

COMPLEXITY OF RAMSEY NULL SETS

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ABSTRACT. We show that the set of codes for Ramsey positive analytic sets is Σ_2^1 -complete. This is a one projective-step higher analogue of the well-known Hurewicz theorem saying that the set of codes for uncountable analytic sets is Σ_1^1 -complete. This shows a close resemblance between the Sacks forcing and the Mathias forcing. In particular, we get that the σ -ideal of Ramsey null sets is not ZFC-correct. This solves a problem posed by Ikegami, Pawlikowski and Zapletal.

1. INTRODUCTION

Ramsey measurability was introduced by Galvin and Prikry [4] to prove a Ramsey theorem for Borel colorings of the plane. Shortly after, their result was generalized by Silver [10] to those colorings of the plane which are in the σ -algebra generated by analytic sets. Ellentuck [3] has later pointed out that Ramsey measurable sets are precisely the sets with the Baire property in a certain topology on $[\omega]^\omega$, called today the *Ellentuck topology*. The basic open sets in the Ellentuck topology are of the form $[\sigma, s] = \{x \in [\omega]^\omega : x \upharpoonright \max(\sigma) = \sigma \wedge x \setminus \max(\sigma) \subseteq s\}$ for $\sigma \in [\omega]^{<\omega}$, $s \in [\omega]^\omega$ such that $\max \sigma < \min s$. Of crucial importance is the fact that analytic subsets of $[\omega]^\omega$ have the Baire property in the Ellentuck topology. This leads to the Silver theorem, saying that every analytic set $A \subseteq [\omega]^\omega$ is *Ramsey measurable*, i.e. for any basic open set $[\sigma, s]$ as above there is an infinite set $s' \subseteq s$ such that $[\sigma, s']$ is either disjoint from A , or contained in A . If for any $[\sigma, s]$ there is an infinite $s' \subseteq s$ such that $[\sigma, s]$ is disjoint from A , then we say that A is *Ramsey null*. A set is *Ramsey positive* if it is not Ramsey null. Note that, by the Silver theorem, an analytic set is Ramsey positive if and only if it contains some $[\sigma, s]$ as above. It is worth noting here that the Silver

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theorem and the notion of Ramsey measurability have found many applications outside of set theory, e.g. in the Banach space theory, cf [7, Section 19.E]. Similar notion appeared also in the early years of forcing as the *Mathias forcing*, which is the forcing with basic open sets in the Ellentuck topology, ordered by inclusion. In an equivalent form, it can be viewed as the quotient Boolean algebra of Borel subsets of $[\omega]^\omega$ modulo the σ -ideal of Ramsey null sets.

Given a (definable) family Φ of analytic sets we say that Φ is *ZFC-correct* if there is a finite fragment ZFC^* of ZFC such that for any $A \in \Sigma_1^1$ and any model M of ZFC^* containing a code for A we have that

$$M \models A \in \Phi \quad \text{if and only if} \quad V \models A \in \Phi.$$

In fact, ZFC-correctness of Φ is equivalent to the fact that the set of codes for analytic sets in Φ is provably Δ_2^1 .

In [11] Zapletal developed a general theory of iteration for idealized forcing. One of the necessary conditions for a σ -ideal to be *iterable* (see [11, Definition 5.1.3]) is its ZFC-correctness. This seems to be very natural assumption since most of the examples share this property. In fact, many of them, including the σ -ideals associated to the Cohen, Sacks or Miller forcing are Π_1^1 on Σ_1^1 (see [11, Definition 3.8.1] or [7, Definition 25.9]), which is even stronger than ZFC-correctness. Among the few examples which are known to be ZFC-correct but not Π_1^1 on Σ_1^1 is the σ -ideal associated to the Laver forcing.

In [6] Ikegami presented a general framework of generic absoluteness results for *strongly arboreal* [6, Definition 2.4] forcing notions \mathbb{P} . Again, an important assumption (cf. [6, Theorem 4.3], [6, Theorem 4.4]) is that the set of Borel codes for sets in $I_{\mathbb{P}}^*$ (see [6, Definition 2.11]) is Σ_2^1 (for a discussion see also [6, Paragraph 7.2]). In the case \mathbb{P} is the Mathias forcing, $I_{\mathbb{P}}^*$ is the family of Ramsey null sets.

Mathias forcing is a natural example of a forcing notion, for which it was not clear whether the results of [6] and [11] can be applied. This motivated Ikegami, Pawlikowski and Zapletal to ask whether the σ -ideal of Ramsey null sets is ZFC-correct. In this paper we answer this question negatively. In fact, we prove the following stronger result, which seems to be interesting in its own right.

