3.16, which says that if $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal generated by closed sets, then in all forcing extensions $\mathcal{I}$ is still $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal generated by closed sets. To prove Corollary 3.14 and Proposition 3.16 we introduce a Borel structure on the family of $\mathbf{G}_\delta$ subsets of a Polish space and analyze the complexity of the family of $\mathcal{I}$-perfect $\mathbf{G}_\delta$ sets.

In Section 3.6 we study the countable-support iteration of $\mathbb{P}_\mathcal{I}$, which by [**35**, Theorem 5.1.6] can be described in terms of the Fubini power of $\mathcal{I}$. In Proposition 3.29 we show that if $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal generated by closed sets, then any analytic subset of X^ω either belongs to the Fubini power $\mathcal{I}^\omega$, or else contains a $\mathbf{G}_\delta$ $\mathcal{I}$-positive set, which has a "tree" structure (for the exact formulation see Section 3.6). This result is motivated by the following.

THEOREM 3.2 (Solecki, [**30**, Theorem 1]). *If $\mathcal{I}$ is a σ-ideal in a Polish space X generated by closed sets then each analytic set in X is either in $\mathcal{I}$ or else contains an $\mathcal{I}$-positive $\mathbf{G}_\delta$ set.*

In the next two sections we study some examples of σ-ideals generated by closed sets motivated by analysis and measure theory.

In Section 3.7 we study an example from the theory of real functions. To any function f we associate the σ-ideal $\mathcal{I}^f$ generated by closed sets on which f is continuous. The σ-ideal $\mathcal{I}^f$ is nontrivial if and only if f cannot be decomposed into countably many continuous functions with closed domains, i.e. f is not *piecewise continuous*. Piecewise continuity of Baire class 1 functions has been studied by several authors (Jayne and Rogers [**12**], Solecki [**31**], Andretta [**2**]). In Corollary 3.39 we show that if f is Baire class 1, not piecewise continuous, then the forcing $\mathbb{P}_{\mathcal{I}^f}$ is equivalent to the Miller forcing. In Corollary 3.46 we establish a Banach-Mazur game characterization for the σ-ideal $\mathcal{I}^f$.

In Section 3.8 we study an example from measure theory. Let $\mathcal{E}$ denote the σ-ideal generated by closed null sets in the Cantor space. $\mathcal{E}$ has been investigated by Bartoszyński and Shelah [**4**]. In Proposition 3.43 we show that $\mathbb{P}_\mathcal{E}$ does not add Cohen reals, which implies that any forcing extension with $\mathbb{P}_\mathcal{E}$ is minimal. In Corollary 3.46 we establish a Banach-Mazur game characterization for the σ-ideal $\mathcal{E}$.

In Section 3.9 we study the following question [**35**, Section 3.7] of Zapletal, loosely related to the previous topics. Is it true that if $\mathcal{I}$ is a ccc σ-ideal, then every Borel set contains a positive closed set modulo $\mathcal{I}$? We answer this question negatively.

3.2 Notation

We denote by $\mathbb{Q}$ the set of all points in 2^ω which are eventually equal to 0. The Cantor set 2^ω is equipped with the standard metric defined by $d(x,y) = 2^{-n(x,y)}$ where $n(x,y)$ is the least m such that $x(m) \neq y(m)$. We equip 2^ω with the standard Haar measure, that is the measure μ such that $\mu([\tau]) = 2^{-n}$, if $|\tau| = n$.

Let X, Y, X', Y' be Polish spaces. For two functions $f : X \to Y$ and $f' : X' \to Y'$ we write $f \sqsubseteq f'$ if there are topological embeddings $\varphi : X \to X'$ and $\psi : Y \to Y'$ such that $f' \circ \varphi = \psi \circ f$.

We write $F : A \multimap B$ to indicate that F is a partial function from A to B.

3.3 Generalized Banach-Mazur games

Recall the classical Banach-Mazur game [**15**, Section 8.H] which "decides" whether a Borel set is meager nor not, in terms of existence of a winning strategy for one of the players. Suppose we have a game scheme G together with a family of payoff sets $p(A)$ for each $A \subseteq X$ such that

(i) $p(A) \subseteq \lim G$ is Borel, for each Borel set $A \subseteq X$,
(ii) $p(A) \subseteq p(B)$ for each $B \subseteq A$,
(iii) $p(\bigcup_{n<\omega} A_n) = \bigcap_{n<\omega} p(A_n)$ for each sequence $\langle A_n : n < \omega \rangle$.

For each $A \subseteq X$ the game $G(A)$ is a game in the game scheme G with the payoff set $p(A)$. We denote by $G(\cdot)$ the game scheme G together with the function p. We call $G(\cdot)$ a *Banach-Mazur scheme* if

(iv) the moves of Adam code (in a prescribed way, in terms of a fixed enumeration of the basis) basic open sets U_n such that $\overline{U_{n+1}} \subseteq U_n$ and $\operatorname{diam}(U_n) < 1/n$,
(v) for each Borel set $A \subseteq X$ and each play g in $G(A)$ if Adam wins g, then the single point in the intersection of U_n's (as above) is in A.

Notice that if the family of sets $q \subseteq \lim G$ such that Eve has a winning strategy in the game (G, q) is closed under countable intersections, then the family of sets $A \subseteq X$ such that Eve has a winning strategy in $G(A)$ forms a σ-ideal.

The concept of considering σ-ideals defined in terms of a winning strategy in a game scheme occurs in a paper of Mycielski [**22**] and also in a work of Schmidt [**28**].

If the family of sets $A \subseteq X$ for which Eve has a winning strategy in $G(A)$ forms a σ-ideal $\mathcal{I}$, then we say that $G(\cdot)$ is a *Banach-Mazur representation of* $\mathcal{I}$.

Suppose $X = \lim T$ for some countable tree T. Suppose also that the game scheme G is such that the possible n-th moves of Adam correspond to $\operatorname{Lev}_n(T)$ (i.e. to the basic clopen sets $[\tau]_T$ for $\tau \in \operatorname{Lev}_n(T)$). Let $G_{\mathcal{I}}(\cdot)$ be a Banach-Mazur represenattion of a σ ideal $\mathcal{I}$ and let τ be a partial play in the game scheme $G_{\mathcal{I}}$. Let U be the basic clopen set coded by the last move of Adam in τ. Recall that the relativized game scheme $(G_{\mathcal{I}})_\tau$ consists of the continuations of τ in $G_{\mathcal{I}}$. The game scheme $(G_{\mathcal{I}})_\tau$ together with the function $p_\tau(A) = p(A) \cap U$ defines a *relativized Banach-Mazur scheme*. Using the property (v) we easily get the following.

PROPOSITION 3.3. *Let $X = \lim T$, $G_{\mathcal{I}}$, τ and U be as above. Let $A \subseteq X$. Eve has a winning strategy in $(G_{\mathcal{I}})_\tau(A)$ if and only if Eve has a winning strategy in $G_{\mathcal{I}}(A \cap U)$.*

3.4 Tree representation of the forcing $\mathbb{P}_{\mathcal{I}}$

THEOREM 3.4. *Let X be a Polish space. If $\mathcal{I}$ is a σ-ideal on X generated by of closed sets, then the forcing $\mathbb{P}_{\mathcal{I}}$ is equivalent to a forcing with trees satisfying Axiom A.*

PROOF. In the proof we will use an "unfolded" variant of the Banach-Mazur game, which will be used to determine whether an analytic set is $\mathcal{I}$-positive. To simplify notation we assume that the underlying space X is the Baire space ω^ω. Pick a bijection $\rho : \omega \to \omega \times \omega$.

We will use the following notation. Let Y be an arbitrary set. If $\tau \in (\omega \times Y)^{<\omega}$, then by $\bar{\tau} \in \omega^{<\omega}$ we denote the sequence of the first coordinates of the elements of τ. The map $\mathrm{p}_Y : \lim T \to \omega^\omega$ is defined as follows: if $t \in \lim T$ and $t{\restriction}n = \tau_n$, then $\mathrm{p}_Y(t) = \bigcup_{n<\omega} \bar{\tau}_n$. Let z be arbitrary. If $\tau \in (\omega \times Y)^{<\omega}$ and $\tau = \langle a_i : i < |\tau| \rangle$, then by $\tau^z \in (\omega \times Y \times \{z\})^{<\omega}$ we denote the sequence $\langle (a_i, z) : i < |\tau| \rangle$. If $T \subseteq Y^{<\omega}$ is a tree, then by T^z we denote the tree $\{\tau^z : \tau \in T\}$ on $Y \times \{z\}$. If $Y = W \times Z$ and $\tau \in (\omega \times Y)^{<\omega}$, then by $\tau_W \in (\omega \times W)^{<\omega}$ we denote $\langle \pi_{\omega \times W}(a_i) : i < |\tau| \rangle$ (where $\pi_{\omega \times W} : \omega \times W \times Z \to \omega \times W$ is the projection to the first two coordinates). By T_W we denote the tree $\{\tau_W : \tau \in T\}$.

For $Y \in H_{\omega_1}$ and a tree T on $\omega \times Y$ let $G_{\mathcal{I}}(Y, T)$ be the game scheme in which

- in his n-th turn Adam constructs $\tau_n \in T$ such that $\tau_n \supsetneq \tau_{n-1}$ ($\tau_{-1} = \emptyset$),
- in her n-th turn Eve picks a clopen set O_n in ω^ω such that

$$\mathrm{proj}[T(\tau_n)] \notin \mathcal{I} \ \Rightarrow \ O_n \cap \mathrm{proj}[T(\tau_n)] \notin \mathcal{I}.$$

By the end of a play, Adam and Eve have a sequence of closed sets E_k in ω^ω defined as follows:

$$E_k = 2^\omega \setminus \bigcup_{i<\omega} O_{\rho^{-1}(i,k)}.$$

Define $x = \bigcup_{n<\omega} \bar{\tau}_n \in \omega^\omega$. Consider a payoff set in $G_{\mathcal{I}}(Y, T)$ such that Adam wins if and only if

$$x \notin \bigcup_{k<\omega} E_k.$$

The game in the game scheme $G_{\mathcal{I}}(Y, T)$ with the above payoff set will be also denoted by $G_{\mathcal{I}}(Y, T)$.

If S is a subtree of the game scheme $G_{\mathcal{I}}(Y,T)$, then by $\hat{S} \subseteq T$ we denote the tree $\{\tau : \tau$ is a move of Adam in some partial play $\pi \in S\}$. We write $\mathrm{proj}[S]$ for $\mathrm{proj}[\hat{S}]$. If $Y' = Y \times Z$, $z \in Z$ is fixed and T' is a tree on Y' such that $T^z \subseteq T$, then by S^z we denote the subtree of $G_{\mathcal{I}}(Y',T')$, in which the moves τ of Adam are changed to τ^z.

LEMMA 3.5. *The game $G_{\mathcal{I}}(Y,T)$ is determined. Eve has a winning strategy in $G_{\mathcal{I}}(Y,T)$ if and only if*

$$\mathrm{proj}[T] \in \mathcal{I}.$$

PROOF. Suppose first that $\mathrm{proj}[T] \in \mathcal{I}$. Then Eve chooses $\varnothing$ in her first move and wins the game in this way.

On the other hand, suppose that $\mathrm{proj}[T]$ is $\mathcal{I}$-positive. We define a winning strategy for Adam as follows. In his moves, Adam constructs $\tau_n \in T$ so that

- $[\bar{\tau}_n] \subseteq O_{n-1}$,
- $\mathrm{proj}[T(\tau_n)] \notin \mathcal{I}$.

Suppose Adam is about to make his n-th move, his previous move is τ_{n-1} and the last move of Eve is O_{n-1} ($O_{-1} = \varnothing$). Using $O_{n-1} \cap \mathrm{proj}[T(\tau_{n-1})] \notin \mathcal{I}$, Adam picks $\tau_n \in T$ extending τ_{n-1} such that $[\bar{\tau}_n] \subseteq O_{n-1}$ and $\mathrm{proj}[\tau_n] \notin \mathcal{I}$. This is the strategy for Adam. It is winning since after each play we have that $x \in O_n$ for each $n < \omega$, so in particular $x \in \bigcup_{k<\omega} E_k$. □

REMARK 3.6. Note that if S is a winning strategy for Adam in the game $G_{\mathcal{I}}(Y,T)$ then for each partial play $\pi \in S$, we have $\mathrm{proj}[S(\pi)] \notin \mathcal{I}$. This is because otherwise we could construct a counterplay to the strategy S. In particular, if τ is the last move of Adam in π, then we have $\mathrm{proj}[T(\tau)] \notin \mathcal{I}$.

Let π be a partial play in $G_{\mathcal{I}}(Y,T)$ of length $2l$, in which Eve chooses clopen sets O_i, for $i < l$, and Adam picks $\tau_{l-1} \in T$ in his last move. Suppose that $Y' = Y \times Z$, $Z \in H_{\omega_1}$, $z \in Z$ is fixed and T' is a tree on $\omega \times Y'$ such that $(\tau_{l-1})^z \in T'$. By the *relativized unfolded game* $G_{\mathcal{I}}(Y',T')^z_\pi$ we mean the game, in which

- in his n-th move Adam picks $\tau'_{n+l} \in T'$, $\tau'_{n+l} \supsetneq \tau'_{n+l-1}$ ($\tau'_{l-1} = (\tau_{l-1})^z$),
- in her n-th move Eve picks a clopen set O_{n+l} in ω^ω such that

$$\mathrm{proj}[T'(\tau'_{n+l})] \notin \mathcal{I} \;\Rightarrow\; O_{n+l} \cap \mathrm{proj}[T'(\tau'_{n+l})] \notin \mathcal{I}.$$

The payoff set is the same as in the unrelativized case, i.e. we use all $\langle O_n : n < \omega\rangle$ to define a sequence of closed sets $\langle E_k : k < \omega\rangle$, we put $x = \bigcup_{n<\omega} \bar{\tau}'_n$ and Eve wins if and only if $x \in \bigcup_{k<\omega} E_k$.

With an analogous proof as in Lemma 3.5 we get the following.

LEMMA 3.7. *Suppose π is a partial play in $G_{\mathcal{I}}(Y,T)$ and τ is the last move of Adam in π. Let $G_{\mathcal{I}}(Y',T')^z_\pi$ be a relativized unfolded game. Eve has a winning strategy in $G_{\mathcal{I}}(Y',T')^z_\pi$ if and only if*

$$\mathrm{proj}[T'(\tau^z)] \in \mathcal{I}.$$

LEMMA 3.8. *If S is a winning strategy for Adam in $G_{\mathcal{I}}(Y,T)$ then*
$$\operatorname{proj}[S] \notin \mathcal{I}.$$

PROOF. Let $A = \operatorname{proj}[S]$. If $A \in \mathcal{I}$ then we have closed sets $E_k \in \mathcal{I}$ such that $A \subseteq \bigcup_{k<\omega} E_k$. Let U_k^m be clopen sets such that $U_k^m \subseteq U_k^{m+1}$ and $\omega^\omega \setminus E_k = \bigcup_{m<\omega} U_k^m$ for each $k < \omega$. We construct an Eve's counterplay to the strategy S in the following way. Suppose she is to make her n-th move and let τ_n be the last move of Adam. By Remark 3.6, $\operatorname{proj}[T(\tau_n)] \notin \mathcal{I}$. Let $\rho(n) = (i,k)$. She chooses $m \geq n$ big enough so that
$$U_k^m \cap \operatorname{proj}[T(\tau_n)] \notin \mathcal{I}.$$
Let her n-th move be $O_n = U_k^m$. If she plays in this way then $\bigcup_{i<\omega} O_{\rho^{-1}(i,k)} = \omega^\omega \setminus E_k$, i.e. the closed sets she gets are precisely the sets E_k. If $x = \bigcup_{n<\omega} \bar{\tau}_n$ is the point in ω^ω constructed by Adam, then by the definition of A, $x \in A \subseteq \bigcup_{k<\omega} E_k$, which shows that Eve wins. □

Note that it follows from Lemmas 3.5 and 3.8 that any analytic $\mathcal{I}$-positive set $A \subseteq \omega^\omega$ contains an analytic $\mathcal{I}$-positive subset of the form $\operatorname{proj}[S]$ for a winning strategy S for Adam in a game $G_{\mathcal{I}}(Y,T)$ (where $Y = \omega$ and T is a tree on $\omega \times \omega$ such that $A = \operatorname{proj}[T]$).

Let $\mathbb{T}_{\mathcal{I}}$ be the set of all triples (Y,T,S) where $Y \in H_{\omega_1}$, T is a tree on $\omega \times Y$ and S is a winning strategy for Adam in $G_{\mathcal{I}}(Y,T)$. $\mathbb{T}_{\mathcal{I}}$ is a forcing notion with the following ordering: for $(Y',T',S'),(Y,T,S) \in \mathbb{T}_{\mathcal{I}}$ let
$$(Y',T',S') \leq (Y,T,S) \quad \text{iff} \quad \operatorname{proj}[S'] \subseteq \operatorname{proj}[S].$$
Notice that $(Y,T,S) \mapsto \operatorname{proj}[S]$ is a dense embedding from $\mathbb{T}_{\mathcal{I}}$ to $\mathbb{Q}_{\mathcal{I}} = (\mathbf{\Sigma}^1_1 \setminus \mathcal{I}, \subseteq)$. Indeed, suppose that $(Y',T',S') \perp (Y,T,S)$. If $\operatorname{proj}[S']$ and $\operatorname{proj}[S]$ were compatible in $\mathbb{Q}_{\mathcal{I}}$, then we would find an $\mathcal{I}$-positive $\mathbf{\Sigma}^1_1$ set $A \subseteq \omega^\omega$ such that $A \subseteq \operatorname{proj}[S'] \cap \operatorname{proj}[S]$. Take any tree T on $\omega \times \omega$ such that $\operatorname{proj}[T] = A$ and find a winning strategy S'' for Adam in $G_{\mathcal{I}}(\omega,T)$. Then $(\omega,T,S'') \leq (Y',T',S'),(Y,T,S)$, a contradiction.

By Theorem 3.2, $\mathbb{P}_{\mathcal{I}}$ is dense in $\mathbb{Q}_{\mathcal{I}}$. Therefore the three forcing notions $\mathbb{T}_{\mathcal{I}}$, $\mathbb{Q}_{\mathcal{I}}$ and $\mathbb{P}_{\mathcal{I}}$ are equivalent. We will show that the forcing $\mathbb{T}_{\mathcal{I}}$ satisfies Axiom A.

Take $(Y,T,S) \in \mathbb{T}_{\mathcal{I}}$ and recall that for each play $p \in \lim S$ ending with $t \in \omega^\omega \times Y^\omega$, $x \in \omega^\omega$ (defined from the moves of Adam) and a sequence of closed sets E_n (defined from the moves of Eve), we have $x \notin \bigcup_k E_k$. Note that for each $k \in \omega$ there is $n \in \omega$ (even) such that (the partial play) $t{\restriction}n$ already determines that $x \notin E_k$ (i.e. $[\bar{\tau}_n] \subseteq O_m$ for some $m < \omega$ such that $\rho(m) = (i,k)$ for some $i < \omega$). Let $n_0(p) \in \omega$ be the minimal such n for $k = 0$. Put
$$F_0(S) = \{p{\restriction}n_0(p) : p \in \lim S\}.$$
Note that $F_0(S)$ is a front in S. Analogously we define $F_k(S)$, for each $k < \omega$ (instead of E_0 take E_k and put $n_k(p) > n_{k-1}(p)$ minimal even number such that $p{\restriction}n_k(p)$ determines $x \notin E_k$).

Define $(Y',T',S') \le_k (Y,T,S)$ iff

(i) $(Y',T',S') \le (Y,T,S)$,
(ii) there is $Z \in H_{\omega_1}$ such that $Y' = Y \times Z$,
(iii) there is $z \in Z$ such that $T^z \subseteq T'$,
(iv) $(T')_Y \subseteq \hat{S}$,
(v) $F_k(S') = F_k(S)^z$.

We will prove that $\mathbb{T}_{\mathcal{I}}$ satisfies Axiom A with the inequalities $\le_k$.

1. Fix $k < \omega$. Suppose that $(Y,T,S) \in \mathbb{T}_{\mathcal{I}}$ and $\dot{\alpha}$ is a name for an ordinal. We will find $(Y',T',S') \le_k (Y,T,S)$ and a countable set of ordinals A such that $(Y',T',S') \Vdash_{\mathbb{T}_{\mathcal{I}}} \dot{\alpha} \in \check{A}$.

For each $\pi \in F_k(S)$ find an ordinal α_π and an $\mathcal{I}$-positive analytic set $A_\pi \subseteq \mathrm{proj}[S(\pi)]$ (recall that $\mathrm{proj}[S(\pi)]$ is $\mathcal{I}$-positive by Remark 3.6) such that

$$A_\pi \Vdash_{\mathbb{Q}_{\mathcal{I}}} \dot{\alpha} = \check{\alpha}_\pi.$$

Let $\tau_\pi \in T$ be the last move of Adam in π. Next, pick $Z_\pi \in H_{\omega_1}$ such that $0 \in Z_\pi$ and find a pruned tree T_π on $\omega \times Y \times Z_\pi$ such that

- ${\tau_\pi}^0 \in T_\pi$ and ${\tau_\pi}^0$ is a stem of T_π,
- $(T_\pi)_Y \subseteq \hat{S}$,
- $A_\pi = \mathrm{proj}[T_\pi]$

(T_π is such that its projection to $\omega^\omega \times Y^\omega$ is the analytic set $\lim \hat{S} \cap \mathrm{p}_Y^{-1}[A_\pi]$). Ensure also that

- for each $\pi, \pi' \in F_k(S)$, if $\pi \neq \pi'$, then $Z_\pi \cap Z_{\pi'} = \{0\}$,
- for each $\tau \in T_\pi$ if $\tau \supsetneq (\tau_\pi)^0$, then $\tau(|\tau|-1) \in \omega \times Y \times (Z_\pi \setminus \{0\})$,

Put

$$Z = \bigcup_{\pi \in F_k(S)} Z_\pi, \quad z = 0, \quad Y' = Y \times Z.$$

For each $\pi \in F_k(S)$ let S_π be a winning strategy for Adam in $G_{\mathcal{I}}(Y \times Z_\pi, T_\pi)^0_\pi$. Such a strategy exists by Lemma 3.7 since $\mathrm{proj}[T_\pi({\tau_\pi}^0)] = \mathrm{proj}[T_\pi] = A_\pi \notin \mathcal{I}$. Let

$$T' = T^0 \cup \bigcup_{\pi \in F_k(S)} T_\pi$$

and consider the game $G_{\mathcal{I}}(Y',T')$. The tree

$$S' = \bigcup_{\pi \in F_n(S)} (\pi^0)^\frown S_\pi$$

is a strategy in $G(Y',T')$ since all π^0, for $\pi \in F_k(S)$, are partial plays in $G(Y',T')$ (because $T^0 \subseteq T'$). Moreover, it is a winning strategy for Adam since each S_π is a winning strategy for Adam in $G_{\mathcal{I}}(Y \times Y_\pi, T_\pi)_\pi$. Therefore $(Y',T',S') \in \mathbb{T}_{\mathcal{I}}$. By the construction we have $(Y',T',S') \le_k (Y,T,S)$. Moreover,

$$(Y',T',S') \Vdash_{\mathbb{T}_{\mathcal{I}}} \dot{\alpha} \in \{\alpha_\tau : \tau \in F_k(S)\}$$

because the set $\{\mathrm{proj}[S'(\pi)] : \pi \in F_k(S)\}$ is predense below $\mathrm{proj}[S']$ and we have $\mathrm{proj}[S'(\pi)] \Vdash_{\mathbb{Q}_{\mathcal{I}}} \dot{\alpha} = \check{\alpha}_\pi$ (since $\mathrm{proj}[S'(\pi)] \subseteq A_\pi$).

