

New a-T-menable HNN-extensions

Światosław R. Gal
Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50-384 Wrocław
<http://www.math.uni.wroc.pl/~sgal/>

Tadeusz Januszkiewicz
Instytut Matematyczny Polskiej Akademii Nauk
and
Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50-384 Wrocław
<http://www.math.uni.wroc.pl/~tjan/>

Abstract The Baumslag-Solitar groups are a-T-menable. This is proved by embedding them into topological groups and studying representation theoretic properties of the latter. The paper is motivated by a question of A. Valette.

We are mainly interested in Baumslag-Solitar groups which are discrete. However, topological groups (Lie groups and the automorphism group of a tree) arise in the proof.

Definition 1 (M. Gromov). *A locally compact, second countable group G is a-T-menable iff there exists a metrically proper affine isometric action of G on some Hilbert space.*

a-T-menability is often referred to as the *Haagerup approximation property*. It is equivalent to the existence in $C_0(G)$ of an approximate unit of positive definite functions on G . For ample discussion one may consult [CCJJV].

Let $G \subset \mathfrak{N}$ be a closed subgroup of a locally compact compactly generated topological group \mathfrak{N} . Let $i_k: H \rightarrow G$ ($k = 1, 2$) be two inclusions onto finite index open subgroups, which are conjugated by an automorphism ϕ of \mathfrak{N} .

The main case of interest is when G is discrete and \mathfrak{N} is a Lie group.

Definition 2. *A \mathfrak{N} -BS group is the group derived from such (G, H, i_1, i_2) by the (topological) HNN construction. In other words, if G is given by the presentation $\langle S | R \rangle$, Γ has the presentation $\langle S, t | R, t i_1(g) t^{-1} = i_2(g) \forall g \in H \rangle$.*

Through this paper let Γ be a \mathfrak{N} -BS group as above.

Since we are working with topological presentations, recall that the topology in the HNN extension Γ is given by the basis $\mathcal{B} = \{\gamma U \gamma' : U \text{ open in } G, \gamma, \gamma' \in \Gamma\}$. It is clear, that Γ is a topological group with respect to \mathcal{B} .

2000 *Mathematics Subject Classification*: 20f65

Both authors artially supported by a KBN grant 2 P03A 023 14.

To see that \mathcal{B} is in fact a basis consider U open in G . We can decompose U with respect to $i_1(H)$ as follows $U = \bigcup i_1(U_n)g_n$, where g_n are representatives of cosets of $i_1(H)$. Then $tU = \bigcup i_2(U_n)tg_n$. Therefore any set of \mathcal{B} can be written uniquely in form $\bigcup U_\gamma\gamma$ where U_γ runs over open subsets of G and γ runs over the chosen set of representatives of $G \setminus \Gamma$. Thus \mathcal{B} is closed under the intersections and in particular \mathcal{B} is a basis. The same argument shows that the topology on G induced from Γ coincides with the original one (i.e. $G \subset \Gamma$ is an open embedding).

Examples:

1. ($\mathfrak{N} = \mathbb{R}$) Baumslag-Solitar group with parameters p and q is given by a following presentation $BS_q^p = \langle x, t \mid x^p = tx^qt^{-1} \rangle = HNN(\mathbb{Z}, \mathbb{Z}, p\cdot, q\cdot)$.
2. ($\mathfrak{N} = \mathbb{R}^n$) Torsion free, finitely generated, abelian-by-cyclic groups are exactly ascending ($i_1 = id$) HNN extensions of \mathbb{Z}^n with i_2 given by a n by n matrix with nonzero determinant [BS].
3. \mathfrak{N} a homogeneous nilpotent group (i.e. one admitting a dilating automorphism ϕ) with a discrete subgroup $G \subset \mathfrak{N}$ such that $\phi(G) \subset G$.

An obvious adaptation of the Bass-Serre theory [S] to the topological context shows that for a topological HNN extension Γ of a group G there is a tree T with an edge-transitive Γ -action such that the vertex stabilizers are conjugated to G and edge stabilizers are conjugated to H .