Theorem 1. *The set of codes for Ramsey positive analytic sets is Σ_2^1 -complete.*

Until now, not so many examples of Σ_2^1 -complete sets have been known. One of the first sources of such sets is the paper of Becher, Kahane and Louveau [2], where the authors study examples appearing in harmonic analysis. More recently, Σ_2^1 -complete sets have received

considerable attention through the work of Adams and Kechris [1] (in one of the main results [1, Theorem 2] the authors show that certain sets naturally associated to the reducibility and bireducibility of countable Borel equivalence relations are Σ_2^1 -complete). On the other hand, one level below in the projective hierarchy there are plenty of natural examples of Σ_1^1 -complete sets (cf [7, Section 27]). Theorem 1 should be compared to the Hurewicz theorem [7, Theorem 27.5] saying that the set of codes for uncountable analytic (or even closed) sets is Σ_1^1 -complete. Together, these two results show that on two consecutive levels of the projective hierarchy we observe a very similar phenomenon. This reveals an analogy between the Sacks forcing and the Mathias forcing.

In the proof of Theorem 1, we show a one projective-step higher version of a theorem of Kechris [8], saying that Σ_1^1 -completeness with respect to continuous maps and Borel maps are equivalent. The precise formulation is stated in Theorem 6. This result seems to be of independent interest.

This paper is organized as follows. In Section 3 we show that for a certain universal \mathbf{G}_δ set $G \subseteq 2^\omega \times [\omega]^\omega$ the set $\{x \in 2^\omega : G_x \text{ is Ramsey positive}\}$ is Σ_2^1 -complete with respect to $\Sigma_1^1 \cup \Pi_1^1$ -submeasurable maps (for a definition see Section 2). In Section 4 we prove that any Σ_2^1 set which is Σ_2^1 -complete with respect to $\Sigma_1^1 \cup \Pi_1^1$ -submeasurable maps is actually Σ_2^1 -complete. Hence $\{x \in 2^\omega : G_x \text{ is Ramsey positive}\}$ is Σ_2^1 -complete and therefore the same is true for any *good* (cf. [9, Section 3.H.1]) \mathbf{G}_δ -universal (or good Σ_1^1 -universal) in place of G .

2. NOTATION

For a tree $T \subseteq \omega^{<\omega}$ we write $\lim T$ for $\{x \in \omega^\omega : \forall n \in \omega \ x \upharpoonright n \in T\}$. If $\tau \in \omega^{<\omega}$, then we denote by $[\tau]$ the set $\{x \in \omega^\omega : \tau \subseteq x\}$. Similarly, for $\tau \in [\omega]^{<\omega}$ we write $[\tau]$ for $\{x \in [\omega]^\omega : x \upharpoonright \max(\tau) = \tau\}$. For each $n < \omega$ and $i \in 2$ we write $[(n, i)]$ for $\{x \in 2^\omega : x(n) = i\}$. For a tree $T \subseteq \omega^{<\omega}$ we write $P(T)$ (respectively $R(T)$) for the set of all perfect (resp. pruned) subtrees of T . $P(T)$ and $R(T)$ are endowed with Polish topologies induced via the natural embeddings into 2^ω . In particular $P(2^{<\omega})$ stands for the space of all perfect binary trees.

If $D \subseteq \omega^\omega \times \omega^\omega$ and $F \subseteq \omega^\omega$ are closed, then we write $f : F \xrightarrow{c} D$ to denote that f is a continuous function from F to Y whose graph is contained in D . Recall [7, Proposition 2.5] that if T and S are trees such that $F = \lim T$ and $D = \lim S$, then we can code f by a monotone map from T to S , and any monotone map from T to S gives rise to a continuous function defined on a comeager subset of F .

By the *standard topology on $[\omega]^\omega$* we mean the one induced from the Baire space ω^ω via the standard embedding of $[\omega]^\omega$ into ω^ω . Unless stated otherwise, $[\omega]^\omega$ is always considered as a topological space with the standard topology. In special cases we will indicate when we refer to the Ellentuck topology on $[\omega]^\omega$.

For a sequence of Polish spaces $\langle X_i : i \in I \rangle$ (I countable) we write $\bigsqcup_{i \in I} X_i$ for the disjoint union of the spaces X_i with the natural Polish topology.

For a Polish space X we write $K(X)$ for the space of compact subsets of X with the Vietoris topology (cf. [7, Section 4.F]) and $F(X)$ for the Polish space of all closed subsets of X (usually $F(X)$ is considered only as a standard Borel space, but we make use of the fact that there is an appropriate Polish topology on it, cf. [7, Theorem 12.3]). Note that if X is the Baire space ω^ω (or $[\omega]^\omega$), then the natural coding of closed sets by pruned trees gives a homeomorphism of $F(\omega^\omega)$ and $R(\omega^{<\omega})$ (see also [7, Exercise 12.10]).