2. Let $\langle (Y_k, T_k, S_k) : k < \omega\rangle$ be a fusion sequence. For each $k < \omega$ let $Z_k \in H_{\omega_1}$ be such that $Y_{k+1} = Y_k \times Z_k$ and let $z_k \in Z_k$ be as in the definition of $\leq_k$. Let $\vec{z}_k = \langle z_k, z_{k+1}, \ldots\rangle$. Put $Y = \bigcup_{k<\omega}(Y_k)^{\vec{z}_k}$ and $T = \bigcup_{k<\omega}(T_k)^{\vec{z}_k}$. T is a tree on Y. Notice that for each $k < \omega$, for each $\tau \in T_k$ we have

$$(*) \qquad \mathrm{proj}[T_k(\tau)] = \mathrm{proj}[T(\tau^{\vec{z}_k})].$$

Indeed, $\mathrm{proj}[T_k(\tau)] \subseteq \mathrm{proj}[T(\tau^{\vec{z}_k})]$ follows from (iii) and $\mathrm{proj}[T(\tau^{\vec{z}_k})] \subseteq \mathrm{proj}[T_k(\tau)]$ from (iv) (because for each $t \in \lim T(\tau^{\vec{z}_k})$ a sequence of its initial coordinates is in $\lim \hat{S}_k$ and hence in $\lim T_k$)

Consider the game $G_{\mathcal{I}}(Y,T)$ and let

$$S = \bigcup_{k<\omega} F_k(S_k)^{\vec{z}_k}.$$

Note that it follows from $(*)$ that S is a strategy for Adam in $G_{\mathcal{I}}(Y,T)$. Moreover, for each $k < \omega$ we have $F_k(S) = F_k(S_k)^{\vec{z}_k}$, by the definition of F_k. Since for each $p \in \lim S$ we have

$$\forall k < \omega\ \exists m < \omega \quad p{\restriction}m \in F_k(S),$$

it follows that S is a winning strategy for Adam in $G_{\mathcal{I}}(Y,T)$. Therefore $(Y,T,S) \in \mathbb{T}_{\mathcal{I}}$.

To see that $(Y,T,S) \leq (Y_k,T_k,S_k)$ we use the property (iv). Indeed, if $x \in \mathrm{proj}[S]$, then there is a play in $\lim S$, in which x is defined. By (iv), however, we can extract from this play a play in $\lim S_k$, in which x is defined.

To check that $(Y,T,S) \leq_k (Y_k,T_k,S_k)$ we put $Z = \prod_{m\geq k} Z_m$ and $z = \vec{z}_k$.

This ends the proof. □

3.5 Coanalytic families of closed sets

If $\mathcal{K}$ is a family of closed subsets of a Polish space X, then its projective complexity can be defined in terms of the Borel Effros space $F(X)$. Namely, if $\mathbf{\Gamma}$ is a projective pointclass, then we say that $\mathcal{K}$ *is* $\mathbf{\Gamma}$ if it belongs to $\mathbf{\Gamma}$ in $F(X)$.

Recall that if $X = \omega^\omega$, then for each closed set $C \subseteq X$ there is a pruned subtree T of $\omega^{<\omega}$ such that $C = \lim T$. If X is an arbitrary Polish space, then for each closed set $C \subseteq X$ there is a family $\mathcal{U}$ of basic open sets such that

$$(*) \qquad \forall U \text{ basic open} \quad U \subseteq \bigcup\mathcal{U} \;\Rightarrow\; U \in \mathcal{U}$$

and $C = X \setminus \bigcup\mathcal{U}$ (take $\mathcal{U} = \{U \text{ basic open} : U \cap C = \varnothing)\}$). Let us code all families $\mathcal{U}$ satisfying $(*)$ by elements of ω^ω and create a universal closed set $\tilde{C} \subseteq \omega^\omega \times X$ such that if $t \in \omega^\omega$ codes $\mathcal{U}(t)$, then $\tilde{C}_t = X \setminus \bigcup\mathcal{U}(t)$.

Using the property $(*)$ of the coding, we can check that the function

$$\omega^\omega \ni t \mapsto \tilde{C}_t \in F(X)$$

is Borel measurable (i.e. preimages of Borel sets in $F(X)$ are Borel). Therefore, for any projective pointclass $\mathbf{\Gamma}$, a family of closed sets $\mathcal{K}$ is $\mathbf{\Gamma}$ if and only if the set $\{t \in \omega^\omega : \tilde{C}_t \in \mathcal{K}\}$ is $\mathbf{\Gamma}$ in ω^ω.

The projective complexity of families of closed subsets of X can be also generalized to families of sets in other Borel pointclasses — in terms of universal sets. We will now introduce a Borel structure on the family of $\mathbf{G}_\delta$ sets.

Note that for any $\mathbf{G}_\delta$ set $G \subseteq \omega^\omega$ there is a pruned tree $T \subseteq \omega^{<\omega}$ and a family $\langle \sigma_\tau \in \omega^{<\omega} : \tau \in T\rangle$ such that the family of clopen sets $\langle [\sigma_\tau] : \tau \in T\rangle$ forms a Lusin scheme and $G = \bigcap_{n<\omega} \bigcup_{\tau \in T \cap \omega^n} [\sigma_\tau]$. Generalizing this to an arbitrary Polish space X we claim that for any $\mathbf{G}_\delta$ set G in X there is a Souslin scheme $\langle U_\tau : \tau \in \omega^{<\omega}\rangle$ of basic open sets such that

(i) $\mathrm{diam}(U_\tau) < 1/|\tau|$,
(ii) $\overline{U_\tau} \subseteq U_{\tau\restriction(|\tau|-1)}$
(iii) if $U_\tau \neq \emptyset$ then $U_{\tau^\frown n} \neq \emptyset$ for some $n < \omega$

and $G = \bigcap_{n<\omega} \bigcup_{|\tau|=n} U_\tau$. Indeed, if $G = \bigcap_{n<\omega} O_n$ (each O_n open and $O_{n+1} \subseteq O_n$), then we construct a Souslin scheme U_τ by induction on $|\tau|$ as follows. Having all U_τ for $|\tau| \leq n$ we find a family $\{U_\tau : \tau \in \omega^{n+1}\}$ such that

- for each $\tau \in \omega^{n+1}$ we have $U_\tau \cap G \neq \emptyset$,
- for each $\sigma \in \omega^n$ we have $U_\sigma \cap O_{n+1} = \bigcup\{\overline{U_\tau} : \sigma \subseteq \tau, \tau \in \omega^{n+1}\}$.

Let us code all Souslin schemes of clopen sets satisfying (i)–(iii) by elements of the Baire space ω^ω and create a universal $\mathbf{G}_\delta$ set $\tilde{G} \subseteq \omega^\omega \times X$ such that if $t \in \omega^\omega$ codes a Souslin scheme $\langle U_\tau(t) : \tau \in \omega^{<\omega}\rangle$, then $\tilde{G}_t = \bigcap_{n<\omega} \bigcup_{|\tau|=n} U_\tau(t)$.

LEMMA 3.9. *If $U \subseteq X$ is open, then*

$$\{t \in \omega^\omega : \tilde{G}_t \cap U \neq \emptyset\} \text{ is open.}$$

PROOF. Note that by (iii) and (ii), $\tilde{G}_t \cap U \neq \emptyset$ if and only if there is a nonempty basic open set $V \subseteq U$ such that V occurs in the Souslin scheme coded by t. □

If $\mathbf{\Gamma}$ is a projective pointclass and $\mathcal{G}$ is a family of $\mathbf{G}_\delta$ sets, then we say that $\mathcal{G}$ *is* $\mathbf{\Gamma}$ if $\{t \in \omega^\omega : \tilde{G}_t \in \mathcal{G}\}$ is $\mathbf{\Gamma}$ in ω^ω. By Lemma 3.9 the map

$$\mathbf{G}_\delta \ni G \mapsto \overline{G} \in F(X)$$

is Borel (i.e. preimages of Borel sets in $F(X)$ are Borel).

Let $\mathcal{K}$ be a family of closed sets in a Polish space X. We say that $\mathcal{K}$ is *hereditary* if for any two closed sets C, D such that $C \subseteq D$, if $D \in \mathcal{K}$, then $C \in \mathcal{K}$.

Ket $\mathcal{I}$ be a σ-ideal on a Polish space X and $A \subseteq X$. We say that A is *$\mathcal{I}$-perfect* if $A \neq \emptyset$ and for each open set U the set $A \cap U$ is either empty or $\mathcal{I}$-positive. If $\mathcal{K}$ is a family of closed sets on a Polish space X and $D \subseteq X$ is closed, then we say that D is *$\mathcal{K}$-perfect* if the sets from $\mathcal{K}$ have relatively empty interior on D. Note that if $\mathcal{K}$ is hereditary, then a closed set D is $\mathcal{K}$-perfect if and only if for each basic open set U in X, either $U \cap D = \emptyset$, or else $\overline{U \cap D} \notin \mathcal{K}$.

LEMMA 3.10. *Let $\mathcal{I}$ be a σ-ideal generated by closed sets on a Polish space X. If $G \subseteq \omega^\omega$ is a $\mathbf{G}_\delta$ set and $\overline{G}$ is $\mathcal{I}$-perfect, then $G \notin \mathcal{I}$.*

PROOF. Let $C = \overline{G}$ and suppose C is strictly $\mathcal{I}$-positive yet $G \in \mathcal{I}$. If $G \subseteq \bigcup_n F_n$ and F_n are closed sets in $\mathcal{I}$, then each $F_n \cap C$ is a closed nowhere dense subset of C. This contradicts the Baire category theorem. □

LEMMA 3.11. *Let $\mathcal{I}$ be a σ-ideal generated by closed sets on a Polish space X. If $G \subseteq \omega^\omega$ is an $\mathcal{I}$-positive $\mathbf{G}_\delta$ set, then it contains an $\mathcal{I}$-perfect $\mathbf{G}_\delta$ set G'.*

PROOF. Put $G' = G \setminus \bigcup\{U : U \text{ is basic open set and } U \cap G \in \mathcal{I}\}$. □

LEMMA 3.12. *Let X be a Polish space.*

(i) *Let $\mathcal{I}$ be a σ-ideal on X generated by closed sets. If $G \subseteq X$ is a $\mathbf{G}_\delta$ set, then G is $\mathcal{I}$-perfect if and only if $\overline{G}$ is $\mathcal{I}$-perfect.*

(ii) *Let $\mathcal{K}$ be a family of closed subsets of X, let $\sigma(\mathcal{K})$ be the σ-ideal of closed sets generated by $\mathcal{K}$ and let $\mathcal{I}$ be the σ-ideal generated by $\mathcal{K}$. If $D \subseteq X$ is closed, then the following are equivalent*
- *D is $\mathcal{K}$-perfect,*
- *D is $\sigma(\mathcal{K})$-perfect,*
- *if D is $\mathcal{I}$-perfect.*

PROOF. (i) Clearly, if G is $\mathcal{I}$-perfect, then $\overline{G}$ is also $\mathcal{I}$-perfect. Suppose $\overline{G}$ is $\mathcal{I}$-perfect but G is not $\mathcal{I}$-perfect. Then we can find an open set U such that $U \cap G \in \mathcal{I}$ and $U \cap G \neq \varnothing$. Consider $U \cap \overline{G}$, which is $\mathcal{I}$-perfect because $\overline{G}$ is $\mathcal{I}$-perfect. $U \cap \overline{G}$ is a Polish $\mathcal{I}$-perfect space which contains a dense $\mathbf{G}_\delta$ set in $\mathcal{I}$. By Lemma 3.10 we get a contradiction with the Baire category theorem.

(ii) This follows directly from the Baire category theorem. □

LEMMA 3.13. *Let X be a Polish space and let $\mathcal{I}$ be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal generated by closed sets on X. Let $\mathcal{K}$ be a hereditary coanalytic family of closed sets on X. Then*

(i) *the family of $\mathcal{I}$-perfect $\mathbf{G}_\delta$ sets is $\mathbf{\Sigma}^1_1$,*
(ii) *the family of $\mathcal{K}$-perfect closed sets is $\mathbf{\Sigma}^1_1$,*
(iii) *the family of $\mathbf{G}_\delta$ sets with $\mathcal{K}$-perfect closure is $\mathbf{\Sigma}^1_1$.*

PROOF. (i) We see that $G \in \mathbf{G}_\delta$ is $\mathcal{I}$-perfect if and only if

$$G \neq \varnothing \quad \wedge \quad \forall U \text{ basic open } (G \cap U \neq \varnothing \Rightarrow G \cap U \notin \mathcal{I}).$$

This is a $\mathbf{\Sigma}^1_1$ condition by Lemma 3.9 and the assumption that $\mathcal{I}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

(ii) Note that a closed set D is $\mathcal{K}$-perfect if and only if

$$D \neq \varnothing \quad \wedge \quad \forall U \text{ basic open } \ (D \cap U \neq \varnothing \Rightarrow \overline{D \cap U} \notin \mathcal{K}).$$

This is $\mathbf{\Sigma}^1_1$ condition since the closure is a Borel map.

(iii) This follows (ii) and the fact that the closure is a Borel map. □

It is known that if $\mathcal{K}$ is a coanalytic hereditary family of closed sets, then the σ-ideal generated by $\mathcal{K}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$. In [**15**, Theorem 35.38] a proof using derivatives is given. Also, an effective proof of this fact is known (cf. [**8**, Lemma 4.8]). We present a new proof of this fact, which uses idealized forcing.

Corollary 3.14. *Let X be a Polish space. If $\mathcal{K}$ is a coanalytic hereditary family of closed sets in X, then the σ-ideal generated by $\mathcal{K}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

Proof. Let $\mathcal{I}$ be the σ-ideal generated by $\mathcal{K}$ and let $A \subseteq X \times X$ be $\mathbf{\Sigma}^1_1$. Denote by $\mathcal{G}$ the family of $\mathcal{I}$-perfect $\mathbf{G}_\delta$ sets. By Lemmas 3.12 and 3.13, $\mathcal{G}$ is $\mathbf{\Sigma}^1_1$ (we use here the assumptions about $\mathcal{K}$). By Theorem 3.2 and Lemma 3.11, if $x \in X$, then

$$A_x \notin \mathcal{I} \qquad \text{iff} \qquad \exists G \in \mathcal{G} \quad G \subseteq A_x.$$

Let $D \subseteq X^2 \times \omega^\omega$ be a closed set such that $A = \pi[D]$ (π denotes the projection to the first two coordinates). Note that $G \subseteq A_x$ is equivalent to $\forall y \in G\ \exists z \in \omega^\omega\ (y, z) \in D_x$. By $\mathbf{\Sigma}^1_2$-absoluteness we get a name $\dot{z}_x$ such that $G \Vdash (\dot{g}, \dot{z}_x) \in D_x$. Now, by continuous reading of names and properness of $\mathbb{P}_\mathcal{I}$ we get a $G' \in \mathcal{G}$, $G' \subseteq G$ and a continuous function $f_x : G' \to D_x$. Thus, we have shown that

$$A_x \notin \mathcal{I} \qquad \text{iff} \qquad \exists G \in \mathcal{G} \quad \exists f : G \to D_x \text{ continuous}.$$

Using the coding of $\mathbf{G}_\delta$ sets by the Souslin schemes of open sets, one can easily check that

$$\exists f : G \to D_x \text{ continuous}$$

is a $\mathbf{\Sigma}^1_1$ formula. Thus, the whole formula is $\mathbf{\Sigma}^1_1$ and we are done. $\square$

Analytic sets in a Polish space X can be coded by a universal $\mathbf{\Sigma}^1_1$ on $\omega^\omega \times X$. In the remaining part of this Section we fix a universal analytic set $\tilde{A} \subseteq \omega^\omega \times X$ which is Σ^1_1 and *good* (cf. [**20**, Section 3.H.1]). The set $\tilde{A}$ will be used to code analytic sets in X.

We say that a set $S \subseteq \omega^\omega$ *codes a σ-ideal $\mathcal{I}$ of analytic sets* if the family $\mathcal{I} = \{\tilde{A}_t : t \in S\}$ is a σ-ideal of analytic sets. Let $I(v)$ be a $\mathbf{\Pi}^1_1$ formula. Note that the famlily of analytic sets whose codes satisfy $I(v)$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ (because $\tilde{A}$ is good and $I(v)$ is a $\mathbf{\Pi}^1_1$ formula).

Lemma 3.15 (Folklore). *If $\mathcal{A}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ family of analytic sets, then $\mathcal{A}$ is downward closed, i.e. if $A, B \in \mathbf{\Sigma}^1_1$ are such that $A \subseteq B$ and $B \in \mathcal{A}$, then $A \in \mathcal{A}$.*

Proof. Suppose $A \subseteq B$ are $\mathbf{\Sigma}^1_1$ and $B \in \mathcal{A}$. Let $Z \subseteq \omega^\omega$ be such that $Z \in \mathbf{\Sigma}^1_1 \setminus \mathbf{\Pi}^1_1$. Take $L \subseteq \omega^\omega \times X$ such that

$$(t, x) \in L \quad \Leftrightarrow \quad (t \in Z \wedge x \in B) \vee x \in A.$$

As $\{t \in \omega^\omega : L_t \in \mathcal{A}\} \in \mathbf{\Pi}^1_1$, we conclude that $B \in \mathcal{A}$. $\square$

Suppose $V \subseteq W$ is a generic extension and in V we have a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal $\mathcal{I}$. Let $I(v)$ be a $\mathbf{\Pi}^1_1$ formula which codes the σ-ideal of analytic sets $\mathcal{I} \cap \mathbf{\Sigma}^1_1$. By $\mathcal{I}^W$ we denote the family of analytic sets whose codes satisfy $I(v)$ in $V[G]$. This definition does not depend on the formula $I(v)$ since if $I'(v)$ is another such formula, then

$$\forall t \in \omega^\omega \quad I(t) \Leftrightarrow I'(t)$$

is a $\mathbf{\Pi}^1_2$ sentence and hence it is absolute for $V \subseteq W$.

Proposition 3.16. *Let $V \subseteq W$ be a generic extension and let $\mathcal{I}$ be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal generated by closed sets in V. In W, the family $\mathcal{I}^W$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal of analytic sets, generated by closed sets.*

Proof. Let $K(v)$ be a $\mathbf{\Pi}^1_1$ formula defining the set of codes of closed sets in $\mathcal{I}$. By $\mathcal{K}^W$ we denote the family of closed sets in W, whose codes satisfy $K(v)$ (as previously, this does not depend on the formula $K(v)$).

First we show that in W the family $\mathcal{K}^W$ is hereditary. Consider the following sentence

$$\exists t, s \in \omega^\omega \quad (\neg K(s) \ \wedge \ K(t) \ \wedge \ \tilde{C}_s \subseteq \tilde{C}_t).$$

It is routine to check that this is a $\mathbf{\Sigma}^1_2$ sentence and hence it is absolute for $V \subseteq W$. This shows that $\mathcal{K}^W$ is hereditary.

Next we show that $\mathcal{I}^W$ is a σ-ideal of analytic sets. Let $D \subseteq \omega^\omega \times X \times \omega^\omega$ be a closed set such that $\pi[D] = \tilde{A}$ (here π denotes the projection to the first two coordinates). Consider the following formula $I'(v)$:

$$\neg \ (\exists G \in \mathbf{G}_\delta \quad \overline{G} \text{ is } K\text{-perfect} \ \wedge \ \exists f : G \to D_v \text{ continuous})$$

(writing that $\overline{G}$ is K-perfect we mean that $\overline{G}$ is perfect with respect to the family of closed sets defined by $K(v)$). Using Lemma 3.13(iii) we can check that $I'(v)$ is a $\mathbf{\Pi}^1_1$ formula. From the proof of Corollary 3.14 and from Lemmas 3.10, 3.11 and 3.12 we conclude that in V we have

$$\forall t \in \omega^\omega \quad I(t) \Leftrightarrow I'(t).$$

This is a $\mathbf{\Pi}^1_2$ sentence and hence it holds in W. Therefore, it is enough to check that $(I')^W(\omega^\omega)$ codes a σ-ideal of analytic sets. However, it follows from Theorem 3.2 and from Lemmas 3.10, 3.11 and 3.12, that $(I')^W(\omega^\omega)$ codes the σ-ideal generated by $K(\omega^\omega)^W$.

The fact that $\mathcal{I}^W$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ follows now from the remarks preceding this proposition. □

We have also the following

Alternate proof of Proposition 3.16. Throughout this proof we denote the closure of a set A by cl$A$. Withoug loss of generality assume that $\mathcal{I}$ is Π^1_1 on Σ^1_1 and $X = \omega^\omega$.

We will use the following notation. If $\varphi(v)$ is a formula and $t \in \omega^\omega$, then by $\Sigma^1_1(t) \wedge \varphi$ (respectively $\Delta^1_1(t) \wedge \varphi$) we denote the family of $\Sigma^1_1(t)$ (respectively $\Delta^1_1(t)$) sets whose codes satisfy $\varphi(v)$.

Let $I(v)$ be a Π^1_1 formula defining the set of codes of analytic sets in $\mathcal{I}$. Let $\mathcal{K}$ be the family of closed sets in $\mathcal{I}$ and let $K(v)$ be a Π^1_1 formula defining the set of codes of the sets in $\mathcal{K}$ (in terms of the universal set $\tilde{C}$). Consider the formula $\hat{K}(v)$ saying that $\mathrm{cl}\tilde{A}_v \in \mathcal{K}$. Note that $\hat{K}(v)$ can be written as follows

$$\forall s \in \omega^\omega \quad \tilde{C}_s \subseteq \mathrm{cl}\tilde{A}_v \quad \Rightarrow \quad K(s)$$

and notice that it is a Π^1_1 formula. Therefore, the family of analytic sets whose codes satisfy $\hat{K}(v)$ is Π^1_1 on Σ^1_1.

Consider the set $F \subseteq \omega^\omega \times X$ defined as follows:

$$(t,x) \in F \quad \text{iff} \quad x \in \bigcup(\Delta^1_1(t) \wedge \hat{K}).$$

By the usual coding of Δ^1_1 sets we get that F is Π^1_1.

LEMMA 3.17. *For each $t \in \omega^\omega$ we have*

$$\bigcup(\Sigma^1_1(t) \wedge \hat{K}) = \bigcup(\Delta^1_1(t) \wedge \hat{K}) = \bigcup(\Sigma^1_1(t) \wedge I).$$

PROOF. Without loss of generality assume that $t = 0$. The first equalitiy follows from the First Reflection Theorem. Denote $C = \bigcup(\Delta^1_1(t) \wedge \hat{K}) = \bigcup(\Sigma^1_1(t) \wedge \hat{K})$.