Since $i_k(H)$ are of finite index in G , T is locally finite. So, the simplicial automorphism group $\text{Aut}(T)$ carries the natural (compact-open) topology such that the group is locally compact. More precisely, the basis of neighborhoods of the identity is the family $\mathcal{U}_K := \{g \mid gv = v \forall v \in K\}$ where K runs over the family of the compact subsets $K \subset T$. Denote $j_T: \Gamma \rightarrow \text{Aut}(T)$ the homomorphism given by the action.

Define $\tilde{\mathfrak{N}} = \mathbb{Z} \ltimes \mathfrak{N}$, to be the semidirect product given by the \mathbb{Z} -action on \mathfrak{N} via ϕ . There is an obvious homomorphism $j_{\tilde{\mathfrak{N}}}: \Gamma \rightarrow \tilde{\mathfrak{N}}$, which is the identity on G and sends t to the generator of \mathbb{Z} .

Theorem 1. *Let Γ be a \mathfrak{N} -BS group, and let j_T and $j_{\tilde{\mathfrak{N}}}$ be the homomorphisms defined above. Then the homomorphism $j = (j_T, j_{\tilde{\mathfrak{N}}}): \Gamma \rightarrow \text{Aut}(T) \times \tilde{\mathfrak{N}}$ is an embedding onto a closed subgroup which is a topological isomorphism onto its image.*

Proof: Observe that since $j_{\tilde{\mathfrak{N}}}$ restricted to G is an embedding onto a closed subgroup (this follows from the fact that G is a closed subgroup in \mathfrak{N}) the same is true for j restricted to G . Let v be the vertex stabilized by G . Then $j_T(G) = \text{Stab}_v \cap j_T(\Gamma)$.

Let $g \in \Gamma$ such that $j(g) = \mathbf{1}$. Since $j_T(g) = \mathbf{1}$, $gv = v$ i.e. $g \in G$. Since $j_{\tilde{\mathfrak{N}}}$ restricted to G is an embedding; this implies $g = \mathbf{1}$. Thus injectivity follows.

Let $\gamma \notin j(\Gamma)$. Since Γ acts transitively on T , we can multiply γ by some element of $j(\Gamma)$ and assume that $\gamma v = v$. Then $(\text{Stab}_v - G) \times \mathfrak{N}$ is an open neighbourhood of γ omitting $j(\Gamma)$. So $j(\Gamma)$ is closed.

The topologies on G and $j(G)$ coincide. Since G and $j(G)$ are open in Γ and $j(\Gamma)$ respectively, the same is true for Γ and $j(\Gamma)$.

Corollary 1. *If \mathfrak{N} is a-T-menable then so are \mathfrak{N} -BS groups.*

Proof: Since T is locally finite, the group $\text{Aut}(T)$ is a-T-menable by a result of Haagerup [H].

A cyclic extension of a-T-menable group is a-T-menable by [J]. Therefore $\tilde{\mathfrak{N}}$ is a-T-menable.

It is clear that a-T-menability is closed under direct products and under taking closed subgroups. Thus the corollary follows from Theorem 1. \square

Remark 1. According to Chapter 4 of [CCJJV] every a-T-menable connected Lie group \mathfrak{N} is locally isomorphic to a direct product of an amenable group and copies of $SO(1, n)$ and $SU(1, n)$. A Lie group is amenable if it is compact-by-solvable. Thus the examples 1.–3. are a-T-menable.

Remark 2. Another proof of Corollary 1, relying on the study of a-T-menability of amalgams, is given in a paper by the first author [G].

Bibliography

- [BS] R. Bieri and K. Strebel, *Almost finitely presented soluble groups*, Comm. Math. Helv. **53** (1978), pp. 258–278,
- [G] Gal, S. R., *a-T-menability of groups acting on trees*, Bull. of the Australian Math. Soc, **69** (2004) pp. 297–303,
- [CCJJV] P. A. Cherix, M. Cowling, P. Jolissaint, P. Julg and A. Valette, *Groups with the Haagerup property (Gromov’s a-T-menability)*, Birkhäuser Verlag, 2001,
- [H] U. Haagerup, *An example of non-nuclear C^* -algebra which has the metric approximation property*, Invent. Math. **50** (1979), pp. 279–293,
- [J] P. Jolissaint, *Borel cocycles, approximation properties and relative property T*, Ergodic Th. Dyn. Syst. **20** (2000), pp. 483–499,
- [S] J. P. Serre, *Trees*, Springer-Verlag, 1980.