All Polish spaces considered in this paper are assumed to be endowed with a fixed topology subbase. For the Cantor space 2^ω we fix the subbase consisting of the sets $[(n, 0)]$ and $[(n, 1)]$ for $n < \omega$. For zero-dimensional Polish spaces we assume that the fixed subbase is the one inherited from 2^ω via a fixed embedding into 2^ω . In particular, the space of all pruned subtrees of $\omega^{<\omega}$ inherits its subbase from 2^ω and this subbase consists of the sets $\{T \in R(\omega^{<\omega}) : \sigma \in T\}$ and $\{T \in R(\omega^{<\omega}) : \sigma \notin T\}$. Similarly, the subbase for $F([\omega]^\omega)$ consists of the sets $\{D \in F([\omega]^\omega) : D \cap [\sigma] \neq \emptyset\}$ and $\{D \in F([\omega]^\omega) : D \cap [\sigma] = \emptyset\}$ for $\sigma \in \omega^{<\omega}$. Note that this topology is strictly weaker than the natural analogue of the Vietoris topology, since the sets $\{D \in F([\omega]^\omega) : D \subseteq U\}$ are in general not open for any open $U \subseteq [\omega]^\omega$ (cf. [7, Remark 12.12]).

By a *pointclass* we mean one of the classes $\Sigma_\alpha^0, \Pi_\alpha^0$ for $\alpha < \omega_1$ or Σ_n^1, Π_n^1 for $n < \omega$. If \mathcal{B} is a Boolean combination of pointclasses, X and Y are Polish spaces, \mathcal{U} is the fixed subbase for Y , and $f : X \rightarrow Y$ is a function, then we say that f is \mathcal{B} -*submeasurable* if $f^{-1}(U) \in \mathcal{B}$ for each $U \in \mathcal{U}$. If \mathcal{A} is a pointclass, $A \subseteq X$ is in \mathcal{A} and $f : A \rightarrow Y$ is a function, then we say that f is \mathcal{A} -*measurable* if for each open set $V \subseteq Y$ there is $B \in \mathcal{A}$ such that $f^{-1}(V) = A \cap B$. If Y is zero-dimensional and $A \subseteq Y$ is in \mathcal{A} , then we say that A is $(\mathcal{A}, \mathcal{B})$ -*complete* if for any zero-dimensional Polish space Z and $A' \subseteq Z$ in \mathcal{A} there is a \mathcal{B} -submeasurable function $f : Z \rightarrow Y$ such that $f^{-1}(A) = A'$. Note that the notion of $(\mathcal{A}, \Sigma_1^0)$ -completeness coincides with the usual notion of \mathcal{A} -completeness.

Given a pointclass \mathcal{A} and a Polish space X we code the \mathcal{A} -subsets of X using the *standard universal \mathcal{A} -set* $A \subseteq 2^\omega \times X$ (cf. [7, Theorem 22.3,

Section 26.A]). Recall that the standard universal sets for pointclasses are good. If Φ is a family of \mathcal{A} -subsets of X , then we refer to $\{x \in 2^\omega : A_x \in \Phi\}$ as to the set of *codes for \mathcal{A} -sets in Φ* .

3. CORRECTNESS

In this section we show that the σ -ideal of Ramsey null sets is not ZFC-correct.

Construct a universal \mathbf{G}_δ set $G \subseteq 2^\omega \times [\omega]^\omega$ in such a way that if $x \in 2^\omega$ codes a sequence of closed subsets $\langle D_n : n < \omega \rangle$ of $[\omega]^\omega$, then

$$G_x = [\omega]^\omega \setminus \bigcup_{n < \omega} D_n.$$

More precisely, realize this using $\prod_{n < \omega} F([\omega]^\omega)$ as the set of codes for sequences of closed subsets of $[\omega]^\omega$. The space $\prod_{n < \omega} F([\omega]^\omega)$ is embedded (as a \mathbf{G}_δ set) into 2^ω using the pruned trees. We will show that the set $\{x \in 2^\omega : G_x \text{ is Ramsey positive}\}$ is $(\Sigma_2^1, \Sigma_1^1 \cup \Pi_1^1)$ -complete.

Notice that this result is optimal, i.e. the set of codes for Ramsey positive closed sets (and hence also \mathbf{F}_σ sets) is Σ_1^1 . This follows from the fact that a closed set $C \subseteq [\omega]^\omega$ is Ramsey positive if and only if there is a basic open set $[\sigma, s]$ in the Ellentuck topology such that

$$[\sigma, s] \subseteq C.$$

The latter condition is arithmetical, since both sets $[\sigma, s]$ and C are closed in the standard topology on $[\omega]^\omega$.

Note also that if B is a \mathbf{G}_δ set (or even Borel), then the condition

$$\exists [\sigma, s] \quad [\sigma, s] \subseteq B$$

is Σ_2^1 and hence the set of codes for Ramsey positive \mathbf{G}_δ sets is Σ_2^1 .

Since any $\Sigma_1^1 \cup \Pi_1^1$ -submeasurable function is Δ_2^1 -measurable, we immediately get that the set $\{x \in 2^\omega : G_x \text{ is Ramsey positive}\}$ is not Δ_2^1 . This implies that the σ -ideal of Ramsey null sets is not ZFC-correct, for otherwise we could express the fact that G_x is Ramsey null as

$$\exists M \text{ c.t.m. of ZFC}^* \quad x \in M \wedge M \models G_x \text{ is Ramsey null}$$

or as

$$\forall M \text{ c.t.m. of ZFC}^* \quad x \in M \Rightarrow M \models G_x \text{ is Ramsey null,}$$

where ZFC* is a fragment of ZFC recognizing the correctness of the σ -ideal of Ramsey null sets.