In the second equality, the left-to-right inclusion is obvious since $\hat{K}(s)$ implies $I(s)$, for each $s \in \omega^\omega$. We need to prove that if $A \in \Sigma^1_1$ is not contained in C, then $A \notin \mathcal{I}$. Suppose $A \in \Sigma^1_1$ and $A \nsubseteq C$. Since $C \in \Pi^1_1$, we may assume that $A \cap C = \varnothing$. Let T be a recursive pruned tree on $\omega \times \omega$ such that $A = \mathrm{proj}[T]$. If $A \in \mathcal{I}$, then there is a sequence of closed sets $\langle D_n : n < \omega\rangle$ such that each $D_n \in \mathcal{I}$ and $A \subseteq \bigcup_{n<\omega} D_n$. By induction we construct a sequence of $\langle \tau_n \in \omega^{<\omega}\rangle$ and $\sigma_n \in T$ such that for each $n < \omega$ the following hold

- $\sigma_{n+1} \supsetneq \sigma_n$ and $\tau_{n+1} \supsetneq \tau_n$,
- $\mathrm{proj}[T(\sigma_n)] \subseteq [\tau_n]$,
- $\mathrm{proj}[T(\sigma_{n-1})] \cap [\tau_n] \cap D_n = \varnothing$

We take $\sigma_{-1} = \varnothing$. Suppose σ_n and τ_n are constructed. Notice that $\mathrm{proj}[T(\sigma_n)]$ is Σ^1_1. Since $A \cap C = \varnothing$ we see that $\mathrm{cl}(\mathrm{proj}[T(\sigma_n)]) \notin \mathcal{K}$. Consequently, $\mathrm{proj}[T(\sigma_n)] \nsubseteq D_n$ and hence there is $\tau_{n+1} \supsetneq \tau_n$, $[\tau_{n+1}] \subseteq \tau_n$ such that

(i) $\mathrm{proj}[T(\sigma_n)] \cap [\tau_{n+1}] \neq \varnothing$,
(ii) $\mathrm{proj}[T(\sigma_n)] \cap \cap[\tau_{n+1}] \cap D_n = \varnothing$.

Using (i) find $\sigma_{n+1} \supsetneq \sigma_n$ such that $\sigma_{n+1} \in T$ and $\mathrm{proj}[T(\sigma_{n+1})] \subseteq [\tau_{n+1}]$.

Now, if $s = \bigcup_{n<\omega} \sigma_n$, then $s \in \lim T$, so $\pi(s) \in A$, but $\pi(s) \notin \bigcup_{n<\omega} D_n$. This ends the proof of the lemma. □

Consider the following formula $I'(v)$ (v is a variable):

$$\forall z \in X \quad z \in \tilde{A}_v \Rightarrow z \in F_v.$$

Note that I' is a Π^1_1 formula and

$$V \vDash \forall t \in \omega^\omega \quad I(t) \Leftrightarrow I'(t).$$

This is a Π^1_2 sentence, so by absoluteness we see that I and I' define the same set of codes of analytic sets in W.

Now we show that $\mathcal{I}^W$ is a σ-ideal generated by closed sets. The fact that $\mathcal{I}^W$ is closed under taking analytic subsets follows from Lemma 3.15 because $\mathcal{I}^W$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

Let us show that $\mathcal{I}^W$ is closed under countable unions. Pick a recursive bijection $\lceil\cdot\rceil : (\omega^\omega)^\omega \to \omega^\omega$. The following sentence

$$(*) \qquad \forall \langle t_n : n<\omega\rangle \in (\omega^\omega)^\omega \quad ((\forall n<\omega\ I'(t_n)) \Rightarrow I'(\lceil t_n : n<\omega\rceil))$$

is Π^1_2 and hence it is absolute. Note that for any $\langle t_n : n<\omega\rangle \in (\omega^\omega)^\omega$ we have $F_{t_k} \subseteq F_{\lceil t_n : n<\omega\rceil}$ for each $k<\omega$ (because $t_k \in \Delta^1_1(\lceil t_n : n<\omega\rceil)$). Therefore $(*)$ holds in V and hence also in W. This shows that the family of analytic sets coded by $(I')^W(\omega^\omega)$ is closed under countable unions.

To see that $\mathcal{I}^W$ is generated by closed sets, take any $t \in W \cap \omega^\omega$ such that $(\tilde{A}^W)_t \in \mathcal{I}^W$. This means that $W \vDash I'(t)$, so $(\tilde{A}^W)_t \subseteq (F^W)_t$. Let $\langle t_n : n<\omega\rangle \in W$ be the the sequence of all elements of ω^ω in W which are $\Delta^1_1(t)$ and satisfy $\hat{K}(v)$. By the definition of F we see that

$$W \vDash (\tilde{A}^W)_t \subseteq \bigcup_{n<\omega} (\tilde{A}^W)_{t_n}$$

is satisfied in W. Let $\langle s_n : n<\omega\rangle \in W$ be a sequence of elements of $W\cap\omega^\omega$ such that $W \vDash (\tilde{C}^W)_{s_n} = \mathrm{cl}(\tilde{A}^W)_{t_n}$ for each $n<\omega$. Now $W \vDash \hat{K}(t_n)$ implies $W \vDash K(s_n)$. Therefore $W \vDash (\tilde{C}^W)_{s_n} \in \mathcal{I}^W$ because

$$\forall t,s \in \omega^\omega \quad (\tilde{A}_s \subseteq \tilde{C}_t \ \wedge\ K(t)) \Rightarrow I(s)$$

is $\mathbf{\Pi}^1_2$ and holds in V. Since

$$W \vDash (\tilde{A}^W)_t \subseteq \bigcup_{n<\omega} (\tilde{C}^W)_{s_n},$$

we conclude that $\mathcal{I}^W$ is generated by closed sets. ☐

REMARK 3.18. It is worth noting that analogously as in the alternate proof of Proposition 3.16 we can get the following. If $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal and $V \subseteq W$ is a generic extension, then $\mathcal{I}^W$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal in W.

Zapletal defines in [**35**] the class of *iterable* σ-ideals (see [**35**, Definition 5.1.3] for a definition without large cardinals, and [**35**, Definition 5.1.2] for a definition under large cardinal assumptions). By Proposition 3.16 we get the following corollary.

COROLLARY 3.19. *If $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal generated by closed sets in a Polish space X, then $\mathcal{I}$ is iterable.*

3.6 Products and iterations

Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals on Polish spaces X and Y, respectively. Recall that the Fubini product of $\mathcal{I}$ and $\mathcal{J}$, denoted by $\mathcal{I} * \mathcal{J}$, is the σ-ideal of those

$A \subseteq X \times Y$ such that $\{x \in X : A_x \notin \mathcal{J}\} \in \mathcal{I}$. For each $n < \omega$ we also define $\mathcal{I}^n$ such that $\mathcal{I}^{n+1} = \mathcal{I} * \mathcal{I}^n$ and $\mathcal{I}^1 = \mathcal{I}$.

LEMMA 3.20 (Folklore). *Suppose $\mathcal{I}$ and $\mathcal{J}$ are $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideals on Polish spaces X and Y, respectively. Let $A \subseteq X \times Y$ be a $\mathbf{\Sigma}^1_1$ set in $\mathcal{I} * \mathcal{J}$. There is a $\mathbf{\Sigma}^1_1$ set D such that $A \cap D = \varnothing$ and*

$$\forall^{\mathcal{I}} x \in X\, \forall^{\mathcal{J}} y \in Y \quad (x,y) \in D.$$

PROOF. To simplify notation suppose that $X = Y = \omega^\omega$, $A \in \Sigma^1_1$ and$\mathcal{I}$ and $\mathcal{J}$ are Π^1_1 on Σ^1_1. Put

$$U^1 = \bigcup(\Sigma^1_1 \cap \mathcal{I})$$

and let $U^2 \subseteq \omega^\omega \times \omega^\omega$ be such that for each $t \in \omega^\omega$ we have

$$(U^2)_t = \bigcup(\Sigma^1_1(t) \cap \mathcal{J}).$$

By the First Reflection Theorem we have $U^1 = \bigcup(\Delta^1_1 \cap \mathcal{I})$ and $(U^2)_t = \bigcup(\Delta^1_1(t) \cap \mathcal{J})$ for each $t \in \omega^\omega$. Therefore, by the usual coding of Δ^1_1 sets, we get that U^1 and U^2 are Π^1_1. Put

$$C = (U^1 \times Y) \cup U^2.$$

Notice that $A \subseteq C$ (since otherwise we get that $A \notin \mathcal{I}*\mathcal{J}$). Now $B = X \times Y \setminus C$ is as needed. □

Generalizing the finite Fubini powers, one can define the α-th Fubini power of $\mathcal{I}$ (denoted by $\mathcal{I}^\alpha$) for any $\alpha < \omega_1$. A game-theoretic definition of $\mathcal{I}^\alpha$ is given in [**35**, Definition 5.1.1]. Definition 3.21 below (equivalent to [**35**, Definition 5.1.1]) appears in [**7**, p. 74]. If $\beta < \alpha$ are countable ordinals, then we write $\pi_{\alpha,\beta}$ for the projection to the first β coordinates from X^α to X^β.

DEFINITION 3.21. A set $B \subseteq X^\alpha$ belongs to $\mathcal{I}^\alpha$ if and only if there is a set (arbitrary) $D \subseteq X^\alpha$ such that D is disjoint from B and

(i) for each $\beta < \alpha$ and for each $x \in \pi_{\alpha,\beta}[D]$ we have

$$X \setminus (\pi_{\alpha,\beta+1}[D])_x \in \mathcal{I}$$

(ii) for each limit $\beta < \alpha$ and $x \in X^\beta$,

$$x \in \pi_{\alpha,\beta}[D] \quad \Leftrightarrow \quad \forall \gamma < \beta \quad x{\restriction}\gamma \in \pi_{\alpha,\gamma}[D].$$

REMARK 3.22. Let X be a Polish space and let $\mathcal{I}$ be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on X, generated by closed sets. Suppose $A \subseteq X^2$ is $\mathbf{\Sigma}^1_1$. We will show that either A belongs to $\mathcal{I}^2$, or else A contains an $\mathcal{I}^2$-positive $\mathbf{G}_\delta$ set.

Let $D \subseteq X^2 \times \omega^\omega$ be a closed set such that $\pi[D] = A$ (here π denotes the projection to the first two coordinates). Since $\mathcal{I}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$, the family of $\mathcal{I}$-positive $\mathbf{G}_\delta$ sets is $\mathbf{\Sigma}^1_1$ (in the sense of Section 3.5, in terms of $\tilde{G}$). Denote this family by $\mathcal{G}$. Put $A' = \{x \in X : A_x \notin \mathcal{I}\}$. If $A' \in \mathcal{I}$, then clearly $A \in \mathcal{I}^2$. Suppose that $A' \notin \mathcal{I}$. By Theorem 3.2, for each $x \in A'$ there is an $\mathcal{I}$-positive $\mathbf{G}_\delta$ set G contained in A_x. Pick $x \in A'$ and such a $G \subseteq A_x$. Using $\mathbf{\Sigma}^1_2$-absoluteness, we get a $\mathbb{P}_{\mathcal{I}}$-name $\dot{y}$ for an element of D such that

$G \Vdash \pi(\dot{y}) = \dot{g}$ ($\dot{g}$ is the name for the generic point). By properness and continuous reading of names for $\mathbb{P}_{\mathcal{I}}$, there is an $\mathcal{I}$-positive $\mathbf{G}_\delta$ set $G' \subseteq G$, and a continuous function $f : G' \to D$. Therefore, for each $x \in A'$ the following holds

$$\exists G \in \mathcal{G} \quad \exists f : G \to D_x \text{ continuous.}$$

This is a $\mathbf{\Sigma}^1_1$ formula, so by $\mathbf{\Sigma}^1_2$-absoluteness we have

$$A' \Vdash \exists G \in \mathcal{G} \quad \exists f : G \to D_{\dot{g}} \text{ continuous.}$$

Again, by properness and continuous reading of names for $\mathbb{P}_{\mathcal{I}}$ we get an $\mathcal{I}$-positive $\mathbf{G}_\delta$ set $G' \subseteq A'$ and a continuous function $g : G' \to \mathcal{G}$ such that for each $x \in G'$ we have $g(x) \subseteq A_x$. Let $G = \{(x,y) \in X^2 : x \in G' \wedge y \in g(x)\} = (g, id)^{-1}[\tilde{G}]$. This is an $\mathcal{I}^2$-positive $\mathbf{G}_\delta$ set contained in A.

Definition 3.23. Let $\mathbf{\Gamma}$ be a projective pointclass. Let X be a Polish space, $\mathcal{I}$ be a σ-ideal on X and let α be a countable ordinal. We say that a set $A \subseteq X^\alpha$ is an $(\mathcal{I}, \alpha)$ $\mathbf{\Gamma}$ *tree* if

- for each $\beta \leq \alpha$ the set $\pi_{\alpha,\beta}[A] \in \mathbf{\Gamma}(X^\beta)$,
- for each $\beta < \alpha$ and for each $x \in \pi_{\alpha,\beta}[A]$ we have $(\pi_{\alpha,\beta+1}[A])_x \notin \mathcal{I}$
- for each limit $\beta < \alpha$ and $x \in X^\beta$,

$$x \in \pi_{\alpha,\beta}[A] \quad \Leftrightarrow \quad \forall \gamma < \beta \quad x{\restriction}\gamma \in \pi_{\alpha,\gamma}[A].$$

We say that A is an $(\mathcal{I}, \alpha)$ $\mathbf{\Gamma}$ *full tree* if A is an $(\mathcal{I}, \alpha)$ $\mathbf{\Gamma}$ tree and

- for each $\beta < \alpha$ and for each $x \in \pi_{\alpha,\beta}[A]$ the set $(\pi_{\alpha,\beta+1}[A])_x$ is $\mathcal{I}$-full.

Definition of an $(\mathcal{I}, \alpha)$ **Bor** tree appears in [**35**, Definition 5.1.5]. Note that the following lemma immediately follows from Lemma 3.20

Lemma 3.24 (Folklore). *Suppose $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space X. Let $A \subseteq X^n$ be a $\mathbf{\Sigma}^1_1$ set in $\mathcal{I}^n$. There is an $(\mathcal{I}, n)$ $\mathbf{\Sigma}^1_1$ full tree D disjoint from A.*

If $\mathcal{I}$ is an iterable σ-ideal and $\alpha < \omega_1$, then we denote by $(\mathbb{P}_{\mathcal{I}})^{*\alpha}$ the countable support iteration of $\mathbb{P}_{\mathcal{I}}$ of length α. If $A \subseteq X^\alpha$ is an $(\mathcal{I}, \alpha)$ **Bor** tree, then we associate with A a condition $p_\alpha(A)$ in $(\mathbb{P}_{\mathcal{I}})^{*\alpha}$ as follows. If $\beta < \alpha$, then $p_\alpha(A)(\beta)$ is a $(\mathbb{P}_{\mathcal{I}})^{*\beta}$-name $\dot{Y}_\beta$ (for an $\mathcal{I}$-positive Borel set) such that

$$p_\beta(\pi_{\alpha,\beta}[A]) \Vdash \dot{Y}_\beta = A_{\dot{g}_\beta},$$

where $\dot{g}_\beta$ is the name for the $(\mathbb{P}_{\mathcal{I}})^{*\beta}$-generic point in X^β. Zapletal [**35**, Theorem 5.1.6] proved the following.

Theorem 3.25 (Zapletal, [**35**, Theorem 5.1.6]). *If $\mathcal{I}$ is an iterable σ-ideal on a Polish space X and $\alpha < \omega_1$, then the function p_α is a dense embedding from the poset of $(\mathcal{I}, \alpha)$ **Bor** trees, ordered by inclusion, into $(\mathbb{P}_{\mathcal{I}})^{*\alpha}$. In particular, any $\mathcal{I}^\alpha$-positive Borel set in X^α contains an $(\mathcal{I}, \alpha)$ **Bor** tree and the forcing $\mathbb{P}_{\mathcal{I}^\alpha}$ is equivalent to the countable support iteration of $\mathbb{P}_{\mathcal{I}}$ of length α.*

Together with Kanovei, Zapletal extended Theorem 3.25 under the assumption that $\mathcal{I}$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.

THEOREM 3.26 (Kanovei, Zapletal, [**35**, Theorem 5.1.9]). *Let $\mathcal{I}$ be an iterable $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on a Polish space X and α be a countable ordinal. If $A \subseteq X^\alpha$ is $\mathbf{\Sigma}^1_1$, then either $A \in \mathcal{I}^\alpha$, or else A contains an $(\mathcal{I},\alpha)$* **Bor** *tree.*

In the proof of Theorem 3.26, Kanovei and Zapletal generalized Lemma 3.24 in the following way.

THEOREM 3.27 (Kanovei, Zapletal, [**35**, proof of Theorem 5.1.9]). *Let $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal, and let α be a countable ordinal. If $A \subseteq X^\alpha$, is $\mathbf{\Sigma}^1_1$ and $A \in \mathcal{I}^\alpha$, then there is a $(\mathcal{I},\alpha)$ $\mathbf{\Sigma}^1_1$ full tree D disjoint from A.*

The following (unpublished) corollary was communicated to me by Pawlikowski.

COROLLARY 3.28 (Pawlikowski, [**23**]). *If X is a Polish space and α is a countable ordinal, then $\mathcal{M}(X)^\alpha \cap \mathbf{\Sigma}^1_1(X^\alpha) \subseteq \mathcal{M}(X^\alpha)$.*

The next proposition is motivated by Theorems 3.2, 3.26 and Corollary 3.28.

PROPOSITION 3.29. *Let X be a Polish space and let $\mathcal{I}$ be a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal on X generated by closed sets. If $A \subseteq X^\omega$ is an analytic set, then either $A \in \mathcal{I}^\omega$, or else A contains an $(\mathcal{I},\omega)$ $\mathbf{G}_\delta$ tree G such that*

$$(\mathbf{\Sigma}^1_1(X^\omega) \cap \mathcal{I}^\omega)\restriction G \subseteq \mathcal{M}(G).$$

PROOF. Suppose $A \subseteq X^\omega$ is an analytic $\mathcal{I}^\omega$-positive set. By Theorem 3.26 we may assume that A is an $(\mathcal{I},\omega)$ **Bor** tree. For each $n < \omega$ write A_n for $\pi_{\omega,n}[A]$ and let $E_n \subseteq X^n \times \omega^\omega$ be a closed set projecting to A_n. Let $\mathcal{G} \subseteq \omega^\omega$ be the analytic set from Lemma 3.13 consisting of codes of all $\mathcal{I}$-perfect $\mathbf{G}_\delta$ sets. In this proof we denote $\pi_{n+1,n}$ by π_n.

We construct a sequence of $\mathbf{G}_\delta$ sets $G_n \subseteq X^n$ such that

(i) $\pi_n[G_n] \subseteq G_{n-1}$ is comeager in G_{n-1},
(ii) for each $x \in \pi_n[G_n]$ the set $(G_n)_x$ is $\mathcal{I}$-perfect,
(iii) $\pi_n\restriction G_n : G_n \to \pi_n[G_n]$ is an open map,
(iv) if $D \subseteq X^n$ is $(\mathcal{I},n)$ $\mathbf{\Sigma}^1_1$ full tree, then $D \cap G_n$ is comeager in G_n

Note that, by Lemma 3.24, (iv) implies

(v) $(\mathbf{\Sigma}^1_1(X^n) \cap \mathcal{I}^n)\restriction G_n \subseteq \mathcal{M}(G_n)$.

For $n = 0$ use Lemma 3.11 to find an $\mathcal{I}$-perfect $\mathbf{G}_\delta$-set $G_0 \subseteq A_0$. Notice that $(\mathbf{\Sigma}^1_1(X) \cap \mathcal{I})\restriction G_0 \subseteq \mathcal{M}(G_0)$ follows from the fact that G_0 is $\mathcal{I}$-perfect.

Suppose the set $G_n \subseteq X^n$ is constructed. Similarly as in Remark 3.22 we conclude that by Lemma 3.11, $\mathbf{\Sigma}^1_2$-absoluteness and continuous reading of names for $\mathbb{P}_\mathcal{I}$, for each $x \in G_n$ there is a code $c(x) \in \mathcal{G}$ for an $\mathcal{I}$-perfect $\mathbf{G}_\delta$ set $\tilde{G}_{c(x)}$ and a continuous function $f : \tilde{G}_{c(x)} \to (E_{n+1})_x$. Consider the set

$$W = \{(x,c) \in X^n \times \omega^\omega : \quad x \in G_n,\ c \in \mathcal{G}\ \wedge\ \exists f : \tilde{G}_c \to (E_{n+1})_x \text{ continuous}\}.$$

W is analytic and all vertical sections of W are nonempty. Hence, by the Jankov-von Neumann theorem, W has a $\sigma(\mathbf{\Sigma}^1_1)$-measurable uniformization $g : G_n \to \mathcal{G}$. In particular, g is Baire measurable and hence it is continuous on a dense $\mathbf{G}_\delta$ set $G'_n \subseteq G_n$. Let

$$G_{n+1} = \{(x,y) \in X^n \times X : x \in G'_\gamma \wedge y \in \tilde{G}_{g(x)}\}.$$

G_{n+1} is a $\mathbf{G}_\delta$ set since $G_{n+1} = (g, id)^{-1}[\tilde{G}]$. Moreover, $\pi_{n+1}[G_{n+1}] = G'_n$ is comeager in G_n.

Note that the function $\pi_{n+1}\restriction G_{n+1} : G_{n+1} \to G'_n$ is open by Lemma 3.9 and the fact that g is continuous on G'_n.

Now, let $D \subseteq X^{n+1}$ be an $(\mathcal{I}, n+1)$ $\mathbf{\Sigma}^1_1$ tree. The set $D_n = \pi_{n+1}[D]$ is an $(\mathcal{I}, n)$ $\mathbf{\Sigma}^1_1$ tree, so, by the inductive hypothesis, $D_n \cap G_n$ is comeager in G_n. Therefore, $D_n \cap G'_n$ is comeager in G'_n. Moreover, if $x \in G'_n \cap D_n$, then $D_x \cap (G_{n+1})_x$ is comeager in $(G_{n+1})_x$, since $(G_{n+1})_x$ is $\mathcal{I}$-perfect. Now, D has the Baire property, so by Theorem 2.1 (for the function $\pi_{n+1}\restriction G_{n+1} : G_{n+1} \to G'_n$) we have that $D \cap G_{n+1}$ is comeager in G_{n+1}.

This ends the construction.

For each $n, k < \omega$ consider the set

$$H^k_n = \{x_n \in G_n :\ \forall^{\mathcal{M}} y_{n+1} \in (G_{n+1})_{x_n}\ \ldots\ \forall^{\mathcal{M}} y_{n+k} \in (G_{n+k})_{(x_n, y_{n+1}, \ldots, y_{n+k-1})}$$
$$\exists y_{n+k+1} \in (G_{n+k+1})_{(x_n, y_{n+1}, \ldots, y_{n+k})}\quad (x_n, y_{n+1}, \ldots y_{n+k+1}) \in G_{n+k}\}$$

Applying $(k+1)$-many times Theorem 2.1 we conclude that H^k_n is comeager in G_n for each $k < \omega$. Put

$$H_n = \bigcap_{k<\omega} H^k_n.$$

Each H_n is also a comeager subset of G_n. Notice that $\pi_{n+1}[H_{n+1}] \subseteq H_n$ and $\pi_{n+1}[H_{n+1}]$ is comeager in G_n for each $n < \omega$. Moreover, for each $n < m < \omega$

$$(*) \qquad \pi_{m,n}[H_m] \text{ is comeager in } G_n$$

(by repeatedly applying Theorem 2.1). Let

$$G = \bigcap_{n<\omega} \pi^{-1}_{\omega,n}[G_n].$$

G is a $\mathbf{G}_\delta$ set and it is contained in A since A is an $(\mathcal{I}, \omega)$ **Bor** tree.

Notice that for each $n < \omega$

$$\bigcap_{n<m<\omega} \pi_{m,n}[H_m] \subseteq \pi_{\omega,n}[G].$$

Consequently, by $(*)$ we have that $\pi_{\omega,n}[G]$ is comeager in G_n. Therefore, it is $\mathcal{I}^n$-positive, by (v). For each $k < \omega$ and $x \in \pi_{\omega,k}[G]$ we may repeat the above argument in the space $(X^k)_x$ and conclude that the set $(\pi_{\omega,k+1}[G])_x$ is comeager in $(G_{k+1})_x$ and hence is $\mathcal{I}$-positive (since $(G_{k+1})_x$ is $\mathcal{I}$-perfect). Therefore G is $(\mathcal{I}, \omega)$ $\mathbf{G}_\delta$ tree.

Now we prove that $(\mathbf{\Sigma}^1_1(X^\omega) \cap \mathcal{I}^\omega)\restriction G \subseteq \mathcal{M}(G)$. By Theorem 3.27 it is enough to prove that if $D \subseteq X^\omega$ is $(\mathcal{I}, \omega)$ $\mathbf{\Sigma}^1_1$ full tree, then $D \cap G$ is comeager in G. Write $D_n = \pi_{\omega,n}[D]$. Using (iv) we see that D_n is comeager in G_n. Find a dense (in G_n) $\mathbf{G}_\delta$ set $G''_n \subseteq D_n$ such that $G''_{n+1} \subseteq \pi_n^{-1}[G''_n]$. Let $G'' = \bigcap_{n<\omega} \pi_{\omega,n}^{-1}[G''_n]$. Note that $G'' \subseteq D \cap G$ and G'' is a $\mathbf{G}_\delta$ set. We will prove that G'' is dense in G.

Repeatedly applying Theorem 2.1 and the property (iv) we see that for each $n < \omega$ the following holds

$$\forall^{\mathcal{M}} y_0 \in G_0 \ \forall^{\mathcal{M}} y_1 \in (G_1)_{y_0} \ \ldots \forall^{\mathcal{M}} y_n \in (G_n)_{(y_0,\ldots,y_{n-1})} \quad (y_0,\ldots,y_n) \in G''_n.$$

Using this we can easily show G'' is nonempty, and, in fact, that if $U_n \subseteq X^n$ is open, then $G'' \cap \pi_{\omega,n}^{-1}[U_n]$ is nonempty. But this implies that G'' is dense.

This ends the proof. □

Let X and Y be Polish spaces and $F : X \to \mathcal{P}(Y)$ be a multifunction. If $\mathcal{A}$ is a family of subsets of X, then we say that F is *$\mathcal{A}$-measurable* if for each open set $U \subseteq Y$ the set $F^{-1}(U) = \{x \in X : F(x) \cap U \neq \varnothing\}$ belongs to $\mathcal{A}$. We say that F is an *analytic multifunction* if its graph, i.e $\bigcup_{x\in X} \{x\} \times F(x)$, is analytic in $X \times Y$. The following result is motivated by the Kuratowski-Ryll Nardzewski theorem.

Proposition 3.30. *Let X be a Polish space and $\mathcal{I}$ an ideal on X generated by closed sets. If $F : X \to \mathcal{P}(\omega^\omega)$ is an analytic multifunction then there is an $\mathcal{I}$-positive $\mathbf{G}_\delta$ set G such that $F \restriction G$ is $\mathbf{\Sigma}^0_3$-measurable.*

Proof. Denote the graph of F by A and let a be such that $A \in \Sigma^1_1(a)$. Let $A(v,w)$ be a $\Sigma^1_1(a)$ formula defining the set A. Take $M \prec H_\kappa$ (for a big enough κ) containing a and $\mathbb{P}_\mathcal{I}$. Let $Gen(M) \subseteq X$ be the set of $\mathbb{P}_\mathcal{I}$-generic reals over M. $Gen(M)$ is an $\mathcal{I}$-positive Borel set by properness of $\mathbb{P}_\mathcal{I}$. Find an $\mathcal{I}$-positive $\mathbf{G}_\delta$ set $G \subseteq Gen(M)$. We will show that $F \restriction G$ is $\mathbf{\Sigma}^0_3$-measurable. Notice that if $\tau \in \omega^{<\omega}$ and $x \in X$, then

$$x \in F^{-1}([\tau]) \quad \text{iff} \quad \exists y \in [\tau] \ A(x,y).$$

This is a $\Sigma^1_1(a)$ formula, so it is absolute for $M[x] \subseteq V$. Therefore, by a usual forcing argument and the fact that $\mathcal{I}$-positive $\mathbf{G}_\delta$ sets are dense in $\mathbb{P}_\mathcal{I}$ we get

$$F^{-1}\big[[\tau]\big] = \bigcup\{G \in \mathbb{P}_\mathcal{I} \cap M : \quad G \in \mathbf{G}_\delta \ \wedge \ G \Vdash \exists y \in [\tau] \ A(\dot{g},y)\}.$$

This is a $\mathbf{\Sigma}^0_3$ set. □

3.7 Decomposing Baire class 1 functions

Let X be a Polish space and $f : X \to \omega^\omega$ be a Borel function. We say that f is *piecewise continuous* if X can be covered by a countable family of closed sets on each of which f is continuous.

Recall that a function is $\mathbf{G}_\delta$-measurable if preimages of $\mathbf{G}_\delta$ sets are $\mathbf{G}_\delta$ or, equivalently, preimages of open sets are $\mathbf{G}_\delta$. If $f : X \to Y$ is a $\mathbf{G}_\delta$-measurable function and X, Y are Polish spaces then f is of Baire class 1, preimages of closed sets are $\mathbf{F}_\sigma$ and therefore preimages of open sets are in fact $\mathbf{\Delta}^0_2$.

The following characterization of piecewise continuity has been given by Jayne and Rogers.

THEOREM 3.31 (Jayne, Rogers, [**12**, Theorem 5]). *Let X and Y be Polish spaces. A function $f : X \to Y$ is piecewise continuous if and only if it is $\mathbf{G}_\delta$-measurable.*

A nice and short proof of the Jayne-Rogers theorem can be found in [**21**]. Classical examples of Borel functions which are not piecewise continuous are the Lebesgue functions $L, L_1 : 2^\omega \to \mathbb{R}$ (for definitions see [**31**, Section 1]). In [**31**] Solecki strengthened Theorem 3.31 proving the following result.

THEOREM 3.32 (Solecki, [**31**, Theorem 3.1]). *Let X, Y be Polish spaces and $f : X \to Y$ be Baire class 1. Then*

- *either f is piecewise continuous,*
- *or $L \sqsubseteq f$, or $L_1 \sqsubseteq f$.*

Theorem 3.31 follows from Theorem 3.32 because neither L nor L_1 is $\mathbf{G}_\delta$-measurable.

From now now until the end of this section we fix a Polish space X and a Baire class 1, not piecewise continuous function $f : X \to \omega^\omega$ (the assumption about the range is made only to simplify notation). Consider the σ-ideal $\mathcal{I}^f$ on X generated by closed sets on which f is continuous. We will prove that the forcing $\mathbb{P}_{\mathcal{I}^f}$ is equivalent to the Miller forcing (see Corollary 3.39).

Suppose $C \subseteq X$ is a compact set and $c : 2^\omega \to C$ is a homeomorphism. We call (c, C) *a copy of the Cantor space* and denote it by $c : 2^\omega \hookrightarrow C \subseteq X$.

PROPOSITION 3.33. *Suppose that $G \subseteq X$ is a $\mathbf{G}_\delta$ set such that $G \notin \mathcal{I}^f$. There exist an open set $U \subseteq \omega^\omega$ and a copy of the Cantor space $c : 2^\omega \hookrightarrow C \subseteq X$ such that*

- *$f^{-1}[U] \cap C = c[\mathbb{Q}]$,*
- *$C \setminus c[\mathbb{Q}] \subseteq G$.*

PROOF. Denote $\{\tau \in 2^{<\omega} : \tau = \varnothing \vee \tau(|\tau| - 1) = 0\}$ by Q.

DEFINITION 3.34. A *Hurewicz scheme* is a Cantor scheme of closed sets $F_\tau \subseteq X$ for $\tau \in 2^{<\omega}$ together with a family of points $x_\tau \in X$ and clopen sets $U_\tau \subseteq \omega^\omega$ for $\tau \in Q$ such that:

- $x_{\tau^\frown 0} = x_\tau$, $U_{\tau^\frown 0} = U_\tau$,
- $x_\tau \in F_\tau \cap f^{-1}[U_\tau]$.

Suppose that $G = \bigcap_n G_n$ with each G_n open and $G_{n+1} \subseteq G_n$. We will construct a Hurewicz scheme such that for each $\tau \in 2^n$ the following two conditions hold:

- $\left(F_\tau \setminus f^{-1}\left[\bigcup_{\sigma \in 2^n \cap Q} U_\sigma\right]\right) \cap G \notin \mathcal{I}^f$,
- $F_{\tau^\frown 1} \subseteq \left(F_\tau \setminus f^{-1}\left[\bigcup_{\sigma \in 2^n \cap Q} U_\sigma\right]\right) \cap G_{|\tau|}$.

We need the following lemma (its special case can be found in the proof of the Jayne-Rogers theorem in [**21**]).

LEMMA 3.35. *If F is a closed set in X and $F \cap G \notin \mathcal{I}^f$ then there is $x \in F$ and a clopen set $U \subseteq \omega^\omega$ such that $f(x) \in U$ and for each open neighborhood V of x*

$$(V \setminus f^{-1}[U]) \cap G \notin \mathcal{I}^f.$$

Moreover, if $W \subseteq \omega^\omega$ is a clopen set such that $F \cap f^{-1}[W] \notin \mathcal{I}^f$, then we may require that $U \subseteq W$.

PROOF. First let $W = \omega^\omega$. Without loss of generality assume that for each nonempty open set $V \subseteq F$ we have $V \cap G \notin \mathcal{I}^f$. Supppose that the conclusion is false. We show that f is continuous on F, contradicting the fact that $F \cap G \notin \mathcal{I}^f$. Pick arbitrary $x \in F$ and a clopen set U such that $f(x) \in U$. By the assumption there is an open neighborhood $V \ni x$ such that $(V \setminus f^{-1}[U]) \cap G \in \mathcal{I}^f$. We claim that $V \subseteq f^{-1}[U]$. Suppose otherwise, then there is $y \in V$ such that $f(y) \notin U$. Pick a clopen set U' such that $U' \cap U = \varnothing$ and $f(y) \in U'$. Again, by the assumption there is an open neighborhood V' of y such that $(V' \setminus f^{-1}[U']) \cap G \in \mathcal{I}^f$. Now $V'' = V \cap V'$ is a nonempty open set and since $U' \cap U = \varnothing$ we have that

$$V'' \cap G \ \subseteq \ (V \setminus f^{-1}[U]) \cap G \ \cup \ (V' \setminus f^{-1}[U']) \cap G.$$

This shows that $V'' \cap G \in \mathcal{I}^f$, a contradiction.

Now, if $W \subseteq \omega^\omega$ is a clopen set such that $F \cap f^{-1}[W] \notin \mathcal{I}^f$ then $f^{-1}[W] \cap F$ is an $\mathbf{F}_\sigma$ set since f is Baire class 1. So there is a closed set $F' \subseteq F$ such that $F' \notin \mathcal{I}^f$ and $f[F'] \subseteq W$. Applying the previous argument to F' we get a clopen set U such that $U \subseteq W$. This ends the proof. □

Now we construct a Hurewicz scheme. First use Lemma 3.35 to find $x_\varnothing$, $U_\varnothing$ and put $F_\varnothing = X$. Suppose the scheme is constructed up to the level $n-1$. For each $\sigma \in 2^{n-1} \setminus Q$ find a nonempty, perfect closed set $C_\sigma \subseteq \omega^\omega$ such that $F_\sigma \cap f^{-1}[V] \notin \mathcal{I}^f$ for each nonempty relatively clopen set $V \subseteq C_\sigma$ (this is done by removing from ω^ω the collection of clopen sets U such that $F_\tau \cap f^{-1}[U] \in \mathcal{I}^f$).

Next, for each $\tau \in 2^n \setminus Q$ find a nonempty relatively clopen set $W_\tau \subseteq C_\tau$ such that for each $\sigma \in 2^{n-1} \cap Q$ and for each open neighborhood V of x_σ the following holds:

$$\Bigg(\big(V \setminus \bigcup_{\sigma' \in 2^{n-1} \cap Q} f^{-1}[U_{\sigma'}]\big) \setminus \bigcup_{\tau \in 2^n \cap Q} f^{-1}[W_\tau]\Bigg) \cap G \notin \mathcal{I}^f.$$

Do this as follows. Enumerate $2^n \cap Q$ in a sequence $\langle \tau_i : i < 2^{n-1} \rangle$ and construct the sets W_{τ_i} by induction on $i < 2^{n-1}$. Suppose that W_{τ_j} are already defined for $j < i < 2^{n-1}$. Notice that if ω^ω is divided into two clopen sets O_0 and O_1 such that $C_{\tau_i} \cap O_k \neq \varnothing$ for each $k \in \{0,1\}$ (recall that C_{τ_i} is a perfect set) then for each $\sigma \in 2^{n-1} \cap Q$ there exists $k \in \{0,1\}$ such that for each open neighborhood V of x_σ the following holds

$$(*)\qquad \Big(\big(V\setminus\bigcup_{\sigma\in 2^{n-1}\cap Q} f^{-1}[U_\sigma]\big)\setminus\bigcup_{j<i} f^{-1}[W_{\tau_j}]\Big)\cap G\setminus f^{-1}[O_k]\notin\mathcal{I}^f.$$

Enumerating $2^{n-1}\cap Q$ in a sequence $\langle\sigma_k : k<2^{n-2}\rangle$ find a decreasing sequence of nonempty clopen sets $O_k\subseteq O_{k-1}\subseteq\omega^\omega$ for $k<2^{n-2}$ such that $O_k\cap C_{\tau_i}\neq\varnothing$ for each $k<2^{n-2}$ and $(*)$ holds for each open neighborhood V of x_{σ_k}. Put $W_{\tau_i}=O_{2^{n-2}-1}$.

By the assumption on C_τ we have $W_\tau\notin\mathcal{I}^f$, for each $\tau\in 2^n\cap Q$. Using Lemma 3.35, for each $\sigma\in 2^{n-1}\setminus Q$ find a clopen $U_{\sigma^\frown 0}\subseteq W_{\sigma^\frown 0}$ and a point $x_{\sigma^\frown 0}$ such that the assertion of Lemma 3.35 holds.

For each $\sigma\in 2^{n-1}$ find two disjoint $\mathcal{I}^f$-positive closed sets $F_{\sigma^\frown 0},F_{\sigma^\frown 1}\subseteq F_\sigma$ of diameters less than $1/n$ such that

$$F_{\sigma^\frown 1}\subseteq\big(F_\sigma\setminus\bigcup_{\tau\in 2^n\cap Q} f^{-1}[U_\tau]\big)\cap G_n$$

and $F_{\sigma^\frown 0}$ contains $x_{\sigma^\frown 0}$. Do this as follows. For each $\sigma\in 2^{n-1}$ take an open neighborhood V_σ of $x_{\sigma^\frown 0}$ of diameter $<1/n$. The set

$$V_\sigma\setminus f^{-1}\big[\bigcup_{\tau\in 2^n\cap Q} U_\tau\big]$$

is an $\mathbf{F}_\sigma$ set which has $\mathcal{I}^f$-positive intersection with G. Thus it has a closed subset F such that F also has $\mathcal{I}^f$-positive intersection with G. Now, the set $F\cap G_n$ is $\mathbf{F}_\sigma$, so find $F_{\sigma^\frown 1}$ which is a closed subset of $F\cap G_n$ and has $\mathcal{I}^f$-positive intersection with $F\cap G$. Let $F_{\sigma^\frown 0}$ be a closed neighborhood of $x_{\sigma^\frown 0}$, disjoint from $F_{\sigma^\frown 1}$.

This ends the construction of the Hurewicz scheme. To finish the proof, we put $U=\bigcup_{\tau\in 2^{<\omega}} U_\tau$, $C=\bigcap_{n<\omega}\bigcup_{\tau\in 2^n} F_\tau$ and $c:2^\omega\hookrightarrow C\subseteq X$ such that $c(x)\in\bigcap_{n<\omega} F_{x\restriction n}$ for each $x\in 2^\omega$. □

Proposition 3.36. *The σ-ideal $\mathcal{I}^f$ is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$.*

Proof. This follows from Proposition 3.14 since the family of closed sets on which f is continuous is hereditary and $\mathbf{\Pi}^1_1$. □

Remark 3.37. Using Proposition 3.33 we can explicitly write the formula defining the set of closed sets in $\mathcal{I}^f$. Let $\tilde{K}\subseteq\omega^\omega\times X$ be the universal closed set. Notice that $\tilde{K}_x\notin\mathcal{I}^f$ if and only if

$$\exists U\subseteq 2^\omega\text{ open}\quad \exists c:2^\omega\to\tilde{K}_x\text{ topological embedding}$$
$$\big(c[2^\omega]\cap f^{-1}[U]\text{ is dense in }c[2^\omega]\big)\wedge\big(c[2^\omega]\setminus f^{-1}[U]\text{ is dense in }c[2^\omega]\big).$$

Indeed, the left-to-right implication follows from Proposition 3.33 (when $G=X$). The right-to-left implication holds because the set $f^{-1}[U]$ is an $\mathbf{F}_\sigma$ set which is dense and meager on $c[2^\omega]$, therefore it cannot be a relative $\mathbf{G}_\delta$ set on $c[2^\omega]$. Hence, by the Jayne-Rogers theorem we have that f is not piecewise continuous on $c[2^\omega]$ and $\tilde{K}_x\notin\mathcal{I}^f$.

Now, the above formula is $\mathbf{\Sigma}^1_1$. Indeed, it is routine to write a $\mathbf{\Sigma}^1_1$ formula saying that $c : 2^\omega \to K_x$ is a topological embedding. The first clause of the conjunction can be written as

$$\forall \tau \in 2^{<\omega}\ \exists x \in [\tau] \quad f(c(x)) \in U,$$

which is $\mathbf{\Sigma}^1_1$, and analogously we can rewrite the second clause.

If $G \subseteq X$ is a $\mathbf{G}_\delta$ set and $b : \omega^\omega \to G$ is a homeomorphism, then we call (b, G) *a copy of the Baire space* and denote it by $b : \omega^\omega \hookrightarrow G \subseteq X$.

Proposition 3.38. *For any $B \in \mathbb{P}_{\mathcal{I}^f}$ there is an $\mathcal{I}^f$-positive $\mathbf{G}_\delta$ set $G \subseteq B$ and a copy of the Baire space $b : \omega^\omega \hookrightarrow G \subseteq X$ such that*

$$\mathcal{I}^f \restriction G = \{b[A] : A \subseteq \omega^\omega, A \in \mathbf{K}_\sigma\}.$$

Proof. By Theorem 3.2 and the continuous reading of names we may assume that B is of type $\mathbf{G}_\delta$ and f is continuous on B. Applying Proposition 3.33 we get a copy of the Cantor space $c : 2^\omega \hookrightarrow C \subseteq X$ and an open set $U \subseteq \omega^\omega$ such that $f^{-1}[U] \cap C = c[\mathbb{Q}]$ and $C \setminus c[\mathbb{Q}] \subseteq B$. Let $G = C \setminus c[\mathbb{Q}]$. Via the natural homeomorphism $\iota : \omega^\omega \to 2^\omega \setminus \mathbb{Q}$ we get a copy of the Baire space $b : \omega^\omega \hookrightarrow G \subseteq X$, for $b = c\restriction(2^\omega \setminus \mathbb{Q}) \circ \iota$. Note that $C \notin \mathcal{I}^f$ (by Theorem 3.33, since $f^{-1}[U] \cap C$ is not $\mathbf{G}_\delta$ in C) and hence also $G \notin \mathcal{I}^f$.

The σ-ideal $\mathcal{I}^f \restriction G$ is generated by the sets $D \cap G$ for $D \subseteq C$ closed such that $f \restriction D$ is continuous. The σ-ideal $\{b[A] : A \subseteq \omega^\omega, A \in \mathbf{K}_\sigma\}$ is generated by compact subsets of G. We need to prove that these two families generate the same σ-ideals on G.

If $D \subseteq G$ is compact, then D is closed in C and f is continuous on D because $f \restriction G$ is continuous. Hence $D = D \cap G \in \mathcal{I}^f \restriction G$.

If $D \subseteq C$ such that $f \restriction D$ is continuous, then $D \cap G = (f \restriction D)^{-1}[\omega^\omega \setminus U]$ is closed in D subset of G, so compact.

This ends the proof. □

As an immediate consequence of Proposition 3.38 we get the following corollary.

Corollary 3.39. *The forcing $\mathbb{P}_{\mathcal{I}^f}$ is equivalent to the Miller forcing.*

Recall a theorem of Kechris, Louveau and Woodin [**16**, Theorem 7], which says that any coanalytic σ-ideal of compact sets in a Polish space is either a $\mathbf{G}_\delta$ set, or else is $\mathbf{\Pi}^1_1$-complete. If X is compact, then $\mathcal{I}^f \cap K(X)$ is a coanalytic σ-ideal of compact sets by Proposition 3.36, so it satisfies the assumption of the theorem of Kechris, Louveau and Woodin. Below we prove that it $\mathbf{\Pi}^1_1$-complete.

Proposition 3.40. *$\mathcal{I}^f \cap F(X)$ is a $\mathbf{\Pi}^1_1$-complete set in $F(X)$.*

Proof. As in Proposition 3.38 take $c : 2^\omega \hookrightarrow C \subseteq X$ a copy of the Cantor space and $b : \omega^\omega \hookrightarrow G \subseteq X$ a copy of the Baire space, $G \subseteq C$ a dense $\mathbf{G}_\delta$ set in C. Recall that the Borel structure on $F(C)$ is induced from the topology of the hyperspace.

It is known (see [**15**, Exercise 27.9]) that the set $\mathbf{K}_\sigma \cap F(\omega^\omega)$ is a $\mathbf{\Pi}^1_1$-complete set in $F(\omega^\omega)$. In other words, this means that if $F \subseteq \omega^\omega \times G$ is the standard universal closed set, then $\{x \in \omega^\omega : F_x \in \mathbf{K}_\sigma\}$ is a $\mathbf{\Pi}^1_1$-complete set. Now let $\varphi : \omega^\omega \to K(C)$ be the the function $\omega^\omega \ni x \mapsto \overline{F_x} \in K(C)$, where $\overline{A}$ denotes the closure of A in C (for $A \subseteq G$). It is routine to check that φ is continuous. By Proposition 3.38 we have $\varphi^{-1}[\mathcal{I}^f] = \mathbf{K}_\sigma$. This proves that $\mathcal{I}^f$ is $\mathbf{\Pi}^1_1$-complete. □

Piecewise continuity of functions from ω^ω to ω^ω has already been investigated from the game-theoretic point of view. In [**33**] Van Wesep introduced the Backtrack Game $G_{\mathrm{B}}(g)$ for arbitrary function $g : \omega^\omega \to \omega^\omega$. Andretta [**2**, Theorem 21] characterized piecewise continuity of a function g in terms of existence of a winning strategy for one of the players in the game $G_{\mathrm{B}}(g)$. The assumption that the domain of g is ω^ω is made only to simplify notation. For the same reason, we also consider only functions $g : \omega^\omega \to \omega^\omega$.

If g is not piecewise continuous, then the Backtrack Game can be used to represent the σ-ideal $\mathcal{I}^g$. We shall introduce another Banach-Mazur game scheme, which seems maybe more natural.

We need to have a representation of partial continuous functions $h : \omega^\omega \rightharpoonup \omega^\omega$ with closed domains. Notice that for each such function there is a partial function $H : \omega^{<\omega} \rightharpoonup \omega^{<\omega}$ such that

- for $\sigma, \tau \in \mathrm{dom}(H)$ if $\sigma \subseteq \tau$, then $H(\sigma) \subseteq H(\tau)$,
- for each $x \in \omega^\omega$ we have $x \in \mathrm{dom}(h)$ if and only if the set $\{n : x{\restriction}n \in \mathrm{dom}(H)\}$ is infinite,
- for $\sigma \in \omega^{<\omega}$ we have $h(x) \supseteq \sigma$ if and only if there is $n < \omega$ such that $x{\restriction}n \in \mathrm{dom}(H)$ and $H(x{\restriction}n) = \sigma$.