Theorem 2. *The set $\{x \in 2^\omega : G_x \text{ is Ramsey positive}\}$ is $(\Sigma_2^1, \Sigma_1^1 \cup \Pi_1^1)$ -complete.*

Proof. Consider the following set

$$Z = \{C \in K(2^\omega) : \exists a \in [\omega]^\omega \forall x \in C \lim_{n \in a} x(n) = 0\}$$

and recall that Z is Σ_2^1 -complete, by a result of Becker, Kahane and Louveau [2, Theorem 3.1]. We will find a $\Sigma_1^1 \cup \Pi_1^1$ -submeasurable reduction from Z to $\{x \in 2^\omega : G_x \text{ is Ramsey positive}\}$.

For $C \in K(2^\omega)$ and $\tau \in [\omega]^{<\omega}$ we define $F_\tau(C) \subseteq [\omega]^\omega$ as follows. Put

$$F_\tau(C) = \{a \in [\omega]^\omega : [\neg(\exists x \in C \forall n \in a \setminus \max(\tau) \ x(n) = 1)] \vee a \notin [\tau]\}.$$

Lemma 3. *For each $C \in K(2^\omega)$ and $\tau \in 2^{<\omega}$ the set $F_\tau(C)$ is open in the standard topology on $[\omega]^\omega$.*

Proof. Write

$$\bar{C} = \{(a, x) \in [\omega]^\omega \times 2^\omega : x \in C \wedge x \upharpoonright (a \setminus \max(\tau)) = 1\}$$

and let π be the projection to $[\omega]^\omega$ from $[\omega]^\omega \times 2^\omega$. Since \bar{C} is closed in $[\omega]^\omega \times 2^\omega$, the set $\pi''(\bar{C})$ is closed in $[\omega]^\omega$. Now we have

$$[\omega]^\omega \setminus F_\tau(C) = [\tau] \cap \pi''(\bar{C}).$$

□

Lemma 4. *For each $\tau \in 2^{<\omega}$ the function*

$$K(2^\omega) \ni C \mapsto [\omega]^\omega \setminus F_\tau(C) \in F([\omega]^\omega)$$

is $\Sigma_1^1 \cup \Pi_1^1$ -submeasurable.

Proof. Recall (see Section 2) that the subbase for the space $F([\omega]^\omega)$ (embedded in 2^ω) consists of the sets

$$\{D \in F([\omega]^\omega) : D \cap [\sigma] \neq \emptyset\}, \quad \{D \in F([\omega]^\omega) : D \cap [\sigma] = \emptyset\}$$

for $\sigma \in \omega^{<\omega}$. It is enough to prove that for each $\sigma \in \omega^{<\omega}$ the preimage A_σ of the set $\{D \in F([\omega]^\omega) : D \cap [\sigma] \neq \emptyset\}$ is Σ_1^1 in $K(2^\omega)$. Moreover, it is enough to show this for $\sigma \supseteq \tau$. Indeed, $[\omega]^\omega \setminus F_\tau(C)$ is always contained in $[\tau]$, so for $\sigma \not\supseteq \tau$ we have $A_\sigma = A_\tau$ if $\sigma \subseteq \tau$ and $A_\sigma = \emptyset$ otherwise. But if $\sigma \supseteq \tau$, then A_σ is equal to

$$\{C \in K(2^\omega) : \pi''(\bar{C}) \cap [\sigma] \neq \emptyset\},$$

which is the same as

$$\{C \in K(2^\omega) : \exists x \in C \exists a \in [\sigma] \ x \upharpoonright (a \setminus \max(\tau)) = 1\}.$$

The latter set is easily seen to be Σ_1^1 . □

Now we define $F : K(2^\omega) \rightarrow \prod_{n < \omega} F([\omega]^\omega)$ so that $F(C) = \langle [\omega]^\omega \setminus F_\tau(C) : \tau \in [\omega]^{<\omega} \rangle$ (we use some fixed recursive bijection between ω and $[\omega]^{<\omega}$). In other words, $F(C)$ is the code for the \mathbf{G}_δ set

$$G_{F(C)} = \bigcap_{\tau \in 2^{<\omega}} [\omega]^\omega \setminus F_\tau(C).$$

Note that, by Lemma 4, the function F is $\Sigma_1^1 \cup \Pi_1^1$ -submeasurable. We will be done once we prove the following lemma.