If $T \subseteq \omega^{<\omega}$ is a subtree and $H : T \rightharpoonup \omega^{<\omega}$ is a partial function such that $\mathrm{dom}(H)$ is contained in the set of splitting nodes of T and H satisfies the following condition:

- for $\sigma, \tau \in \mathrm{dom}(H)$ if $\sigma \subseteq \tau$, then $H(\sigma) \subseteq H(\tau)$,

then we call H a *labelling*. If $H : \rightharpoonup \omega^{<\omega}$ is a labelling and $\tau \in \mathrm{dom}(H)$, then we write $H[\tau]$ for H restricted to $T(\tau)$. We write $D(H) = \{x \in \omega^\omega : \exists^\infty n : x{\restriction}n \in \mathrm{dom}(H)\}$ notice that $D(H)$ is a closed set and H defines a continuous function $h : D(H) \to \omega^\omega$.

If T is a tree and $H : T \to \omega^{<\omega}$ is a finite partial function, then we refer to H as to a *finite labelling*. If $H_0 : T_0 \to \omega^{<\omega}$ and $H_1 : T_1 \to \omega^{<\omega}$ are two labellings then we say that they are *consistent* if $H_0 \cup H_1 : T_0 \cup T_1 \to 2^{<\omega}$ is also a labelling. We say that a finite labelling H_0 *end-extends* a finite labelling H_1 if $H_0 \supseteq H_1$ and $\mathrm{dom}(H_0)$ is a end-extension (in the sense of $\subseteq$ on $\omega^{<\omega}$) of $\mathrm{dom}(H_1)$.

Now, we define the game scheme G_{pc} as follows. In his n-th move, Adam picks $\xi_n \in \omega^n$ such that $\xi_n \supseteq \xi_{n-1}$ ($\xi_{-1} = \varnothing$). The moves of Eve are more complicated. Her intention is to define (by the end of the game) a sequence $\langle h_i : i \in \omega\rangle$ of partial continuous functions with closed domains. In her n-th turn, Eve constructs a sequence of finite partial labellings $H_i^n : \omega^{<\omega} \to \omega^{<\omega}$ (for $i \in \omega$) such that $\forall^\infty i\ H_i^n = \varnothing$ and H_i^n end-extends H_i^{n-1} ($H_i^{-1} = \varnothing$ for each $i < \omega$). The meaning of Eve's moves is the following. In each play in G_{pc}, for each $i < \omega$ we have that $H_i = \bigcup_{n<\omega} H_i^n$ is a labelling which defines a partial continuous function h_i with closed domain.

Let $g : \omega^\omega \to \omega^\omega$ be a not piecewise continuous function and $B \subseteq \omega^\omega$. The game $G_{\mathrm{pc}}^g(B)$ is a game in the game scheme G_{pc} with the following payoff set. Eve wins a play p in $G_{\mathrm{pc}}^g(B)$ if for $x = \bigcup_{n<\omega} \xi_n$ (ξ_n is the n-th move of Adam in p)

$$x \notin B \quad \vee \quad \exists i \in \omega\ (x \in \mathrm{dom}(h_i) \wedge g(x) = h_i(x))$$

(where the functions h_i are computed from Eve's moves as above). Otherwise Adam wins p.

PROPOSITION 3.41. *For any set $A \subseteq \omega^\omega$, Eve has a winning strategy in the game $G_{\mathrm{pc}}^g(A)$ if and only if $A \in \mathcal{I}^g$.*

PROOF. If $A \in \mathcal{I}^g$ then there are closed sets $C_n \subseteq \omega^\omega$ such that $A \subseteq \bigcup_n C_n$ and $g\restriction C_n$ is continuous. Each function $g\restriction C_n$ has its labelling G_n and Eve's strategy is simply to rewrite the G_n's.

On the other hand, suppose that there is a winning strategy for Eve and let T be the tree of this strategy. The nodes of T are determined by Adam's moves so T is in fact isomorphic to $\omega^{<\omega}$. For each $n < \omega$ define a continuous partial function g_n with closed domain in the following way. For $\tau \in T$ let H_n^τ be the labelling H_n defined by Eve in her last move of the partial play τ. Notice that if $\tau \subseteq \sigma$ are in T then $H_n^\sigma[\sigma]$ is consistent with $H_n^\tau[\tau]$. We put $G_n = \bigcup_{\tau\in T} H_n^\tau[\tau]$ and let g_n be the partial continuous function with closed domain determined by the labelling G_n. It follows from that the winning condition for Eve that $g\restriction A \subseteq \bigcup_n g_n$, so $A \in \mathcal{I}^g$. □

COROLLARY 3.42. *If $B \subseteq \omega^\omega$ is Borel and $g : \omega^\omega \to \omega^\omega$ is a Borel, not piecewise continuous function, then $B \in \mathcal{I}^g$ if and only if Eve has a winning strategy in $G_{\mathrm{pc}}^g(B)$.*

3.8 Closed null sets

We denote by $\mathcal{E}$ the σ-ideal generated by closed null sets in 2^ω (with respect to the standard Haar measure μ on 2^ω). The sets in $\mathcal{E}$ are both null and meager. $\mathcal{E}$ is properly contained in $\mathcal{M} \cap \mathcal{N}$ [**3**, Lemma 2.6.1] and in fact, one can show that $\mathcal{E}$ is not ccc.

The family of closed sets in $\mathcal{E}$ coincides with the family of closed null sets and $\mathcal{E} \cap \mathbf{\Pi}_1^0(2^\omega)$ is a $\mathbf{G}_\delta$ set in $K(2^\omega)$ (for each $\varepsilon > 0$ the set $\{C \in K(2^\omega) : \mu(C) < \varepsilon\}$ is open). Therefore, $\mathcal{E}$ is $\mathbf{\Pi}_1^1$ on $\mathbf{\Sigma}_1^1$ by Corollary 3.14.

The forcing $\mathbb{P}_\mathcal{E}$ adds an unbounded real and a splitting real. In fact, one can check that the generic real is splitting. To see that $\mathbb{P}_\mathcal{E}$ adds an unbounded real, recall a theorem of Zapletal [**35**, Theorem 3.3.2], which says that a forcing $\mathbb{P}_\mathcal{I}$ is ω^ω-bounding if and only $\mathbb{P}_\mathcal{I}$ has continuous reading of names and compact sets are dense in $\mathbb{P}_\mathcal{I}$. Let G be a $\mathbf{G}_\delta$ set such that $G \in \mathcal{N}$ and $2^\omega \setminus G \in \mathcal{M}$. G is $\mathcal{E}$-positive but no compact $\mathcal{E}$-positive set is contained in G. In particular compact sets are not dense in $\mathbb{P}_\mathcal{E}$ and hence this forcing is not ω^ω-bounding. $\mathbb{P}_\mathcal{E}$ does not, however, add a dominating real. This follows from another theorem of Zapletal [**35**, Theorem 3.8.15], which says that if $\mathcal{I}$ is a $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ σ-ideal, then the forcing $\mathbb{P}_\mathcal{I}$ does not add a dominating real.

Zapletal proved in [**35**, Theorem 4.1.7] (see the first paragraph of the proof) that if $\mathcal{I}$ is a σ-ideal generated by an analytic collection of closed sets and $V \subseteq V[G]$ is a $\mathbb{P}_\mathcal{I}$-extension, then any intermediate extension $V \subseteq W \subseteq V[G]$ is equal either to V or $V[G]$, or is an extension by a single Cohen real. Therefore it is natural to ask if $\mathbb{P}_\mathcal{E}$ adds Cohen reals.

Recall that a closed set $D \subseteq 2^\omega$ is *self-supporting* if for any clopen set U the set $D \cap U$ is either empty or not null. Notice that a closed set is self-supporting if and only if it is $\mathcal{E}$-perfect.

Proposition 3.43. *The forcing $\mathbb{P}_\mathcal{E}$ does not add Cohen reals.*

Proof. Suppose $B \in \mathbb{P}_\mathcal{E}$ and $\dot{x}$ is a name for a real such that $B \Vdash \dot{x}$ is a Cohen real. By Lemma 3.11 and continuous reading of names we find a $\mathbf{G}_\delta$ set $G \subseteq B$ such that $D = \overline{G}$ is self-supporting and a continuous function $f : G \to \omega^\omega$ such that $G \Vdash \dot{x} = f(\dot{g})$. Pick a continuous, strictly positive measure ν on G. For each $\tau \in \omega^{<\omega}$ the set $C_\tau = f^{-1}\big[[\tau]\big]$ is a relative clopen in G. Find open sets $C'_\tau \subseteq D$ such that $C_\tau = C'_\tau \cap G$. D is zero-dimensional, so by the reduction property for open sets we may assume that

- $C'_{\tau_0} \subseteq C'_{\tau_1}$ for $\tau_0 \subseteq \tau_1$,
- $C'_{\tau_0} \cap C'_{\tau_1} = \varnothing$ for $\tau_0 \perp \tau_1$.

We will find a tree $T \subseteq \omega^{<\omega}$ such that $\lim T$ is nowhere dense in ω^ω and the closure of the set $f^{-1}[\lim T]$ is self-supporting.

Enumerate all nonempty clopen sets in D in a sequence $\langle V'_n : n < \omega\rangle$ and all nonempty clopen sets in G in a sequence $\langle V_n : n < \omega\rangle$, and elements of $\omega^{<\omega}$ in a sequence $\langle \sigma_n : n < \omega\rangle$. If $\tau \in \omega^{<\omega}$, then $\langle C'_{\tau^\frown n} : n < \omega\rangle$ is a sequence of disjoint open sets in D and $\langle C_{\tau^\frown n} : n < \omega\rangle$ is a sequence of disjoint open sets in G. Thus for each $\varepsilon > 0$ there is $n \in \omega$ such that $\mu(C'_{\tau^\frown n}) < \varepsilon$ as well as $\nu(C_{\tau^\frown n}) < \varepsilon$. Moreover, for each $m \in \omega$ there is $n \in \omega$ such that $\mu_{V'_m}(V'_m \cap C'_{\tau^\frown n}) < \varepsilon$ and $\nu_{V_m}(V_m \cap C_{\tau^\frown n}) < \varepsilon$.

By induction, we find a collection of nodes $\tau_n \in \omega^{<\omega}$ such that the tree

$$T = \{\tau \in \omega^{<\omega} : \forall n\ \tau_n \notin \tau\}$$

is such that $\lim T$ is nowhere dense, and for each $m < \omega$ we have

$$\text{either} \quad V'_m \subseteq \bigcup_{n<\omega} C'_{\tau_n} \quad \text{or} \quad \mu(V_m \setminus \bigcup_{n<\omega} C'_{\tau_n}) > 0$$

and

$$\text{either} \quad V_m \subseteq \bigcup_{n<\omega} C_{\tau_n} \quad \text{or} \quad \nu(V_m \setminus \bigcup_{n<\omega} C_{\tau_n}) > 0.$$

Along the induction we also construct sequences of reals $\varepsilon_n \geq 0$ and $\delta_n \geq 0$.

At the n-th step of the induction consider the sets $U'_n = V'_n \setminus \bigcup_{i<n} C'_{\tau_i}$ and $U_n = V_n \setminus \bigcup_{i<n} C_{\tau_i}$, which are either empty or of positive measure (μ or ν, respectively) by the inductive assumption. Put $\varepsilon_n = \mu(U'_n)$, $\delta_n = \mu(U_n)$. Find $\tau_n \in \omega^{<\omega}$ such that $\tau_n = \sigma_n{}^\frown k$ for some $k < \omega$ and for each $i \leq n$

- if $\varepsilon_i > 0$, then $\mu_{V'_i}(C'_{\tau_n} \cap V'_i) < 2^{-n-1}\varepsilon_i$,
- if $\delta_i > 0$, then $\nu_{V_i}(C_{\tau_n} \cap V_i) < 2^{-n-1}\delta_i$.

The set

$$A = G \setminus \bigcup_n C_{\tau_n}$$

is of type $\mathbf{G}_\delta$. Moreover, it follows from the construction that $\overline{A} = D \setminus \bigcup_n C'_{\tau_n}$ and that $\overline{A}$ is self-supported, so $A \notin \mathcal{E}$ by Lemma 3.10. On the other hand, $A \Vdash \dot{x} \in \lim T$, which gives a contradiction, since $\lim T$ is nowhere dense. □

Corollary 3.44. *If G is $\mathbb{P}_{\mathcal{E}}$-generic over V, then the extension $V \subseteq V[G]$ is minimal.*

There is a natural game representation of the σ-ideal $\mathcal{E}$. Denote by $G_{\mathcal{E}}$ the following game scheme. In his n-th turn, Adam picks $\xi_n \in 2^n$ such that $\xi_n \supsetneq \xi_{n-1}$ ($\xi_{-1} = \varnothing$). In her n-th turn, Eve picks a basic clopen set $C_n \subseteq [\xi_n]$ such that

$$\mu_{[\xi_n]}(C_n) < \frac{1}{n}.$$

For a set $A \subseteq 2^\omega$ we define the game $G_{\mathcal{E}}(A)$ in the game scheme $G_{\mathcal{E}}$ as follows. Eve wins a play in $G_{\mathcal{E}}(A)$ if

$$x \notin A \quad \vee \quad \forall^\infty n \; x \in C_n$$

(where $x \in 2^\omega$ is the union of the ξ_n's picked by Adam). Otherwise Adam wins.

Proposition 3.45. *For any set $A \subseteq 2^\omega$, Eve has a winning strategy in $G_{\mathcal{E}}(A)$ if and only if $A \in \mathcal{E}$.*

Proof. Suppose first that Eve has a winning strategy S in $G_{\mathcal{E}}(A)$. For each $\sigma \in 2^{<\omega}$ consider a partial play τ_σ in which Adam picks successively $\sigma \restriction k$ for $k \leq |\sigma|$. Let C_σ be the Eve's next move, according to S, after τ_σ. Put $E_n = \bigcup_{\sigma \in 2^n} C_\sigma$. Clearly E_n is a clopen set and $\mu(E_n) \leq 1/n$. Let $D_n = \bigcap_{m \geq n} E_n$. Now, each D_n is a closed null set and $A \subseteq \bigcup_n D_n$ since S is a winning strategy. Therefore $A \in \mathcal{E}$.

Conversely, assume that $A \in \mathcal{E}$. There are closed null sets D_n such that $A \subseteq \bigcup_n D_n$. Without loss of generality assume $D_n \subseteq D_{n+1}$. Let $T_n \subseteq \omega^{<\omega}$ be a tree such that $D_n = \lim T_n$. We define a strategy S for Eve as follows. Suppose Adam has picked $\sigma \in 2^n$ in his n-th move and consider the tree $T_n(\sigma)$. Since $\lim T_n(\sigma)$ is of measure zero, there is $k < \omega$ such that

$$\frac{|T_n(\sigma) \cap 2^k|}{2^k} < \frac{1}{n}.$$

Let Eve's answer be the set $\bigcup_{\tau \in T_n(\sigma) \cap 2^k} [\tau]$. One can readily check that this defines a winning strategy for Eve in $G_{\mathcal{E}}(A)$. □

COROLLARY 3.46. *If $B \subseteq 2^\omega$ is Borel, then $B \in \mathcal{E}$ if and only if Eve has a winning strategy in $G_{\mathcal{E}}(B)$.*

3.9 Closed sets in ccc forcings

In this section we study ccc σ-ideals on a Polish space X such that for each $\mathcal{I}$-positive Borel set $B \subseteq X$ there is a closed set $C \subseteq X$ such that $C \setminus B \in \mathcal{I}$. Expressed in the forcing language, this says that $\{[C]_{\mathcal{I}} : C \subseteq X \text{ closed}\}$ is dense in $\mathbf{Bor}(X)/\mathcal{I}$.

Recall if (X, μ) is a measure Polish space, then every $\mathcal{N}(X, \mu)$-positive Borel set contains a $\mathcal{N}(X, \mu)$-positive compact set. Also, for every $\mathcal{M}(X)$-positive Borel B set there is a nonmeager closed set $C \subseteq X$ such that $C \setminus B \in \mathcal{M}(X)$. Zapletal [**35**, Question 7.2.7] asked if this can be generalized to all ccc σ-ideals.

QUESTION ([**35**, Question 7.2.7]). Suppose that $\mathcal{I}$ is a ccc σ-ideal. Is it true that every $\mathcal{I}$-positive Borel set contains an $\mathcal{I}$-positive closed set modulo the ideal $\mathcal{I}$?

We answer this question negatively. Take $\mathcal{J} = \mathcal{M} \cap \mathcal{N}$ on 2^ω. It is clear that $\mathcal{J}$ is ccc, so we only need to prove the following proposition.

PROPOSITION 3.47. *There exists an $\mathcal{J}$-positive Borel set $A \subseteq 2^\omega$ which does not contain modulo $\mathcal{J}$ any $\mathcal{J}$-positive closed set.*

PROOF. Let A be any Borel set such that $A \in \mathcal{N} \setminus \mathcal{M}$. Such a set can be obtained by decomposing 2^ω into two sets $A \in \mathcal{N}$ and $B \in \mathcal{M}$. In particular A is $\mathcal{J}$-positive. Take any closed set C such that $C \setminus A \in \mathcal{J}$. Note that this implies that $C \in \mathcal{N}$. But also $C \in \mathcal{M}$ because any set of measure zero must have empty interior. This shows that $C \in \mathcal{J}$. □

4

Continuous reading of names

4.1 Introduction

In examining forcing effects on the real line we often need a representation of names for reals in the generic extension. In most of known examples such a representation given by Borel reading of names. Zapletal proved [**34**, Proposition 3.2.1] that Borel reading of names holds if the forcing $\mathbb{P}_{\mathcal{I}}$ is proper. Theorem 3.1 says that if a σ-ideal $\mathcal{I}$ is generated by closed sets on a Polish space X, then the forcing $\mathbb{P}_{\mathcal{I}}$ has continuous reading of names in the topology of X.

There is a "canonical" example of a an idealized forcing which is proper but fails to have continuous reading of names in the natural topology of the space. This is the *Steprāns forcing*, defined below. In this chapter we study properties of the Steprāns forcing and find other examples of forcing notions without continuous reading of names.

The Pawlikowski function $P : (\omega+1)^\omega \to \omega^\omega$ is a Baire class 1 function, which is not σ-continuous (for definitions see Section 4.2). We consider the following σ-ideal: $\mathcal{I}_P = \{A \subseteq (\omega+1)^\omega : P \restriction A \text{ is } \sigma\text{-continuous}\}$. The *Steprāns forcing* is the forcing $\mathbb{P}_{\mathcal{I}_P}$. We denote it by $\mathbb{S}$. In this form, $\mathbb{S}$ was first introduced by Zapletal in [**34**, Section 2.3.13]. Steprāns [**32**] considers the σ-ideal $\mathcal{I}_P$ and a different forcing notion, which is used to increase the cardinal characteristic $\mathrm{cov}(\mathcal{I}_P)$ in the generic extension.

In [**34**, Lemma 2.3.45] Zapletal showed that the Steprāns forcing is proper. It does not have continuous reading of names because the function P is a Borel function which is not continuous on any $\mathcal{I}_P$-positive Borel set.

Notice that a specific topology giving a Borel structure is irrelevant for the forcing $\mathbb{P}_{\mathcal{I}}$ itself, but continuous reading of names depends (at least formally) on the topology. It is natural to try to change the presentation of the forcing in order to get continuous reading of names in another topology.

In [**11**, Question 5.5] Hrušak and Zapletal ask whether for any proper idealized forcing $\mathbb{P}_{\mathcal{I}}$ there is a presentation in which $\mathbb{P}_{\mathcal{I}}$ has continuous reading of names. In case of the Steprāns forcing a natural candidate for this presentation is the least topology on $(\omega+1)^\omega$ in which P is continuous. We call this topology *the Baire topology on* $(\omega+1)^\omega$. In [**11**, Example 2.7] the authors show that the Steprāns forcing has continuous reading of names in the Baire topology on $(\omega+1)^\omega$. To this end they prove that the σ-ideal $\mathcal{I}_P$ is generated by closed sets in the Baire topology on $(\omega+1)^\omega$ and apply Theorem 3.1.

In [**32**, §4, p. 1273] we read that the author does not know whether the σ-ideal $\mathcal{I}_P$ is generated by closed sets in the Baire topology on $(\omega+1)^\omega$. Hrušak and Zapletal [**11**, Example 2.7] give a positive answer to that question but there is a mistake in their proof. Therefore the proof of continuous reading of names for the Steprāns forcing in the extended topology is also incorrect.

In Section 4.3 we introduce the notion of a wide tree and in Proposition 4.4 we prove that the Steprāns forcing has a dense subset isomorphic to the forcing with wide trees.

In Section 4.4 we use wide trees to show some properties of the Steprāns forcing. In Proposition 4.9 we prove that the Steprāns forcing has continuous reading of names in the Baire topology on $(\omega+1)^\omega$. In Proposition 4.10 we show that the Steprāns forcing satisfies Axiom A.

In Section 4.5 we investigate the σ-ideal $\mathcal{I}_P$. In Proposition 4.11 we answer a question of Steprāns [**32**, §4, p. 1273] and show that $\mathcal{I}_P$ is not generated by closed sets in the Baire topology on $(\omega+1)^\omega$. This disproves a claim of Hrušak and Zapletal [**11**, Example 2.7].

In Section 4.6 we produce other examples of idealized forcings without continuous reading of names. In Proposition 4.14 we answer a question of Hrušak and Zapletal [**11**, Question 5.5], whether there are idealized forcings which do not have continuous reading of names in any presentation. We construct a family of such forcing notions.

4.2 Notation

We say that a Borel function $f : X \to Y$ (X, Y are Polish spaces) is *σ-continuous* if there exist a countable cover of the space $X = \bigcup_n X_n$ (with arbitrary sets X_n) such that $f{\restriction}X_n$ is continuous for each n. If $f : X \to Y$ is not σ-continuous, then we denote by $\mathcal{I}_f$ the σ-ideal of sets on which f is σ-continuous.

Remark 4.1. The sets X_n above can be chosen Borel. Indeed, by the Kuratowski extension theorem each $f{\restriction}X_n$ can be extended to a continuous function g_n defined on a $\mathbf{G}_\delta$ set $G_n \supseteq X_n$. Note that the sets $Y_n = \{x \in G_n : f(x) = g_n(x)\}$ are Borel and $Y_n \supseteq X_n$ for each $n < \omega$. So the sets Y_n form a cover of the space X by Borel sets on each of which f is continuous.

The space $(\omega+1)^\omega$ is endowed with the product topology of order topologies on $\omega+1$. It is homeomorphic to the Cantor space. We fix a metric ρ on $(\omega+1)^\omega$ which gives the above topology. For $x,y \in (\omega+1)^\omega$ let $\rho(x,y) = \sum_n \frac{1}{2^n}\rho'(x(n),y(n))$ where ρ' metrizes $\omega+1$ with its order-topology, i.e. $\rho'(n,\omega) = \frac{1}{2^n}$ and $\rho'(n,m) = |\frac{1}{2^n} - \frac{1}{2^m}|$ for $n,m<\omega$. All metric notions on $(\omega+1)^\omega$, like diameter, distance, etc., will be relative to the metric ρ.

The *Pawlikowski function* $P:(\omega+1)^\omega \to \omega^\omega$ is defined as follows:

$$P(x)(n) = \begin{cases} x(n)+1 & \text{if} \quad x(n)<\omega, \\ 0 & \text{if} \quad x(n)=\omega. \end{cases}$$

Note that the least topology on $(\omega+1)^\omega$ in which P is continuous is the one with basic clopen sets of the form $[\sigma]$ for $\sigma \in (\omega+1)^{<\omega}$. With this topology $(\omega+1)^\omega$ is homeomorphic to the Baire space ω^ω. We will refer to the two topologies: the original and the extended one as to the *Cantor topology on* $(\omega+1)^\omega$ and the *Baire topology on* $(\omega+1)^\omega$, respectively.

4.3 Representation of the Steprāns forcing

Note that for every closed set C in the Cantor topology of $(\omega+1)^\omega$ there is a tree $T \subseteq (\omega+1)^{<\omega}$ such that $C = \lim T$. It is not true, however, that the limit of any tree in $(\omega+1)^{<\omega}$ is a closed set in the Cantor topology. In general, if $T \subseteq (\omega+1)^{<\omega}$ is a tree, then $\lim T$ is a $\mathbf{G}_\delta$ set in the Cantor topology (since its complement is the union of sets $[\tau]$ for $\tau \notin T$).

The following theorem follows from a result of Solecki [**31**, Theorem 4.1].

THEOREM 4.2 (Solecki, [**31**, Theorem 4.1]). *Any $\mathcal{I}_P$-positive Borel set in $(\omega+1)^\omega$ contains an $\mathcal{I}_P$-positive compact set in the Cantor topology of $(\omega+1)^\omega$.*

In the forcing language this means that compact sets (in the Cantor topology of $(\omega+1)^\omega$) are dense in the Steprāns forcing.

Our aim now is to give a tree representation of the Steprāns forcing. First, however, we show that the following claim (which occured in a preliminary version of [**35**]) is false: a Borel set A is in $\mathbb{P}_{\mathcal{I}_P}$ iff it contains limit of a tree $T \subseteq (\omega+1)^{<\omega}$ such that every $\tau \in S$ has an extension $\tau' \in S$ which splits into infinitely many immediate successors including $\tau'^\frown\omega$.

EXAMPLE. We construct a tree S with the property that every $\tau \in S$ has an extension $\tau' \in S$ which splits into infinitely many immediate successors including $\tau'^\frown\omega$, yet P is σ-continuous on $\lim S$. We build the tree inductively on its levels. For any node $\tau \in S$ we also define a set $A_\tau \subseteq \omega$. Begin with $\varnothing$ and put $A_\varnothing = \omega$. Suppose that we have the tree S built up to level k. Now let each node τ split into $\tau^\frown\omega$ as well as $\tau^\frown n$ for $n \in A_\tau$. Define sets $A_{\tau^\frown i}$ for $i \in A_\tau \cup \{\omega\}$ so that they form a partition of A_τ into infinitely many infinite subsets. Note that if $s \in \lim S$ and $s(n)<\omega$, then $s{\restriction}n$ is uniquely determined.

Claim. *The function P is σ-continuous on* $\lim S$.

Proof. Define $X_n = \{s \in \lim S : \forall m \geq n \quad s(m) = \omega\}$ and $X_\infty = \lim S \setminus \bigcup_n X_n$. Note that P is continuous on X_∞. Indeed, take any convergent sequence $s_n \to s$ such that $s_n, s \in X_\infty$ and notice that if $s(m) < \omega$, then $s_n(m) = s(m)$ implies also $s_n \restriction m = s \restriction m$. Since $[(m, s(m))]$ is a neighborhood of s, there exists $m' < \omega$ such that $s_n \restriction m = s \restriction m$ for $n > m'$. Thus, if s has infinitely many values $< \omega$, then the sequence s_n eventually stabilizes on each coordinate. This shows that also $P(s_n) \to P(s)$. Since all sets X_n are countable, P is σ-continuous on $\lim S$. □

Throughout the rest of this section all topological notions concerning the space $(\omega+1)^\omega$ will be relative to the Cantor topology.

Definition 4.3. Call a tree $T \subseteq (\omega+1)^{<\omega}$ *wide* if every node $\tau \in T$ has an extension $\tau' \in T$ such that the set $[\tau']_T$ is nowhere dense in $[\tau]_T$. Say that a subset of $(\omega+1)^\omega$ is *wide* if it is the limit of a wide tree.

Note that the node τ' above must be of the form $\tau''^\frown\omega$ for some $\tau'' \supseteq \tau$ which splits in T into infinitely many immediate successors. We denote by $\mathbb{W}$ the poset of wide trees ordered by inclusion.

Theorem 4.4. *The Steprāns forcing is equivalent to the forcing* $\mathbb{W}$.

Proof. The theorem follows from Proposition 4.5 and Proposition 4.8 given below. □

Proposition 4.5. *Assume $B \subseteq (\omega+1)^\omega$ is Borel such that $P \restriction B$ is not σ-continuous. Then there exists a wide tree T such that* $\lim T = D \subseteq B$.

Proof. We begin with a claim and a definition.

Claim. *Let $E \subseteq (\omega+1)^\omega$ be closed such that $P \restriction E$ is not continuous. There exists a sequence of disjoint relative clopen sets $C_n, n < \omega$ (each of the form $[\tau_n] \cap E$) such that $\bigcup_n C_n$ is dense in E.*

Proof. Note that any relative open set in E contains a relative clopen set of the form $[\sigma] \cap E$, where $\sigma \in (\omega+1)^{<\omega}$. This follows from the fact that any basic open set (in E) of the form $[\tau_1^\frown[n,\omega]^\frown\tau_2] \cap E$ either has a nonempty (in E) clopen subset of the form $[\tau_1^\frown m^\frown\tau_2] \cap E$ for some $m \geq n$ or is equal to the relative clopen set $[\tau_1^\frown\omega^\frown\tau_2] \cap E$.

Thus, if we take a maximal antichain of sets of the form $[\tau] \cap E$, then its union is dense. We will be done if we take an infinite maximal antichain.

Note that E is the limit of a tree which is not finitely-branching (otherwise P would be continuous on E). We first pick an infinite antichain given by infinitely many immediate successors (by numbers less than ω) of a node in the tree. Then we take any extension of this antichain to a maximal antichain. □

Definition 4.6. A *fusion system* in $(\omega+1)^\omega$ is a wide tree $T \subseteq (\omega+1)^{<\omega}$ together with a family of trees $T_\tau, \tau \in T$ such that

- each $\lim T_\tau$ is closed,
- T_τ has stem τ,
- for $\tau \subseteq \tau' \in T$ $T_{\tau'} \subseteq T_\tau$.

By Theorem 4.2 we may assume that $B = \lim T_\varnothing$ is closed. We may also assume that P is not σ-continuous on any basic clopen set. We will construct a fusion system $T \subseteq (\omega+1)^{<\omega}$, $T_\tau, \tau \in T$ and denote $D_\tau = \lim T_\tau$ and $D = \lim T$.

The construction is done inductively (beginning with $T_\varnothing$) in such a way that having constructed τ and T_τ we find infinitely many (pairwise incomparable) extensions of τ and appropriate family of subtrees of T_τ.

Suppose we have constructed a node τ and a tree T_τ. By the above Claim we can find an antichain $\langle \tau_n : n < \omega \rangle$ of extensions of τ such that $\{D_\tau \cap [\tau_n] : n < \omega\}$ is a maximal antichain of relative clopen sets in D_τ. Let T_{τ_n} be the end-extension of τ_n in T. Look now at the closed set $E = D_\tau \setminus \bigcup_n D_{\tau_n}$. If P is σ-continuous on this set, then we extend τ by τ_n's only and call this extension *regular*. If P is not σ-continuous on E, then we first shrink it to E' by cutting off all relative clopen sets on which P is σ-continuous and then we take any $\tau_\omega \in (\omega+1)^{<\omega}$ which gives a nonempty relative clopen set in E' (of length $> |\tau|$). Now extend τ additionally by τ_ω as well as define T_{τ_ω} as the tree of $E' \cap [\tau_\omega]$. The extension of this form is called *irregular* extension and we will refer to τ_ω as to the *irregular node*. The nodes τ_n are called *regular nodes*.

Once the tree has been constructed, note that each node τ has an irregular extension τ'. Indeed, for otherwise D_τ would be the union of the set $[\tau]_T$ and countably many sets on each of which P is σ-continuous. On the set $[\tau]_T$, however, P would be continuous, since for $\sigma \in \omega^{<\omega}$ the set $P^{-1}[[\sigma]] \cap [\tau]_T$ is open by the assumption that there are no irregular nodes extending τ. This would imply that P is σ-continuous on D_τ, a contradiction.

What is left is to show that if τ' is an irregular extension of τ, then the set $[\tau']_T$ is nowhere dense in $[\tau]_T$. Let S be the tree formed by all σ such that

- $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$,
- for each σ', if $\tau \subseteq \sigma' \subseteq \sigma$, then σ' is a regular node.

The set $\lim S$ is an intersection of a sequence of dense open sets in $[\tau]_T$ (namely the unions of levels of S). By the Baire category theorem $\lim S$ is a dense $\mathbf{G}_\delta$ set in $[\tau]_T$ (note that $[\tau]_T$ is a $\mathbf{G}_\delta$ set). Moreover, $\lim S$ is disjoint from $[\tau']_T$ and hence the latter set has empty interior in $[\tau]_T$. Since $[\tau']_T$ is closed in $[\tau]_T$, we have that $[\tau']_T$ is nowhere dense in $[\tau]_T$. This ends the proof. □

REMARK 4.7. The fusion method from the above proof will be further used to establish Axiom A and continuous reading of names. Notice, however, that Proposition 4.5 can be also proved without fusion, using the following method, similar to the Cantor-Bendixson analysis. Call a tree $T \subseteq (\omega+1)^{<\omega}$ *small* if for each $\tau \in T$ the set $[\tau^\frown\omega]_T$ is relatively open in $\lim T$. Notice that if T is small, then P is continuous on $\lim T$. Then use a procedure in the fashion of the Cantor-Bendixson analysis to cut off from T all nodes (and their extensions) such that the end-extension of them is small. The remaining is wide.

PROPOSITION 4.8. *Assume $D \subseteq (\omega+1)^\omega$ is a wide set. Then there are topological embeddings φ and ψ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\psi} & P[D] \\ \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle P\restriction D} \\ (\omega+1)^\omega & \xrightarrow{\varphi} & D \end{array}$$

A proof of Proposition 4.8 is given in Section 5.4.

4.4 Continuous reading of names

Recall that Steprāns forcing does not have continuous reading of names in the Cantor topology of $(\omega+1)^\omega$. We will now show that it has continous reading of names in the Baire topology on $(\omega+1)^\omega$.

PROPOSITION 4.9. *The forcing $\mathbb{S}$ has continuous reading of names in the Baire topology on $(\omega+1)^\omega$.*

Recall that all metric notions on $(\omega+1)^\omega$ (like diameter, distance, etc.) are relative to the metric ρ (see Section 4.2) on $(\omega+1)^\omega$.

PROOF. Let B be any Borel $\mathcal{I}_P$-positive set in $(\omega+1)^\omega$ and $\dot{x}$ be an $\mathbb{S}$-name for a real. By Proposition 4.5 we may assume B is a limit of a wide tree.

CLAIM. *Let T be a tree and $\sigma \in T$ be such that $[\sigma^\frown\omega]_T$ is nowhere dense in $[\sigma]_T$. Then for each $\tau \in T$ such that $\sigma^\frown\omega \subseteq \tau$, any $\varepsilon > 0$ and $n < \omega$ there is $m > n$ and $\sigma^\frown m \subseteq \tau' \in T$ such that*

- *$[\tau']_T$ is a relative clopen set,*
- $\mathrm{diam}([\tau']_T) < \varepsilon/2$,
- *dist*$([\tau]_T, [\tau']_T) < \varepsilon/2$.

PROOF. Consider the family of relative clopen sets of the form $[\tau']_T$ with τ' extending some $\sigma^\frown m, m > n$ and having diameters $< \varepsilon/2$. Put also $\delta = \inf_{i\le n} \mathrm{dist}([\tau]_T, [\sigma^\frown i])$. If the assertion of this lemma were false, then the open ball around $[\tau]_T$ with radius $\min\{\delta, \varepsilon/2\}$ would exhibit that $\sigma^\frown\omega$ has nonempty interior. □

We say that a set of nodes of a tree S is a spanning set if S is the smallest tree containing those nodes. We will find a wide subtree $S \subseteq T$ and a spanning set $\{\tau_\sigma : \sigma \in \omega^{<\omega}\}$ (with $\sigma \mapsto \tau_\sigma$ being order isomorphism) of nodes of S together with a set of natural numbers k_τ such that for τ in the spanning set

$$[\tau]_S \Vdash_{\mathbb{S}} \dot{x}(|\tau|) = k_\tau.$$

This will show that on $\lim S$ the name $\dot{x}$ is read by a function continuous in the Baire topology on $(\omega+1)^\omega$ relativized to $\lim S$.

The construction is by induction on $|\tau|$, using fusion method, that is we define additionally subtrees T_τ with stems τ. Suppose we have built the trees up to level n. We show how to extend a single node. First find an extension $\tau \subseteq \tau' \in T_\tau$ such that $[\tau'^\frown\omega]_{T_\tau}$ is nowhere dense in $[\tau]_{T_\tau}$. Without loss of generality assume that $\{n < \omega : \tau'^\frown n \in T_\tau\}$ is the whole ω. Find a wide tree $T_{\tau^\frown 0}$ such that $\lim T_{\tau^\frown 0} \subseteq [\tau'^\frown\omega]_{T_\tau}$ and for some $k_{\tau^\frown 0}$

$$\lim T_{\tau^\frown 0} \Vdash_{\mathbb{S}} \dot{x}(|\tau|+1) = k_{\tau^\frown 0}.$$

Next, using the Claim and bookkeeping find extensions $\tau'^\frown n \subseteq \tau'_n, n \in \omega$ so that for any $\sigma \in T_{\tau^\frown 0}$ and $n < \omega$ there is $m \in \omega$ such that $\mathrm{dist}([\tau'_m]_{T_\tau}, [\sigma]_{T_{\tau^\frown 0}}) < 1/n$ and $\mathrm{diam}([\tau'_m]_{T_\tau}) < 1/n$. Then find wide trees $T_{\tau^\frown m+1}$ such that $\lim T_{\tau^\frown m+1} \subseteq [\tau'_m]_{T_\tau}$ and for some natural numbers $k_{\tau^\frown m+1}$

$$\lim T_{\tau^\frown m+1} \Vdash_{\mathbb{S}} \dot{x}(|\tau|+1) = k_{\tau^\frown m+1}.$$

Notice that if

$$\mathrm{diam}([\tau'_m]_{T_\tau}),\ \mathrm{dist}([\tau'_m]_{T_\tau}, [\sigma]_{T_{\tau^\frown 0}}) < 1/2n$$

then

$$\mathrm{dist}(\lim T_{\tau^\frown m+1}, [\sigma]_{T_{\tau^\frown 0}}) < 1/n,$$

so the interior of $\lim T_{\tau^\frown 0}$ remains empty. Moreover, it will remain empty even when we pass to the fusion tree S. Thus after the fusion we get an $\mathcal{I}_P$-positive set and numbers k_τ which define a continuous function in the Baire topology on $(\omega+1)^\omega$. ☐

Using wide trees we can prove quite easily that the forcing $\mathbb{S}$ satisfies Axiom A.

PROPOSITION 4.10. *The Steprāns forcing satisfies Axiom A.*

PROOF. Let $\mathbb{W}'$ be the forcing with trees T satisfying the following conditions:

(1) each $\tau \in T$ either has only one immediate successor or is such that $\tau^\frown\omega \in T$ and $[\tau^\frown\omega]_T$ is nowhere dense in $\lim T$,
(2) whenever $\tau \in T$ is such that $\tau^\frown\omega \in T$ and $[\tau^\frown\omega]_T$ is nowhere dense in $\lim T$, we have the following. For each $n < \omega$ such that $\tau^\frown n \in T$ denote the stem of the tree T above $\tau^\frown n$ by τ_n. For each clopen set C intersecting $[\tau^\frown\omega]_T$ and any $\varepsilon > 0$ there is $n < \omega$ such that $\mathrm{diam}[\tau_n]_T < \varepsilon$ and $\mathrm{dist}([\tau_n]_T, C) < \varepsilon$.

Notice that $\mathbb{W}'$ is dense in $\mathbb{W}$. So it is enough to show that $\mathbb{W}'$ satisfies Axiom A.

Let $\prec$ be a linear order on $(\omega+1)^{<\omega}$ such that each τ occurs later than its initial segments. For a tree $T \in \mathbb{W}'$ let $w(T)$ be the set of those nodes of T which have more than one immediate successor. For $\tau \in w(T)$ the set of its *immediate successors in* $w(T)$ stands for the set $\{\tau' \in w(T) : \neg\exists\tau'' \in w(T) \quad \tau \subsetneq \tau'' \subsetneq \tau'\}$. Now let us denote by $w_n(T)$ the set of n first (with respect to $\prec$) elements of $w(T)$ together with theirs immediate successors in $w(T)$.

For $T, S \in \mathbb{W}'$ let $T \leq_n S$ if $T \leq S$ and $w_n(S) \subseteq T$. It is now easy to see that with these orderings $\mathbb{W}'$ satisfies Axiom A. $\square$

4.5 Generating by closed sets

Both properness and continuous reading of names could be deduced more easily if only we knew that $\mathcal{I}_P$ is generated by closed sets in the Baire topology on $(\omega+1)^\omega$. However, the problem is that if $A \subseteq (\omega+1)^\omega$ is such that $P{\restriction}A$ is continuous, then it its closure in the Baire topology on $(\omega+1)^\omega$ need not be in $\mathcal{I}_P$ (cf. [**11**, Example 2.7]).

PROPOSITION 4.11. *The σ-ideal $\mathcal{I}_P$ is not generated by closed sets in the Baire topology on $(\omega+1)^\omega$.*

Throughout this proof $\overline{X}$ (for $X \subseteq (\omega+1)^\omega$) denotes the closure of X in the Baire topology.

PROOF. Consider the following set $A = \{\alpha_n, \beta_n : n < \omega\} \subseteq (\omega+1)^\omega$, where

$$\alpha_n(0) = n, \quad \alpha_n(k) = \omega \text{ for } k > 0,$$

$$\beta_n(n) = 0, \quad \beta_n(k) = \omega \text{ for } k \neq n.$$

Note that $P{\restriction}A$ is continuous. On the other hand, $\alpha = (\omega, \omega, \dots) \in \overline{A}$, because $\beta_n \to \alpha$ in the Baire topology on $(\omega+1)^\omega$. In the Cantor topology, however, $\alpha_n \to \alpha$, whereas $P(\alpha_n) \not\to P(\alpha)$. This implies that $P{\restriction}\overline{A}$ is not continuous.

Using a bijection from ω to $\omega\times\omega$ we may identify $(\omega+1)^\omega$ with $(\omega+1)^{\omega\times\omega} \simeq ((\omega+1)^\omega)^\omega$. Under this identification P becomes $\prod_{n<\omega} P$, which we denote by P^ω. First note that P^ω is continuous on A^ω as a product of continuous functions, so $A^\omega \in \mathcal{I}_{P^\omega}$. We will prove, however, that A^ω cannot be covered by countably many sets $F_n, n < \omega$ closed in the Baire topology on $(\omega+1)^\omega$, with each $F_n \in \mathcal{I}_{P^\omega}$.

Suppose that $A^\omega \subseteq \bigcup_n F_n$ and F_n are closed in the Baire topology on $(\omega+1)^\omega$. As A is a discrete set in $(\omega+1)^\omega$ (in both topologies), the relative topology (with respect to any of these two) on A^ω is that of the Baire space. $F_n \cap A^\omega$ are relatively closed, so by the Baire category theorem, one of them has nonempty interior. This means that there is $n < \omega$, $k < \omega$ and $\alpha \in A^k$ such that $\alpha^\frown A^{\omega\setminus k} \subseteq F_n$. Without loss of generality $k = 0$ and $\overline{A^\omega} \subseteq F_n$. But $\overline{A^\omega} = (\overline{A})^\omega$ and $\overline{A}$ contains a convergent sequence $\alpha_n \to \alpha$ such that $P(\alpha_n) \not\to P(\alpha)$. So if $A' = \{\alpha, \alpha_n : n < \omega\}$, then $P[A']$ is a discrete set and $(A')^\omega \subseteq F_n$. Notice, however, that $P^\omega{\restriction}(A')^\omega = (P{\restriction}A')^\omega$ is a copy of the function P and it is not σ-continuous. Hence $F_n \notin \mathcal{I}_{P^\omega}$, which ends the proof. $\square$

A natural question that arises after realizing that the Steprāns forcing is described in terms of wide trees is whether this forcing is equivalent to the Miller forcing. A negative answer follows from Proposition 4.13 below. In Corollary 3.39 we proved that if f is a Baire class 1 function which is not piecewise continuous then the forcing $\mathbb{P}_{\mathcal{I}^f}$ (recall that $\mathcal{I}^f$ is the σ-ideal generated by **closed** sets on which f is continuous) is equivalent to the Miller forcing. We would like to mention that if f is the Pawlikowski function, then this fact can be proved as follows.

CLAIM. *The forcing notion $\mathbb{P}_{\mathcal{I}^P}$ is equivalent to the Miller forcing.*

PROOF. P is a Borel isomorphism between ω^ω and $(\omega+1)^\omega$. We claim that it induces an isomorphism between $\mathbf{Bor}(\omega^\omega)/\mathbf{K}_\sigma$ and $\mathbf{Bor}((\omega+1)^\omega)/\mathcal{I}^P$. Indeed, we need only to see that if $A \subseteq (\omega+1)^\omega$, then $P[A]$ is compact if and only if A is a closed set on which P is continuous. This follows from the facts that a continuous image of a compact set is compact, P^{-1} is continuous and a continuous bijection with a compact domain is a homeomorphism. $\square$

The following definition is a generalization of the notions of Cohen and random real.

DEFINITION 4.12. Let M be a model of a fragment of ZFC. We say that $s \in (\omega+1)^\omega$ is a *Steprāns real over M* if $s \notin A$ for any $A \subseteq (\omega+1)^\omega$ such that $A \in \mathcal{I}_P$ and A is coded in M.

The generic real for the Steprāns forcing is a Steprāns real over the ground model.

PROPOSITION 4.13. *The Miller forcing does not add Steprāns reals.*

PROOF. Denote the Miller forcing by $\mathbb{M}$. Suppose that $\dot{s}$ is a $\mathbb{M}$-name for a Steprāns real. Since $(\omega+1)^\omega \simeq 2^\omega \subseteq \omega^\omega$, $\dot{s}$ can be treated as a name for an element of ω^ω. Since Miller forcing has continuous reading of names we have a forcing condition $B \subseteq \omega^\omega$ and a continuous function $f: B \to \omega^\omega$ such that

$$D \Vdash \dot{s} = f(\dot{m}),$$

where $\dot{m}$ is the name for the $\mathbb{M}$-generic real. By another well known property of Miller forcing, there is a stronger condition $D \subseteq B$ such that either $f \restriction D$ is constant or $f \restriction D$ is a topological embedding. We can exclude the first possibility. Let $E = f[D]$. Note that since $P \restriction E$ is a Borel function, there is a dense $\mathbf{G}_\delta$ set $G \subseteq E$ such that $P \restriction G$ is continuous. But then $f^{-1}[G]$ is comeager in D and hence $f^{-1}[G]$ is a condition in the Miller forcing. We have that

$$f^{-1}[G] \Vdash \dot{s} \in G$$

and $G \in \mathcal{I}_P$. This gives a contradition. $\square$

4.6 No continuous reading of names in any presentation

The Steprāns forcing does not have continuous reading of names in one presentation and has continuous reading of names in another presentation. Let us now produce a forcing $\mathbb{P}_{\mathcal{I}}$ which is proper and does not have continuous reading of names in any presentation. This answers a question of Zapletal and Hrušak [**11**, Question 5.5].