Lemma 5. *For $C \in K(2^\omega)$ we have*

$$C \notin Z \text{ if and only if } G_{F(C)} \text{ is Ramsey null.}$$

Proof. (\Leftarrow) Suppose $F(C)$ is a code for a Ramsey null set. We must show that $C \notin Z$. Take any $a \in [\omega]^\omega$. We shall find $x \in C$ such that

$$\lim_{n \in a} x(n) \neq 0.$$

Since $G_{F(C)}$ is Ramsey null, there is $b \subseteq a$, $b \in [\omega]^\omega$ such that

$$[b]^\omega \cap G_{F(C)} = \emptyset.$$

In particular, there is $\tau \in [\omega]^{<\omega}$ such that $b \notin F_\tau(C)$. This means that

$$b \in [\tau] \quad \wedge \quad \exists x \in C \quad \forall n \in b \setminus \max(\tau) \quad x(n) = 1.$$

Hence x is constant 1 on $b \setminus \max(\tau)$, so $\lim_{n \in a} x(n) \neq 0$, as desired.

(\Rightarrow) Suppose now that $C \notin Z$. We must show that $F(C)$ is a code for a Ramsey null set. Take any $\tau \in [\omega]^{<\omega}$ and $a \in [\omega]^\omega$ such that $\max(\tau) < \min(a)$. We shall find $b \in [a]^\omega$ such that

$$[\tau, b] \cap G_{F(C)} = \emptyset.$$

It is enough to find $b \in [a]^\omega$ such that $[\tau, b] \cap F_\tau(C) = \emptyset$. Since it is not the case that

$$\forall x \in C \quad \lim_{n \in a} x(n) = 0,$$

there is $x_0 \in C$ and $b \in [a]^\omega$ such that $x_0 \upharpoonright b = 1$. We shall show that

$$[\tau, b] \cap F_\tau(C) = \emptyset.$$

Suppose not. Take any $y \in [\tau, b] \cap F_\tau(C)$. Then $y \in [\tau]$, $y \setminus \max(\tau) \subseteq b$ and $y \in F_\tau(C)$. So, by the definition of F_τ , we have

$$\neg(\exists x \in C \quad \forall y \setminus \max(\tau) \quad x(n) = 1).$$

But we saw that $x_0 \in C$ and $x_0 \upharpoonright b = 1$, so we have

$$x_0 \upharpoonright (y \setminus \max(\tau)) = 1.$$

This gives a contradiction and shows that $[\tau, b] \cap F_\tau(C) = \emptyset$, as required. \square

This ends the proof of the theorem. \square

4. COMPLETENESS

In this section we show the following result.

Theorem 6. *Any $(\Sigma_2^1, \Sigma_1^1 \cup \Pi_1^1)$ -complete subset of a Polish zero-dimensional space is Σ_2^1 -complete.*

Together with Theorem 2, this will prove Theorem 1. The proof of Theorem 6 will be based on some ideas of Harrington and Kechris from [5] and of Kechris from [8].

We will need the following lemma.

Lemma 7 (Sacks uniformization). *Let Y be a Polish space. If $B \subseteq 2^\omega \times Y$ is Borel and its projection on 2^ω is uncountable, then there is a perfect tree $S \subseteq 2^{<\omega}$ and a continuous function $f : \lim S \rightarrow Y$ such that $f \subseteq B$.*

Zapletal proved [11, Proposition 2.3.4] a general version of the P_I -uniformization for any σ -ideal I for which the forcing P_I is proper. The above lemma follows directly from [11, Proposition 2.3.4] and the fact that the Sacks forcing has *continuous reading of names* (see [11, Definition 3.1.1]).

Since the Mathias forcing also has continuous reading of names, the same uniformization result is true for the Mathias forcing. In particular, this implies that the set of codes for Σ_1^1 Ramsey positive sets is a Σ_2^1 set. Indeed, if $A \subseteq [\omega]^\omega$ is Σ_1^1 and $D \subseteq [\omega]^\omega \times \omega^\omega$ is a closed set projecting to A , then the fact that A is Ramsey positive can be written as

$$\exists[\tau, b] \exists f : [\tau, b] \xrightarrow{c} D \quad f \text{ is total.}$$

Saying that f is total is a Π_1^1 statement, which makes the above Σ_2^1 .

Definition. Let \mathcal{A} be a pointclass and \mathcal{B} be a Boolean combinations of pointclasses. An $(\mathcal{A}, \mathcal{B})$ -*expansion* of a Polish space Y is an \mathcal{A} -subset $E(Y)$ of a Polish space Y' together with an \mathcal{A} -measurable map $r : E(Y) \rightarrow Y$ satisfying the following. For every zero-dimensional Polish space X and \mathcal{B} -submeasurable map $f : Y' \rightarrow X$ there is a closed (in Y') set $F \subseteq E(Y)$ and a continuous map $g : Y \rightarrow X$ such that $r''(F) = Y$ and the following diagram commutes.

$$\begin{array}{ccc} F & & \\ \downarrow r \upharpoonright F & \searrow f \upharpoonright F & \\ Y & \xrightarrow{g} & X \end{array}$$

Note that in this definition we may assume that $X = 2^\omega$.

The above notion is relevant in view of the following.