PROPOSITION 4.14. *There exist a σ-ideal $\mathcal{I} \subseteq \mathbf{Bor}(\omega^\omega)$ such that the forcing $\mathbb{P}_{\mathcal{I}}$ is proper but it does not have continuous reading of names in any presentation.*

PROOF. First notice that any presentation of a Polish space X is given by a Borel isomorphism with a $\mathbf{G}_\delta$ subset of $[0,1]^\omega$. Consider $X = (\omega^\omega)^2$ with its product topology (X is homeomorphic to the Baire space). Note that each $\mathbf{G}_\delta$ set G in $[0,1]^\omega$ as well as a Borel isomorphism from G to X can be coded by a real. Let $x \in \omega^\omega$ code a pair (G_x, f_x) that defines a presentation of X as above, i.e. $f_x : G_x \to X$. For $x \in \omega^\omega$ $f_x^{-1}[X_x]$ (X_x denotes the vertical section of X at x) is an uncountable Borel set in G_x and contains a copy C_x of $(\omega+1)^\omega$. Let $\mathcal{I}_x$ be the transported σ-ideal $\mathcal{I}_P$ from C_x to X_x. We define a σ-ideal $\mathcal{I}$ on $\mathbf{Bor}(X)$ as follows:

$$\mathcal{I} = \{A \in \mathbf{Bor}(X) : \forall x \in \omega^\omega \, A_x \in \mathcal{I}_x\}.$$

$\mathbb{P}_{\mathcal{I}}$ does not have continuous reading of names in any presentation because if (G_x, f_x) defines a presentation then $(P \circ f_x^{-1}) \restriction (f_x[C_x])$ is a counterexample to continuous reading of names in its topology. To show that $\mathbb{P}_{\mathcal{I}}$ is proper, we will prove that $\mathbb{P}_{\mathcal{I}}$ is equivalent to the Steprāns forcing. This implies that $\mathbb{P}_{\mathcal{I}}$ is proper since the Steprāns forcing is proper. Notice that the family $\{X_x : x \in \omega^\omega\}$ forms a maximal antichain in $\mathbb{P}_{\mathcal{I}}$. For each $x \in \omega^\omega$ the forcing $\mathbb{P}_{\mathcal{I}} \restriction X_x$ is equivalent to the Steprāns forcing $\mathbb{S}$. On the other hand, in $\mathbb{S}$ there is also a maximal antichain A of cardinality continuum such that $\mathbb{S}\restriction a$ ($a \in A$) is isomorphic to $\mathbb{S}$. (If we see $(\omega+1)^\omega$ as $(\omega+1)^\omega \times (\omega+1)^\omega$, then the antichain can be taken as $\{((\omega+1)^\omega \times (\omega+1)^\omega)_x : x \in (\omega+1)^\omega\}$.) This proves that $\mathbb{P}_{\mathcal{I}}$ is equivalent to $\mathbb{S}$. □

REMARK 4.15. The σ-ideal constructed above is not intended to be definable. With a little amount of work, however, we can make it such. Below we give a sketch of a construction.

We coded presentations of X by Borel isomorphisms with $\mathbf{G}_\delta$ sets in $[0,1]^\omega$. Notice, however, that we need only to code Polish topologies which extend the original one. Any such topology on X having the same Borel sets as the original one can be coded by a sequence of Borel sets (a clopen subbase of the extended topology).

Recall the usual coding of Borel sets by a $\mathbf{\Delta}^1_1$ formula over a $\mathbf{\Pi}^1_1$ set of codes. In the same way we may choose a $\mathbf{\Pi}^1_1$ set of codes $C \subseteq \omega^\omega$ coding all sequences of Borel sets in $(\omega^\omega)^2$. For $z \in C$ we will refer to the Borel sets in the coded sequence as to $B_n(z)$.

Now for each $z \in P$ we would like to choose compact subset $K(x)$ of X_z such that if z codes a subbase of a Polish topology, then $K(x)$ is a copy of the Cantor set in the topology coded by z. Moreover, we would like to do this effectively.

Let $K(\omega^\omega)$ denote the hyperspace of compact subsets of ω^ω. The elements of $K(\omega^\omega)$ can be seen as subtrees of $\omega^{<\omega}$. The space $P \subseteq K(\omega^\omega)$ of all copies of 2^ω is a $\mathbf{G}_\delta$ set. Consider the set Q of all pairs $(z, K) \in \omega^\omega \times K(X)$ such that $z \in C$, $K \in P$, $K \subseteq X_z$ and

$$\forall n \in \omega\ \exists U \text{ clopen set in } K\ \forall x \in \omega^\omega \quad (x \in B_n(z)_z \Leftrightarrow x \in U).$$

Notice that Q is $\mathbf{\Pi}^1_1$. Therefore, by the Kondô theorem, we find a $\mathbf{\Pi}^1_1$ uniformization of Q. Denote it by k.

As in the previous proof, we define the σ-ideal $\mathcal{I}$ in the following way:

$$A \in \mathcal{I} \quad \text{iff} \quad \forall z\ (z \in C \Rightarrow A_z \cap k(z) \in \mathcal{I}_z),$$

where $\mathcal{I}_z$ is the σ-ideal $\mathcal{I}_P$ computed on $k(z)$. The same argument as before shows that $\mathcal{I}$ satisfies the conclusion of Proposition 4.14.

5

Structure of Borel functions

5.1 Introduction

Recall a question of Lusin whether there exists a real-valued Borel function defined on a Polish space which cannot be decomposed into countably many continuous functions. By now several examples have been given, by Keldiš [**17**] and Adyan and Novikov [**1**] among others. A particularly simple example, however, is the Pawlikowski function (for definition see Section 4.2). The function P was first defined in [**5**], where the authors showed [**5**, Lemma 5.4] that if $A \subseteq \mathrm{dom}(P)$ is such that $P \restriction A$ is continuous, then $P[A]$ is nowhere dense in ω^ω. Since P is a surjection, and ω^ω cannot be covered by countably many nowhere dense sets, it follows that P is not σ-continuous.

Solecki [**31**] proved the following dichotomy:

THEOREM 5.1 (Solecki, [**31**, Theorem 4.1]). *For any Baire class 1 function $f : X \to Y$, where X is a Souslin space and Y is Polish, either f is σ-continuous or else there exist topological embeddings φ and ψ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\psi} & Y \\ \uparrow P & & \uparrow f \\ (\omega+1)^\omega & \xrightarrow{\varphi} & X \end{array}$$

Zapletal [**34**] proved of the following.

THEOREM 5.2 (Zapletal, [**34**, Lemma 2.3.46]). *Suppose $f : \omega^\omega \to \omega^\omega$ is a Borel not σ-continuous function. For every Borel set $A \subseteq \omega^\omega$ either $A \in \mathcal{I}_f$, or A contains a compact subset C such that $C \notin \mathcal{I}_f$.*

It is worth mentioning that the proof of Theorem 5.2 is short and elegant. Using Theorem 5.2, Zapletal [**34**, Corollary 2.3.48] generalized Theorem 5.1 to all Borel functions $f : \omega^\omega \to \omega^\omega$. To this end, however, Zapletal used the dichotomy of Theorem 5.1 for Baire class 1 functions (reducing the Borel case to the Baire class 1 case). It should be noted that the proof of Theorem 5.1 is fairly complicated.

In Section 5.3 we introduce the basic tool of this chapter, the system of projections in $(\omega+1)^{<\omega}$, and we show some basic properties of these functions.

In Section 5.4 we use projections from Section 5.3 to prove Proposition 4.8 from Section 4.3, which says that if $D \subseteq (\omega+1)^\omega$ is wide, then the function $P{\restriction}D$ "contains" the function P.

In Sections 5.5 and 5.6 we give a new proof of the dichotomy of Theorem 5.1 for all Borel functions $f:\omega^\omega \to \omega^\omega$. In Theorem 5.6 we prove a weaker version, and in Theorem 5.8 we use Proposition 4.8 to conclude the original statement. The main ingredient of the proof of Theorem 5.6 are the projections from Section 5.3.

5.2 Notation

In a Polish space space X, a *Lusin scheme* indexed by a tree $T \subseteq Y^{<\omega}$ (Y is an arbitrary set) is a map $T \ni \tau \mapsto D_\tau \subseteq X$ such that if $\tau \subseteq \tau' \in T$, then $D_{\tau'} \subseteq D_\tau$ and if $D_\tau \cap D_{\tau'} = \varnothing$ for $\tau \neq \tau', |\tau| = |\tau'|$.

We say that $\varphi : X \to Y$ is an *open injection* if it is 1-1 and $\varphi^{-1} : \mathrm{rng}(f) \to Y$ is continuous.

For the definition of the function P and metric notions concerning the space $(\omega+1)^\omega$ see Section 4.2.

5.3 Structure of the space $(\omega+1)^\omega$

Unlike in the case of the Baire space and the tree $\omega^{<\omega}$, the topology on $(\omega+1)^\omega$ is not connected to the tree $(\omega+1)^{<\omega}$ in the usual way. The topology on $(\omega+1)^\omega$ is much more complex (although weaker) than that of $\lim(\omega+1)^{<\omega}$. This difference is also the reason why it is pretty hard to construct continuous functions on the space $(\omega+1)^\omega$. We introduce a set of functions (called projections) which will be used for constructing continuous functions on the space $(\omega+1)^\omega$.

For each $n<\omega$ and $0 \le k \le n$ let S^n_k be the set of points in $(\omega+1)^n$ of Cantor-Bendixson rank $\ge n-k$. For each $1 \le k \le n < \omega$ let the function $\pi^n_k : S^n_k \to S^n_{k-1}$ be defined as follows:

- π^n_k is the identity on S^n_{k-1},
- for $\tau \in S^n_k \setminus S^n_{k-1}$ put $\max_{<\omega}(\tau) = \max\{\tau(i) : i \in n \wedge \tau(i) < \omega\}$ and let $i = \min\{j \in n : \tau(j) = \max_{<\omega}(\tau)\}$. Define

$$\pi^n_k(\tau)(i) = \omega, \quad \pi^n_k(\tau)(j) = \tau(j) \quad \text{for } j \neq i.$$

The function π^n_k can be thought of as an analog of the orthogonal projection in the cube $[0,1]^n$.

LEMMA 5.3. *For each $n < \omega$ and $1 \leq k \leq n$ the function $\pi^n_k : S^n_k \to S^n_{k-1}$ is continuous.*

PROOF. Note that all points in $S^n_k \smallsetminus S^n_{k-1}$ are isolated, so we only need to check continuity at points in S^n_{k-1}. Let us take any sequence $\tau_m \to \tau$ such that $\tau \in S^n_{k-1}$ and $\tau_m \in S^n_k$ and let us check that $\pi^n_k(\tau_m) \to \tau = \pi^n_k(\tau)$. We may assume that $\tau_m \notin S^n_{k-1}$ (since π^n_k is the identity on S^n_{k-1}). Notice that for all but finitely many m the following conditions hold:

- for each $i \in n$ if $\tau(i) < \omega$, then $\tau_m(i) = \tau(i)$
- for each $i \in n$ if $\tau(i) = \omega$, then $\tau_m(i) > \max_{<\omega}(\tau)$.

This clearly implies that $\pi^n_k(\tau_m) \to \tau$. □

Note that for each $\tau \in (\omega+1)^{<\omega}$ there exist unique $n < \omega$ and $0 \leq k \leq n$ such that $\tau \in S^n_k \smallsetminus S^n_{k-1}$ (here $S^n_{-1} = \varnothing$). We define the function $\pi : (\omega+1)^{<\omega} \to (\omega+1)^{<\omega}$ as follows:

$$\pi(\tau) = \pi^n_k(\tau)$$

where n and k are as above. The function π will be called projection.

5.4 Wide trees

Now we are ready to prove Proposition 4.8 from Section 4.3. Recall the definition of a wide tree.

DEFINITION 5.4. We call a tree $T \subseteq (\omega+1)^{<\omega}$ *wide* if every node $\tau \in T$ has an extension $\tau' \in T$ such that the set $[\tau']_T$ is nowhere dense in $[\tau]_T$. We say that a subset of $(\omega+1)^\omega$ is *wide* if it is the limit of a wide tree.

Proposition 4.8 says that if $D \subseteq (\omega+1)^\omega$ is wide then there are topological embeddings φ and ψ such that the following diagram commutes:

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\ \psi\ } & P[D] \\ \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle P\restriction D} \\ (\omega+1)^\omega & \xrightarrow{\ \varphi\ } & D \end{array}$$

PROOF OF PROPOSITION 4.8. Let $D = \lim T$ be wide in $(\omega+1)^\omega$. Call a subtree $S \subseteq T$ an *end-subtree* if there is a finite nonempty set $F \subseteq T$ such that $T' = \bigcup_{\tau\in F} T(\tau)$.

Let $\mathbb{C}$ be the forcing with end-subtrees of T ordered by inclusion. $\mathbb{C}$ is equivalent to the Cohen forcing. Let M be a countable elementary submodel of a large enough H_κ such that $P, \mathbb{C}, D \in M$ and let $\mathcal{D}_n, n < \omega$ enumerate all dense subsets of $\mathbb{C}$ in M. For a dense set $\mathcal{D} \subseteq \mathbb{C}$ denote by $\mathcal{D}^*$ the set of all finite unions of elements from $\mathcal{D}$.

We construct only the embedding $\varphi : (\omega+1)^\omega \to D$ since it already determines the function ψ. To this end we definite a family $\{T_\tau \in \mathbb{C} : \tau \in (\omega+1)^{<\omega}\}$ such that

- $T_\tau \subseteq T$ for each $\tau \in (\omega+1)^{<\omega}$,

- the sets $D_\tau = \lim T_\tau$ form a Lusin scheme,
- for each $t \in (\omega+1)^\omega$ the family $\{T_\tau : \tau \subseteq t\}$ generates a $\mathbb{C}$-generic filter over M.

The trees T_τ are constructed by induction on $|\tau|$ and satisfy the following conditions:

(i) $T_\tau \in \mathcal{D}^*_{|\tau|}$,
(ii) $\operatorname{diam}(D_\tau) < 1/2^{|\tau|}$,
(iii) for each n the map $(\omega+1)^n \ni \tau \mapsto D_\tau$ is h-continuous.

Notice that (i) and (ii) implies that each branch generates a generic filter over M. Indeed, by (i) any extension to an ultrafilter must be generic and by (ii) there is precisely one such extension, since it is determined by an appropriate generic real.

Once we have the trees T_τ constructed, we proceed as follows. For $t \in (\omega+1)^\omega$ we define $\varphi(t)$ to be the generic real over M given by the generic filter along the branch t. Thanks to (iii) φ is continuous. Since D_τ form a Lusin scheme, we have that φ is injective, and hence a topological embedding. On the other hand, ψ is open because $P[D_\tau]$ also form a Lusin scheme and $P[D_\tau]$ is open in $P[D]$ (since T_τ is an end-subtree). To see that ψ is continuous we use genericity: a formula of the form $P(\varphi(t))(m) = n$ is absolute and if it holds for t which is generic over M, then it must be forced by some condition in the generic filter.

We need to construct the trees T_τ. Let us say that a sequence $\omega+1 \ni i \mapsto S_i$ of end-subtrees of T is *convergent* if

- the sets $\lim S_i$ are pairwise disjoint for $i \in \omega+1$,
- the map $\omega+1 \ni i \mapsto \lim S_i$ is h-continuous.

In the construction we will use the following lemma which holds in M.

Lemma 5.5. *Let S, S' be end-subtrees of T, $\delta > 0$ and $k < \omega$.*

(1) *There is a convergent sequence $\langle S_i \in \mathbb{C} : i \in \omega+1\rangle$ of subtrees of S such that $\operatorname{diam}(\lim S_i) < \delta$ and $\lim S_i \in \mathcal{D}^*_k$ for each $i \in \omega+1$,*
(2) *If $\langle S_i \in \mathbb{C} : i \in \omega+1\rangle$ is a convergent sequence of end-subtrees of S then there is a convergent sequence $\langle S'_i \in \mathbb{C} : i \in \omega+1\rangle$ of subtrees of S' such that $\lim S'_i \in \mathcal{D}^*_k$, $\operatorname{diam}(\lim S'_i) < 3\delta$ and $h(\lim S_i, \lim S'_i) \le 3h(\lim S, \lim S')$ for each $i \in \omega+1$.*

Before we prove Lemma 5.5, we show how it is used to construct the trees T_τ. We denote $\lim T_\tau$ by D_τ.

Let $T_\varnothing = T$. Having defined T_τ for all $\tau \in (\omega+1)^n$ we define it for $\tau \in (\omega+1)^{n+1}$. This in turn is done by another induction on the sets $S^n_k \times (\omega+1)$ for $0 \le k \le n$. That is, we first define T_τ for $\tau \in S^n_0 \times (\omega+1)$ and then show how to extend the definition from $S^n_k \times (\omega+1)$ to $S^n_{k+1} \times (\omega+1)$. During this construction we ensure that for each k and $\tau \in S^n_k$

$$(*) \qquad \operatorname{diam}(D_\tau) < 1/(3^{n-k} \cdot 2^{n+1})$$

and the map $S_k \times (\omega+1) \ni \tau \mapsto D_\tau$ is h-continuous.

To start with we use Lemma 5.5(1). Suppose we have T_τ defined for $\tau \in S_k \times (\omega+1)$. Let us abbreviate $\pi^n_k : S^n_{k+1} \to S^n_k$ by π. For each $\tau \in S^n_{k+1}$ we use Lemma 5.5(2) for D_τ and $D_{\pi(\tau)}$ to find sets $D_{\tau^\frown i}$ for $i \in \omega+1$ such that

$$(**) \qquad h(D_{\tau^\frown i}, D_{\pi(\tau)^\frown i}) \leq 3h(D_\tau, D_{\pi(\tau)}),$$

$\operatorname{diam}(D_{\tau^\frown i}) < 3\operatorname{diam}(D_\tau)$ and $D_{\tau^\frown i} \in \mathcal{D}^*_{|\tau|}$. Now $(*)$ follows from the inductive assumption. To see h-continuity notice that if $(\tau_n, i_n) \to (\tau, i)$ is a convergent sequence in $S^n_{k+1} \times (\omega+1)$, then either τ_n is eventually constant or $\tau \in S^n_k$ and then $\pi(\tau_n) \to \tau$ thanks to continuity of π. But then the assertion follows from the inductive assumption, $(**)$ and the triangle inequality:

$$\begin{aligned} h(D_{\tau_n^\frown i_n}, D_{\tau^\frown i}) \leq\ & h(D_{\tau_n^\frown i_n}, D_{\pi(\tau_n)^\frown i_n}) + h(D_{\pi(\tau_n)^\frown i_n}, D_{\tau^\frown i_n}) \\ & + h(D_{\tau^\frown i_n}, D_{\tau^\frown i}). \end{aligned}$$

In this way we have constructed the sets T_τ. Let us prove Lemma 5.5.

Proof of Lemma 5.5. (1) Pick any $\tau \in S$ such that $\operatorname{diam}([\tau]_S) < \delta/3$ and $[\tau^\frown\omega]_S$ is nowhere dense in $[\tau]_S$. Let $S_\omega = S(\tau^\frown\omega)$ (and $D_\omega = [\tau^\frown\omega]_S$). Notice that for any $\varepsilon > 0$ there exists a finite set $\{\tau_i : i \leq n\}$ of extensions of $\tau^\frown\omega$ such that $\operatorname{diam}([\tau_i]_S) < \varepsilon$ for each $i \leq n$ and $h(D_\omega, \bigcup_{i\leq n}[\tau_i]_S) < \varepsilon$. Using the fact that $[\tau^\frown\omega]_S$ is nowhere dense in $[\tau]_S$ we can find for each $j \leq n$ node τ'_j (extending nodes $\tau^\frown n_j$ for some $n_j < \omega$) such that $\operatorname{diam}([\tau'_j]_S) < \varepsilon$ and $\operatorname{dist}([\tau'_j]_S, [\tau_j]_S) < \varepsilon$. Now it easily follows that $h(D_\omega, \bigcup_{j\leq n}[\tau'_j]_S) < 3\varepsilon$ (so in particular the set $\bigcup_{j\leq n}[\tau'_i]_S$ has diameter $< \delta$ if ε is small enough). Thus we may inductively on $i < \omega$ define end-subtrees S_i to be of the form $\bigcup_{j\leq n}[\tau'_j]_S$ such that $h(\lim S_i, \lim S_\omega) \to 0$ and $\lim S_i$ are disjoint. Notice that τ'_j can be chosen so that $S(\tau'_j) \in \mathcal{D}_k$ and then $S_i \in \mathcal{D}^*_k$. This ends the first part of the proof.

(2) Let $\gamma = h(D, D')$. First we claim that there is a finite set of nodes $\tau'_i, i \leq n$ in S' such that

- $\operatorname{diam}(D' \cap [\tau'_i]_{S'}) < \gamma$ for each $i \leq n$,
- $h(D_\omega, \bigcup_{i\leq n} D' \cap [\tau'_i]_{S'}) < 2\gamma$,
- $\operatorname{diam}(\bigcup_{i\leq n} D' \cap [\tau'_i]_{S'}) < 3\delta$.

Indeed, if $\gamma < \delta$, then we may first find finitely many nodes $\tau_i, i \leq n$ in S_ω and then appropriate nodes $\tau'_i, i \leq n$ in S' such that $h(D_\omega, \bigcup_{i\leq n} D_\omega \cap [\tau_i]_S) < 2\gamma$, both $\operatorname{diam}(D_\omega \cap [\tau_i]_S), \operatorname{diam}(D' \cap [\tau'_i]_{S'}) < \delta$ and also $\operatorname{dist}(D_\omega \cap [\tau_i]_S, D' \cap [\tau'_i]_{S'}) < \delta$ for $i \leq n$. Then by the triangle inequality $\operatorname{diam}(\bigcup_{i\leq n} D' \cap [\tau'_i]_{S'}) < 3\delta$. If $\gamma \geq \delta$, then we will do by picking in a similar manner just one node τ and τ' in S, S' respectively.

Now above each τ'_i choose a node $\tau^{\omega}_i{}'$ such that $D' \cap [\tau^{\omega}_i{}']_{S'}$ is nowhere dense in D'. Then $\bigcup_{i\leq n} D' \cap [\tau^{\omega}_i{}']_{S'}$ is also nowhere dense and has diameter $< 3\delta$. We may now find conditions in $\mathcal{D}_k$ which are stronger than the end-extensions in S' above the nodes $\tau^{\omega}_i{}'$. And put S'_ω to be the union of these, $D'_\omega = \lim S'_\omega$. It is clear that $h(D_\omega, D'_\omega) < 3\gamma$. Similarly as in (1) we can find a sequence of disjoint subtrees $S'_n \in \mathcal{D}^*_k$ such that if we put $D'_n = \lim S'_n$, then $h(D'_n, D_\omega) < 1/n$. Now it follows from the triangle inequality that $h(D_n, D'_n) < 3\gamma$ as well as $\mathrm{diam}(D'_n) < 3\delta$ holds for n big enough. But we may change those finitely many S_n's (in the same way we have found S'_ω, possibly shrinking the existing sets) to ensure that this holds for all n. □

□

5.5 Dichotomy

In the statement of Theorem 5.1 both functions φ and ψ are to be topological embeddings. However, as we will see below, for the dichotomy it is enough that φ is a continuous injection and ψ is an open injection. We will prove this version of the dichotomy first.