Proposition 8. *Let X and Y be zero-dimensional Polish spaces, $A \subseteq X$ be $(\mathcal{A}, \mathcal{B})$ -complete and $C \subseteq Y$ be \mathcal{A} -complete. If Y has an $(\mathcal{A}, \mathcal{B})$ -expansion, then A is \mathcal{A} -complete.*

Proof. Let Y' , $E(Y)$ and $r : E(Y) \rightarrow Y$ be an $(\mathcal{A}, \mathcal{B})$ -expansion of Y . Put $C' = r^{-1}(C)$ and note that $C' \subseteq Y'$ is also in \mathcal{A} . Let $f : Y' \rightarrow X$ be \mathcal{B} -submeasurable such that $f^{-1}(A) = C'$. Take $F \subseteq E(Y)$ and $g : X \rightarrow Y$ as in the definition of expansion. Note that $g^{-1}(A) = C$. \square

In view of Proposition 8 and the fact that there exists a Σ_2^1 -complete subset of the Cantor space [2, Theorem 3.1], Theorem 6 will follow once we prove the following.

Theorem 9. *There exists a $(\Sigma_2^1, \Sigma_1^1 \cup \Pi_1^1)$ -expansion of the Cantor space.*

We will need the following technical result (cf. [5, Sublemma 1.4.2]).

Proposition 10. *There exists a Σ_2^1 set $R \subseteq 2^\omega$ and a Σ_2^1 -measurable function $T : R \rightarrow P(2^{<\omega})$ such that for each partition of $2^\omega \times 2^\omega$ into $A \in \Sigma_1^1$ and $C \in \Pi_1^1$ there exists $x \in R$ such that*

$$\lim T(x) \subseteq A_x \quad \text{or} \quad \lim T(x) \subseteq C_x.$$

Proof. We begin with a lemma.

Lemma 11. *Given $x \in 2^\omega$, for any partition of $\omega \times 2^\omega$ into $A \in \Sigma_1^1(x)$ and $C \in \Pi_1^1(x)$ there is a $\Delta_2^1(x)$ -recursive function $T : \omega \rightarrow P(2^{<\omega})$ such that for each $n \in \omega$ we have*

$$\lim T(n) \subseteq A_n \quad \text{or} \quad \lim T(n) \subseteq C_n.$$

Proof. Pick a sufficiently large fragment ZFC* of ZFC and consider the set

$$H = \{c \in 2^\omega : \exists M \text{ a countable transitive model of ZFC*} \\ \text{containing } x \text{ and } c \text{ is a Cohen real over } M\}.$$

Since H is $\Sigma_2^1(x)$, it contains a $\Delta_2^1(x)$ element c . For each $n < \omega$ both A_n and C_n have the Baire property and are coded in any model containing x . Hence, if $c \in A_n$, then A_n is nonmeager and if $c \in C_n$, then C_n is nonmeager. Put

$$S = \{n \in \omega : c \in A_n\}, \quad P = \{n \in \omega : c \in C_n\}$$

and note that both sets S and P are $\Delta_2^1(x)$. We shall define the function T on S and P separately.

For each $n \in P$ the set C_n is nonmeager, so in particular contains a perfect set. Consider the set

$$P' = \{(n, T) \in \omega \times P(2^{<\omega}) : n \in P \wedge \lim T \subseteq C_n\}$$

and note that P' is $\Sigma_2^1(x)$. Pick any $\Sigma_2^1(x)$ uniformization T_P of P' and note that T_P is in $\Delta_2^1(x)$.

For each $n \in S$ the set A_n is nonmeager. Let $D \subseteq \omega \times 2^\omega \times \omega^\omega$ be a $\Pi_1^0(x)$ set projecting to A . Since for $n \in S$ the set A_n is uncountable, by Lemma 7 there exists a perfect tree T together with a continuous map $h : \{n\} \times \lim T \xrightarrow{c} D_n$. Note that, by compactness of $\lim T$, we can code a total continuous function on $\{n\} \times \lim T$ using a monotone map. Consider the set

$$S' = \{(n, T) \in \omega \times P(2^{<\omega}) : n \in S \wedge \exists f : \{n\} \times \lim T \xrightarrow{c} D_n\}$$

and note that S' is $\Sigma_2^1(x)$. Pick any $\Sigma_2^1(x)$ uniformization T_S of S' and note that T_S is $\Delta_2^1(x)$.

The function $T = T_P \cup T_S$ is as required. \square

Now we finish the proof of the proposition. Fix a good Σ_2^1 -universal set $U \subseteq \omega \times 2^\omega \times \omega \times \omega$ such that for each $A \subseteq \omega \times \omega$ and $x \in 2^\omega$ if $A \in \Sigma_2^1(x)$, then there is $n < \omega$ such that

$$A = U_{(n,x)}.$$

Let $U^* \subseteq U$ be a Σ_2^1 -uniformization of U treated as a subset of $(\omega \times 2^\omega \times \omega) \times \omega$ and write

$$R' = \{(n, x) \in \omega \times 2^\omega : \forall m < \omega \exists k < \omega (m, k) \in U_{(n,x)}^* \text{ and } U_{(n,x)}^* \text{ codes a characteristic function of a perfect tree}\},$$

where the coding is done via a fixed recursive bijection from ω to $2^{<\omega}$. Note that $R' \in \Sigma_2^1$.

For $(n, x) \in R'$ we write $\{n\}(x)$ for the perfect tree coded by $U_{(n,x)}^*$. Note that

$$(n, x) \mapsto \{n\}(x)$$

is a partial Σ_2^1 -recursive function from $\omega \times 2^\omega$ to $P(2^{<\omega})$.

Now pick a recursive homeomorphism $h : 2^\omega \rightarrow \omega \times 2^\omega$ and write $h(x) = (n_x, x')$. Put $R = h^{-1}(R')$ and $T(x) = \{n_x\}(x')$ for $x \in R$.

We claim that R and T are as required. To see this, pick a partition of $2^\omega \times 2^\omega$ into $A \in \Sigma_1^1$ and $C \in \Pi_1^1$. Let $z \in 2^\omega$ be such that $A \in \Sigma_1^1(z)$ and $C \in \Pi_1^1(z)$. Let $T : \omega \rightarrow P(2^{<\omega})$ be a $\Delta_2^1(z)$ -recursive function as in Lemma 11. For each $n \in \omega$ we have that $T(n)$ is a total $\Sigma_2^1(z)$ -recursive function from ω to ω coding a perfect tree. Therefore, by the

Kleene Recursion Theorem for $\Sigma_2^1(z)$ -recursive functions [9, Theorem 7A.2] there is $n \in \omega$ such that

$$T(n) = U_{(n,z)} = \{n\}(z).$$

Now $x = h^{-1}(n, z)$ has the desired property. \square

Now we are ready to prove Theorem 9.

Proof of Theorem 9. Pick a Σ_2^1 set $R \subseteq 2^\omega$ and a Σ_2^1 -measurable function $T : R \rightarrow P(2^{<\omega})$ as in Proposition 10. For each $x \in R$ let $t(x) \in T(x)$ be the first splitting node of $T(x)$ and let $T^0, T^1 : R \rightarrow P(2^{<\omega})$ be defined as

$$T^i(x) = T(x)_{t(x) \frown i}$$

for $i \in 2$. Note that T^0 and T^1 are also Σ_2^1 -measurable.

For $x \in R$ let

$$s_x^0, s_x^1 : 2^\omega \rightarrow \lim T^i(x)$$

be induced by the canonical isomorphism of $2^{<\omega}$ and $T^i(x)$. It is not difficult to see that for each $i \in 2$ the map $(x, y) \mapsto (x, s_x^i(y))$ is a Σ_2^1 -measurable function from $R \times 2^\omega$ to $R \times 2^\omega$.

For each $n \in \omega$ let $R_n \subseteq (2^\omega)^{n+1}$ be defined as

$$R_n = \{(x_0, \dots, x_n) \in (2^\omega)^{n+1} : x_0 \in R \wedge \dots \wedge x_{n-1} \in R\}.$$

For each $\tau \in 2^{<\omega}$ put $X_\tau = (2^\omega)^{|\tau|+2}$ and write R_τ for a copy of $R_{|\tau|}$ inside X_τ .

Pick a homomorphism $q : 2^\omega \times 2^\omega \rightarrow 2^\omega$. For each $n \in \omega$ let

$$p_{n+1} : \bigsqcup_{\tau \in 2^{n+1}} X_\tau \rightarrow \bigsqcup_{\tau \in 2^n} X_\tau$$

be a partial function such that $\text{dom}(p_{n+1}) = \bigcup_{\tau \in 2^{n+1}} R_\tau$ and if $\tau \in 2^{n+1}$, $\tau = \sigma \frown i$, then p_{n+1} maps R_τ into R_σ as follows:

$$(*) \quad p_{n+1}(x_0, \dots, x_{n-1}, x_n, x_{n+1}) = (x_0, \dots, x_{n-1}, s_{x_n}^i(q(x_n, x_{n+1})))$$

for $(x_0, \dots, x_{n+1}) \in R_\tau$ (the value is treated as a point in R_σ). Note that each p_{n+1} is Σ_2^1 -measurable and 1-1.

We get the following sequence of spaces and partial Σ_2^1 -measurable maps

$$2^\omega \times 2^\omega = X_\emptyset \xleftarrow{p_1} X_{(0)} \sqcup X_{(1)} \xleftarrow{p_2} \dots \xleftarrow{p_n} \bigsqcup_{\tau \in 2^n} X_\tau \xleftarrow{p_{n+1}} \bigsqcup_{\tau \in 2^{n+1}} X_\tau \xleftarrow{p_{n+2}} \dots$$

and we write t_n for $p_1 \circ \dots \circ p_n$ for $n > 0$ and t_0 for the identity function on $2^\omega \times 2^\omega$

Now, let $E(2^\omega) \subseteq 2^\omega \times 2^\omega$ be defined as

$$E(2^\omega) = \bigcap_{n < \omega} \text{rng}(t_n).$$

Notice that $E(2^\omega) \in \Sigma_2^1$. The map $r : E(2^\omega) \rightarrow 2^\omega$ is defined as follows. For $n \in \omega$ and $\tau \in 2^n$ we put

$$r(x) \upharpoonright n = \tau \quad \text{iff} \quad (t_n)^{-1}(x) \in X_\tau.$$

Note that r is Σ_2^1 -measurable.