THEOREM 5.6. *Let $f : \omega^\omega \to 2^\omega$ be a Borel function. Then precisely one of the following conditions holds:*

(1) *either f is σ-continuous*
(2) *or there are an open injection ψ and a continuous injection φ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\ \psi\ } & 2^\omega \\ \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle f} \\ (\omega+1)^\omega & \xrightarrow{\ \varphi\ } & \omega^\omega \end{array}$$

Notice that compactness of $(\omega + 1)^\omega$ implies that the φ above must be a topological embedding.

PROOF. It is straightforward that (2) implies that f is not σ-continuous. We assume that f is not σ-continuous and prove that (2) holds. By Theorem 5.2 we may assume that $f : X \to 2^\omega$, where X is compact.

Notation. First let us introduce some notation.

Let $r : (\omega+1)^{<\omega} \setminus \{\varnothing\} \to (\omega+1)^{<\omega} \setminus \{\varnothing\}$ be defined as $r(\tau^\frown a) = \tau^\frown\omega$ (for $a \in \omega + 1$).

Let $\leq$ be a well-ordering of $(\omega + 1)^{<\omega}$ in type ω such that for any $\tau, \sigma \in (\omega+1)^{<\omega}$ if σ can be obtained from τ by applying π or r, then we have $\sigma \leq \tau$.

For a set $B \in \mathbf{Bor}(X) \setminus \mathcal{I}_f$ let

$$B^* = B \setminus \bigcup\{U : U \text{ clopen set and } U \cap B \in \mathcal{I}_f\}.$$

Strategy of the construction. In order to define functions φ and ψ, we will construct for each $\tau \in (\omega + 1)^{<\omega}$ a clopen set $C_\tau \subseteq 2^\omega$ and a compact set $X_\tau \subseteq X$.

The sets C_τ will form a Lusin scheme, which means that

- $C_{\tau^\frown a} \subseteq C_\tau$,
- $C_{\tau^\frown a} \cap C_{\tau^\frown b} = \varnothing$ for $a \neq b$.

We will also have $X_\tau \subseteq f^{-1}[C_\tau]$, $\mathrm{diam}(X_\tau) < 2^{-|\tau|}$ and $X_{\tau^\frown a} \subseteq X_\tau$.

The construction of the sets X_τ, C_τ will be done by induction along the ordering $\leq$ on $(\omega+1)^{<\omega}$. In fact, we will do something more: at each step n if τ is the n-th element of $(\omega+1)^{<\omega}$, we will construct not only a compact set X_τ but also $\mathcal{I}_f$-positive Borel sets X^n_σ for $\sigma \leq \tau$ such that:

(i) $X_\tau \subseteq X^{n-1}_{\tau\restriction(|\tau|-1)}$,
(ii) $X^n_\sigma \subseteq X^{n-1}_\sigma$ if $\sigma < \tau$,
(iii) $X^n_\sigma \cap f^{-1}[C_{\sigma^\frown a}] = \varnothing$ if $\sigma, \sigma^\frown a \leq \tau$,
(iv) $X^n_{\sigma^\frown\omega} \subseteq \mathrm{cl}(X^n_\sigma)$ if $\sigma, \sigma^\frown\omega \leq \tau$.

The set X^n_σ is to be understood as the space for further construction of sets X_ρ for $\rho \supseteq \sigma$ and $\rho > \tau$. The last condition, as we will see later, will be used to guarantee "continuity" of the family of sets X_τ. For technical reasons we will also make sure that $X^n_\sigma = (X^n_\sigma)^*$.

The fact that $\mathrm{diam}(X_\tau) < 1/2^{|\tau|}$ will follow from the following inductive conditions (recall that $\pi(\tau) \leq \tau$ for any τ):

- $\mathrm{diam}(X_\tau) < 3\,\mathrm{diam}(X_{\pi(\tau)})$,
- $\mathrm{diam}(X_{\tau^\frown\omega}) < 1/(3^{|\tau|+1}2^{|\tau|})$

and the fact that iterating the projection π in $(\omega+1)^n$ stabilizes in at most $n+1$ steps.

The crucial feature of the sets X_τ is that this family should be "continuous". We will require that if τ and $\pi(\tau)$ occur by the n-th step, then

$$(*) \qquad h(X^n_\tau, X^n_{\pi(\tau)}) < 3^{|\tau|}\, d(\tau, \pi(\tau)).$$

This condition is the most difficult. To fulfill it we will construct yet another kind of objects. Notice first that if $h(A,B) < \varepsilon$ for two nonempty sets $A, B \subseteq X$, then there are two finite families (we will refer to them as to *anchors*) A_i and B_i ($i \in I_0$, I_0 a finite set) of nonempty subsets of A and B respectively such that for any $A'_i \subseteq A_i$, $B'_i \subseteq B_i$ we still have $h(\bigcup_i A'_i, \bigcup_i B'_i) < \varepsilon$. Similarly, if $h(A,B) < \varepsilon$ and $C \subseteq A$, then there exist a finite family D_i ($i \in J_0$, J_0 a finite set) of nonempty subsets of B such that for any $D'_i \subseteq D_i$ we have $h(\bigcup_i D'_i, C) < \varepsilon$.

At each step n if τ is the n-the element of $(\omega+1)^{<\omega}$, then we will additionally construct anchors

- for each pair X^n_σ and $X^n_{\pi(\sigma)}$ such that $\sigma, \pi(\sigma) \leq \tau$
- and for each tripple $X^n_\sigma, X^n_{\pi(\sigma)}, X^n_{\sigma^\frown a}$ such that $a \in \omega+1$, $\pi(\sigma^\frown a) \subseteq \pi(\sigma)$ and $\sigma, \pi(\sigma), \sigma^\frown a \leq \tau$.

Completing the diagram. Suppose the sets C_τ, X_τ and X^n_τ are constructed. For each $t \in (\omega+1)^\omega$ the intersection $\bigcap_n X_{t\restriction n}$ has precisely one point and let $\varphi(t)$ be this point. The other function, ψ is defined as $f \circ \varphi \circ P^{-1}$. Let us check that this works. Both functions ψ and φ are injective because the sets C_τ form a Lusin scheme and $X_\tau \subseteq f^{-1}[C_\tau]$. The function ψ is an open injection because C_τ are clopen sets.

To see continuity of φ first notice that since the sets X_τ have diameters vanishing uniformly to 0, it suffices to check that φ is continuous on each $(\omega+1)^n$ (which is treated as a subsets of $(\omega+1)^\omega$ via the embedding $e:\tau\mapsto\tau^\frown(\omega,\omega,\ldots)$). Continuity on $(\omega+1)^n$ is checked inductively on the sets S^n_k for $0\leq k\leq n$.

The set S^n_0 consists of one point, so there is nothing to check. Suppose that $\tau_i\to\tau$, where $\tau,\tau_i\in S^n_k$ for $i\in\omega$. Then either the sequence is eventually constant, or $\tau\in S^n_{k-1}$. Let us assume the latter. Using Lemma 5.3 we get $\pi(\tau_i)\to\tau$ and by the inductive assumption

$$\varphi(\pi(\tau_i))\to\varphi(\tau).$$

Now pick any $\varepsilon>0$. Let m be such that $\mathrm{diam}(X_\sigma)<\varepsilon$ for $\sigma\in(\omega+1)^m$ and let $j\in\omega$ be such that $d(\tau_j,\pi(\tau_j))<3^{-m}\varepsilon$. Write $\rho^\frown\omega^l$ for ρ extended by l many ω's. By $(*)$ and the fact that $\pi(\tau^\frown\omega)=\pi(\tau)^\frown\omega$ we have

$$h(X^l_{\tau_j{}^\frown\omega^{m-n}},X^l_{\pi(\tau_j)^\frown\omega^{m-n}})<\varepsilon,$$

for each l such that both above sets are constructed before the l-th step. This implies that $\varphi(\tau_j)$ and $\varphi(\pi(\tau_j))$ are closer than 3ε. Since $\varphi(\pi(\tau_i))\to\varphi(\tau)$, we have that $\varphi(\tau_j)\to\varphi(\tau)$. This proves continuity of φ.

Key lemma. Now we state the key lemma, which will be used to guarantee "continuity" of the family of sets X_τ.

LEMMA 5.7. *Let Y be a Borel set in X and let $g:Y\to2^\omega$ a Borel, not σ-continuous function. There exist a basic clopen set $C_\omega\subseteq2^\omega$ and a compact set $X_\omega\subseteq g^{-1}[C_\omega]$ such that*

- *$g{\restriction}X_\omega$ is not σ-continuous,*
- *$X_\omega\subseteq\mathrm{cl}\big((g^{-1}[\omega^\omega\setminus C_\omega])^*\big)$.*

The compact set X_ω can be chosen of arbitrarily small diameter.

PROOF. Without loss of generality assume that $g^{-1}[C]=(g^{-1}[C])^*$ for all clopen sets $C\subseteq2^\omega$. Write $W_\tau=g^{-1}\big[[\tau]\big]$ and consider the following tree of open sets, indexed by $2^{<\omega}$

$$U_\tau=\mathrm{int}\big(W_\tau\big).$$

Let $G=\bigcap_n\bigcup_{|\tau|=n}U_\tau$ and $Z_\tau=W_\tau\setminus U_\tau$. Notice that $g{\restriction}G$ is continuous. Since we have $X=G\cup\bigcup_\tau Z_\tau$, there is $\tau\in\omega^{<\omega}$ such that $Z_\tau\notin\mathcal{I}_g$. Observe, however, that $Z_\tau\subseteq\mathrm{cl}\big(\bigcup_{\tau'\neq\tau,|\tau'|=|\tau|}W_{\tau'}\big)$ because: if an open set $U\subseteq W_\tau$ is disjoint from $\bigcup_{\tau'\neq\tau,|\tau'|=|\tau|}W_{\tau'}$, then $U\subseteq U_\tau$ and thus U is disjoint from Z_τ. Now put $C_\omega=[\tau]$ and pick any compact set with small diameter $X_\omega\subseteq Z_\tau$ such that $X_\omega\notin\mathcal{I}_g$. ◻

The construction. We begin with $X_\varnothing = X^0_\varnothing = X$ and $C_\varnothing = \omega^\omega$. Without loss of generality assume that $X = X^*$. Suppose we have done $n-1$ steps of the inductive construction and let $\tau \in (\omega+1)^{<\omega}$ be the n-th element (with respect to $\le$) of $(\omega+1)^{<\omega}$. Let $\sigma = \tau\restriction(|\tau|-1)$. There are three cases.

Case 1. The four points τ, $\pi(\tau)$, $r(\tau)$ and $r(\pi(\tau))$ are equal. So $\tau = (\omega,\dots,\omega)$ and C_σ and X^{n-1}_σ are already constructed. In this case we use Lemma 5.7 to find a clopen set $C_\tau \subseteq f[X^{n-1}_\sigma] \subseteq C_\sigma$ and a compact set $X_\tau \subseteq X^{n-1}_\sigma$ of diameter $< |\tau|/3^{n+1}$. We put $X^n_\tau = X^*_\tau$, $X^n_\sigma = (X^{n-1}_\sigma \setminus f^{-1}[C_\tau])^*$ and $X^n_\rho = X^{n-1}_\rho$ for other $\rho < \tau$. By the assertion of Lemma 5.7 we still have $X^n_\tau \subseteq \mathrm{cl}(X^n_\sigma)$.

Notice that once we shrink the set X^{n-1}_σ to X^n_σ, we must ensure that the elements of the existing anchors associated to X_σ, still have $\mathcal{I}_f$-positive intersection with X^n_σ. In this case, however, no such anchors have been constructed yet (we will have to deal with it in the remaining cases). In this case we do not construct any new anchors either.

Case 2. The two points $\pi(\tau)$ and $r(\tau)$ are equal but distinct from τ. Let $\delta = d(\tau, r(\tau))$. Since $X^{n-1}_{r(\tau)} \subseteq \mathrm{cl}(X^{n-1}_\sigma)$ by the inductive assumption, we may find finitely many sets $B_i \subseteq X^{n-1}_\sigma$ for $i \le m$ such that

- $h(\bigcup_{i\le m} B'_i, X^{n-1}_{r(\tau)}) < \delta$ for any $\varnothing \ne B'_i \subseteq B_i$,
- $B_i \notin \mathcal{I}_f$.

The second condition follows from $X^{n-1}_\sigma = (X^{n-1}_\sigma)^*$. We may assume that for each $i \le m$ and for each clopen set $C \subseteq 2^\omega$ the set $B_i \cap f^{-1}[C]$ is either empty or outside of $\mathcal{I}_f$.

By the inductive assumption we have $X^{n-1}_\sigma \cap f^{-1}[C_{\sigma^\frown a}] = \varnothing$ for each $a \in \omega+1$ such that $\sigma^\frown a \le \tau$. Pick clopen sets C_i for $i \le m$ such that

- $f^{-1}[C_i] \cap B_i \notin \mathcal{I}_f$,
- $C_i \cap C_{\sigma^\frown a} = \varnothing$ for each $a \in \omega+1$ such that $\sigma^\frown a \le \tau$ and each $i \le m$.

Now we will find a clopen set $C_\tau \subseteq \bigcup C_i$, put $X^n_\sigma = (X^{n-1}_\sigma \setminus f^{-1}[C_\tau])^*$ and find $X_\tau \subseteq \left(\bigcup_{i\le k} B_i\right) \cap f^{-1}[C_\tau]$. We will have to carefully define $X^n_{r(\tau)}$ so that we still have $X^n_{r(\tau)} \subseteq \mathrm{cl}(X^n_\sigma)$.

Notice that if $Y, Z \subseteq X$ are such that $Y, Z \notin \mathcal{I}$, $Y = Y^*$, $Z = Z^*$ and $Y \subseteq \mathrm{cl}Z$, then for any $A \subseteq Z$ we have

$$(**)\qquad Y = Y \cap \mathrm{cl}\Big(\big(Z \cap A\big)^*\Big) \ \cup\ Y \cap \mathrm{cl}\Big(\big(Z \cap A^c\big)^*\Big).$$

In particular $(**)$ applies for $Y = X^{n-1}_{\rho^\frown\omega}$ and $Z = X^{n-1}_\rho$ if $\rho, \rho^\frown\omega \le \tau$, (e.g. $\rho = \sigma$ and $\rho^\frown\omega = r(\tau)$) and A of the form $A = f^{-1}\big[\{x \in 2^\omega : x(j) = 0\}\big]$. Recall that for each $i \le m$ we have $B_i \cap f^{-1}[C_i] \notin \mathcal{I}_f$, so $f[B_i] \cap C_i$ is an uncountable analytic set and hence contains a perfect set. Therefore, using $(**)$ in an inductive construction (with respect to the partial ordering $\subseteq$ on $\{\rho < \tau : \rho \supseteq r(\tau)\}$), we can find $k \in \omega$ and binary sequences $\beta_i \in 2^k$ for each $i \le m$ such that $[\beta_i] \subseteq C_i$, $B_i \cap f^{-1}[[\beta_i]] \ne \varnothing$ for each $i \le m$ and and the sets

- $X^n_\rho = \left(X^{n-1}_\rho \setminus f^{-1}\left[\bigcup_{i\le m}[\beta_i]\right]\right)^*$ for $\rho < \tau, \rho \not\supseteq r(\tau)$,
- $X^n_\rho = \left(X^{n-1}_\rho \cap \mathrm{cl}(X^n_\sigma)\right)^*$ (where X^n_σ is as above) for $\rho < \tau, \rho \supseteq r(\tau)$.

satisfy the inductive hypothesis (i)–(iv). Moreover, lenghtening β_i if needed, we may ensure that for each $\rho < \tau$, all elements of the anchors associated so far to X^{n-1}_ρ have $\mathcal{I}_f$-positive intersection with X^n_ρ.

Put $C_\tau = \bigcup_{i\le m}[\beta_i]$ and for each $i \le m$ find an $\mathcal{I}_f$-positive compact sets X_i inside $B_i \cap f^{-1}[[\beta_i]]$, each of diameter $< 1/(3^{n+1}|\tau|)$.

To fulfill

$$\mathrm{diam}(X_\tau) < 3\,\mathrm{diam}(X^{n-1}_\tau)$$

we proceed as follows:

(a) If $\delta > \mathrm{diam}(X^{n-1}_{\pi(\tau)})$, then we can pick one X_i as X_τ and then still

$$h(X_\tau, X^{n-1}_{\pi(\tau)}) \le 3\,h(X^{n-1}_\sigma, X^{n-1}_{\pi(\sigma)}) < 3^{|\tau|}\,\delta.$$

(b) Otherwise, let $X_\tau = \bigcup_{i\le m} X_i$ and notice that

$$\mathrm{diam}(X_\tau) < 3\,\mathrm{diam}(X^{n-1}_{\pi(\tau)}) \le 3\,\mathrm{diam}(X_{\pi(\tau)}).$$

Let $X^n_\tau = X^*_\tau$.

At this step we create new anchors for the pair X^n_τ and $X^n_{r(\tau)}$ as well as for the tripples X^n_σ, X^n_ρ, X^n_τ for $\rho < \tau$.

Case 3. The two points $\pi(\tau)$, $r(\tau)$ are distinct. Then we have $\pi(\tau) \supseteq \pi(\sigma)$. Let $\delta = d(\tau, \pi(\tau))$. By the inductive assumption

$$h(X^{n-1}_\sigma, X^{n-1}_{\pi(\sigma)}) < 3^{|\sigma|}\,\delta.$$

Using the existing anchor for the tripple X^{n-1}_σ, $X^{n-1}_{\pi(\sigma)}$, $X^{n-1}_{\pi(\tau)}$ find finitely many sets $B_i \subseteq X^{n-1}_\sigma$ for $i \le m$ such that

- $h(\bigcup_{i\le m} B'_i, X^{n-1}_{\pi(\tau)}) < 3^{|\sigma|}\,\delta$ for any $\varnothing \ne B'_i \subseteq B_i$,
- $B_i \notin \mathcal{I}_f$.

As before, we assume that for each clopen set $C \subseteq 2^\omega$ if $B_i \cap f^{-1}[C] \in \mathcal{I}_f$, then $B_i \cap f^{-1}[C] = \varnothing$. We have now two subcases.

Subcase 3.1. Suppose $\tau = r(\tau)$. Similarly as in Case 1, we use Lemma 5.7 to find $X_i \subseteq B_i$ and $C_i \subseteq f[B_i]$ for $i \le m$. We choose the sets C_i so that for any element A of the anchors constructed so far we have $A \setminus \bigcup_{i\le m} f^{-1}[C_i] \notin \mathcal{I}_f$. This may be done because we can always shrink the sets C_i if needed and use the fact that for each point $x \in 2^\omega$ $f^{-1}[\{x\}] \in \mathcal{I}_f$. Put $C_\tau = \bigcup_{i\le k} C_i$. Now we proceed as follows:

(a) If $\delta > \mathrm{diam}(X^{n-1}_{\pi(\tau)})$, then we can pick one X_i as X_τ and then

$$h(X_\tau, X^{n-1}_{\pi(\tau)}) \le 3\,h(X^{n-1}_\sigma, X^{n-1}_{\pi(\sigma)}) < 3^{|\tau|}\,\delta.$$

(b) Otherwise, let $X_\tau = \bigcup_{i\le k} X_i$ and then

$$\mathrm{diam}(X_\tau) < 3\,\mathrm{diam}(X^{n-1}_{\pi(\tau)}) \le 3\,\mathrm{diam}(X_{\pi(\tau)}).$$

Again, similarly as in Case 1, we put $X^n_\tau = (X^n_\tau)^*$, $X^n_\sigma = (X^{n-1}_\sigma \smallsetminus f^{-1}[C_\tau])^*$, $X^n_\rho = X^{n-1}_\rho$ for other $\rho < \tau$.

Subcase 3.2. Suppose $\tau \neq r(\tau)$. Similarly as in Case 2 (using $(**)$), we find clopen sets C_i in ω^ω such that

- $X^n_\rho = (X^{n-1}_\rho \smallsetminus \bigcup_{i \le k} f^{-1}[C_i])^*$ for $\rho < \tau, \rho \not\supseteq r(\tau)$,
- $X^n_\rho = X^{n-1}_\rho \cap \mathrm{cl}(X^n_\sigma)$ for $\rho < \tau, \rho \supseteq r(\tau)$.

satisfy the inductive hypothesis (i)–(iv) and for each $\rho < \tau$, all elements of the anchors associated so far to X^{n-1}_ρ have $\mathcal{I}_f$-positive intersection with X^n_ρ.

Next we find $\mathcal{I}_f$-positive compact sets $X_i \subseteq B_i \cap f^{-1}[C_i]$ each of diameter $< 1/(3^{|\tau|+1}|\tau|)$. We proceed as previously:

- If $\delta > \mathrm{diam}(X^{n-1}_{\pi(\tau)})$, then we can pick one X_i as X_τ and then
$$h(X_\tau, X^{n-1}_{\pi(\tau)}) \le 3\, h(X^{n-1}_\sigma, X^{n-1}_{\pi(\sigma)}) < 3^{|\tau|}\,\delta.$$
- Otherwise, let $X_\tau = \bigcup_{i \le k} X_i$ and then
$$\mathrm{diam}(X_\tau) < 3\,\mathrm{diam}(X^{n-1}_{\pi(\tau)}) \le 3\,\mathrm{diam}(X_{\pi(\tau)}).$$

Let $X^n_\tau = (X^n_\tau)^*$. In Case 3 we construct the same anchors as in Case 2.

This ends the construction and the entire proof. □

5.6 Strong version

Here we prove the stronger version of the dichotomy. Below we consider functions with the range contained in ω^ω, instead of 2^ω. This does not change anything since each of the spaces 2^ω and ω^ω can be embedded in the other.

THEOREM 5.8. *If $f : \omega^\omega \to \omega^\omega$ is a Borel function, then f is not σ-continuous if and only if there exist topological embeddings φ and ψ such that the following diagram commutes:*

$$\begin{array}{ccc} \omega^\omega & \xrightarrow{\ \psi\ } & \omega^\omega \\ \uparrow P & & \uparrow f \\ (\omega+1)^\omega & \xrightarrow{\ \varphi\ } & \omega^\omega \end{array}$$

PROOF. We need only to prove the left-to-right implication.

By Theorem 5.6 we have φ and ψ such that φ is a topological embedding and ψ is an open injection. As a Borel function, ψ is continuous on a dense $\mathbf{G}_\delta$ set $G \subseteq \omega^\omega$. We know, however, (cf. [**5**, Lemma 5.4]) that for $X \subseteq (\omega+1)^\omega$, if $X \in \mathcal{I}_P$, then $P[X]$ is meager in ω^ω. This implies that $P^{-1}[G] \notin \mathcal{I}_P$. By Proposition 4.5 there is a wide set $W \subseteq P^{-1}[G]$. By Proposition 4.8, we find topological embeddings $\varphi' : \omega^\omega \to \omega^\omega$ and $\psi' : (\omega+1)^\omega \to W$ and get the following commutative diagram:

$$\begin{array}{ccccc} \omega^\omega & \xrightarrow{\psi'} & \omega^\omega & \xrightarrow{\psi} & \omega^\omega \\ \uparrow{\scriptstyle P} & & \uparrow{\scriptstyle P\restriction W} & & \uparrow{\scriptstyle f} \\ (\omega+1)^\omega & \xrightarrow{\varphi'} & W & \xrightarrow{\varphi} & \omega^\omega. \end{array}$$

This ends the proof. □

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