We need to check that $E(2^\omega)$ and r satisfy the properties of expansion. Let $f : 2^\omega \rightarrow Y$ be $\Sigma_1^1 \cup \Pi_1^1$ -submeasurable, where Y is a zero-dimensional Polish space. Since Y is embedded into 2^ω and inherits its subbase from 2^ω via this embedding, we can assume that $Y = 2^\omega$ and the subbase consists of the sets $[(n, i)]$ for $n \in \omega, i \in 2$.

We shall define two trees $\langle x_\tau : \tau \in 2^{<\omega} \rangle$ and $\langle u_\tau : \tau \in 2^{<\omega} \rangle$ such that for each $\tau \in 2^{<\omega}$ and $i \in 2$ we have

- $x_\tau \in (2^\omega)^{|\tau|+1}$ and $u_\tau \in 2^{|\tau|+1}$
- $u_\tau \subseteq u_{\tau \frown i}$ and $x_\tau \subseteq x_{\tau \frown i}$,

and

$$(**) \quad (f \circ t_n)''(X_\tau | x_\tau) \subseteq [u_\tau]$$

where $X_\tau | x_\tau = \{y \in X_\tau : y \upharpoonright (|\tau| + 1) = x_\tau \wedge y_{n+1} \in T(y_n)\}$.

Suppose this has been done. Note that then for each $n \in \omega$ and $\tau \in 2^n$ the sets $F_\tau = t_n''(X_\tau | x_\tau)$ are closed since, by (*), t_n is a continuous function of the last variable when the remaining ones are fixed. The sets F_τ form a Luzin scheme of closed sets. Put

$$F = \bigcap_{n < \omega} \bigcup_{\tau \in 2^n} F_\tau.$$

We define $g : 2^\omega \rightarrow 2^\omega$ so that

$$g(y) \in \bigcap_{n < \omega} [u_{y \upharpoonright n}].$$

Note that g is continuous. From (**) we get that $g \circ (r \upharpoonright F) = f \upharpoonright F$.

Now we build the trees $\langle x_\tau : \tau \in 2^{<\omega} \rangle$ and $\langle u_\tau : \tau \in 2^{<\omega} \rangle$. We construct them by induction as follows. The two sets

$$f^{-1}([(0, 0)]) \quad \text{and} \quad f^{-1}([(0, 1)])$$

form a partition $2^\omega \times 2^\omega$ into two sets, one of which is Σ_1^1 and the other Π_1^1 , by the assumption that f is $\Sigma_1^1 \cup \Pi_1^1$ -submeasurable. By Proposition 10 there is $x \in R$ and $i \in 2$ such that $T(x) \subseteq f^{-1}([(0, i)])$. Put $x_\emptyset = x$, $u_\emptyset = \langle i \rangle$ and note that (**) is satisfied.

Suppose that $n > 0$ and x_σ and u_σ are constructed for all $\sigma \in 2^{n-1}$. Fix $\tau \in 2^n$ and let $\tau = \sigma \hat{\ } i$ for some $\sigma \in 2^{n-1}$ and $i \in 2$. We must find $x_\tau \in (2^\omega)^{n+1}$ and $u_\tau \in 2^{n+1}$.

Note that the set $\{y \in X_\tau : y \upharpoonright n = x_\sigma\}$ is homeomorphic to $2^\omega \times 2^\omega$. Let $w : 2^\omega \times 2^\omega \rightarrow \{y \in X_\tau : y \upharpoonright n = x_\sigma\}$ denote the canonical homeomorphism $y \mapsto x_\sigma \hat{\ } y$. Consider the partition of $2^\omega \times 2^\omega$ into

$$(f \circ i_{n+1} \circ w)^{-1}([(n-1, 0)]) \quad \text{and} \quad (f \circ i_{n+1} \circ w)^{-1}([(n-1, 1)]).$$

One of them is Σ_1^1 and the other Π_1^1 , so by Proposition 10, there exists $x \in R$ and $i \in 2$ such that

$$T(x) \subseteq (f \circ i_{n+1} \circ w)^{-1}([(n-1, i)]).$$

Put $x_\tau = x_\sigma \hat{\ } x$ and $u_\tau = u_\sigma \hat{\ } i$. To see that $(**)$ holds note that $p_{n+1}''(X_\tau | x_\tau) \subseteq X_\sigma | x_\sigma$ by the definition $(*)$. Therefore, by the inductive assumption we have that $(f \circ t_{n+1})''(X_\tau | x_\tau) \subseteq [u_\sigma] \cap [(n-1, i)] = [u_\tau]$. This ends the construction and the whole proof. \square

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