

Real Root Conjecture fails for five and higher dimensional spheres

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Abstract : A construction of convex flag triangulations of five and higher dimensional spheres, whose h-polynomials fail to have only real roots, is given. We show that there is no such example in dimensions lower than five.

A condition weaker than realrootedness is conjectured and some evidence is provided.

Let the *f-polynomial* f_X of a simplicial complex X be defined by the formula

$$f_X(t) := \sum_{\sigma \in X} t^{\#\sigma}.$$

There is a classical problem: what can be said in general about the f-polynomials of (a certain class of) simplicial complexes?

In particular, it is well known what polynomials appear as f-polynomials of

- general simplicial complexes, or

- triangulations of spheres that are the boundary complexes of convex polytopes (the reader may consult [St1] for ample discussion). The question concerning all triangulations of spheres remains still open. However the answer is conjecturally the same.

What we are interested in is the special case of the latter. Namely, what can be said in general about the f-polynomials of *flag sphere triangulations*?

The paper is organized as follows. In Section 1.1 we recall the definition of and basic facts about flag complexes. In Section 1.2 we discuss the generalized homology spheres (GHS), also called Gorenstein* complexes, a suitable generalization of sphere triangulations. In Section 2.1 we define a substitution in the f-polynomial of a GHS which we call the γ -polynomial and find very useful when the complex is flag. In Sections 2.2 and 2.3 we discuss the relation of γ -polynomial to the well known Charney–Davis Conjecture and the cd-index. In Section 2.4 we explore the construction of edge subdivision which will be used in later proofs. In Section 3.1 we formulate the Real Root Conjecture (a strengthening of the well known Charney–Davis Conjecture) and prove it for generalized homology spheres of dimension at most four. In Section 3.2 we prove a partial converse. This provides an almost complete answer to the question posed in the previous paragraph in low dimensions. In Section 3.4 we construct a flag convex triangulation of S^5 that is a counterexample to the Real Root Conjecture.

Apart from understanding of f-polynomials of flag triangulation of spheres the Real Root Conjecture was important since the affirmative answer, as observed by Reiner and Welker [RW], would provide a partial check for the Stanley–Neggers Conjecture.

Partially supported by a KBN grant 2 P03A 017 25.

2000 *Mathematics Subject Classification*: 20F55.

Key phrases: flag complex, h-vector, cd-index, Charney–Davis Conjecture, Real Root Conjecture.

We conjecture that the coefficients of the γ -polynomial of the flag GHS are nonnegative. We show that this conjecture is stronger than the Charney–Davis Conjecture and weaker than the Real Root Conjecture, thus it could remain true in all dimensions. One should think of the conjecture on γ -polynomial as a lower bound conjecture for flag sphere triangulations. Some evidence is provided.

The author would like to thank Tadeusz Januszkiewicz and Vic Reiner for useful discussions and Andrzej Derdziński and Paweł Goldstein for their extensive help with the final version of this manuscript.

1. PRELIMINARIES

1.1 FLAG COMPLEXES

Definition 1.1.1. *A simplicial complex X with the vertex set S is called flag if for any $T \subset S$ such that T is a clique (i.e., any two distinct vertices of T are joined by an edge), T is a face of X .*

Obviously a flag complex is determined by its one-skeleton. A barycentric subdivision of any polytopial complex or, more generally, regular CW-complex is flag.

A motivation for studying flag complexes is a theorem of Gromov, which states that a cubical complex is locally CAT(0) if and only if the link of any vertex is flag [Gr]. Recall that locally CAT(0) space is in particular aspherical.

Definition 1.1.2. *The link Lk_σ of a simplex σ in a simplicial complex X consists of all $\tau \in X$ such that $\sigma \cup \tau \in X$ and $\sigma \cap \tau = \emptyset$.*

Remark 1.1.3. The link of any simplex in a flag complex is flag itself.

1.2 THE H-POLYNOMIAL

Definition 1.2.1. *A (simplicial) generalized homology sphere of dimension n (GHS ^{n}) is a simplicial complex such that the link of any simplex σ has the homology of a sphere of dimension $(n - \#\sigma)$. We will omit the superscript if not necessary.*

A simplicial GHS is also called a Gorenstein* complex (see [St1, Ch. II, Thm. 5.2] for further reference).

Remark 1.2.2. A triangulation of a (homology) sphere is a generalized homology sphere.

We introduce Definition 1.2.1, since the double suspension of a GHS is a triangulation of a sphere [C]. Thus any GHS may appear as the link of an edge in some triangulation of a sphere. More generally, assume that the dimension of a simplicial (or cubical) complex is greater than 2. Then it is a cellulation of a manifold if and only if the link of any vertex is a simply connected GHS [E,F].

Definition 1.2.3. *We say that X is a convex sphere triangulation if it is the boundary complex of some convex polytope.*

Remark 1.2.4. The link of any simplex of a convex sphere triangulation is also a convex sphere triangulation.

Theorem (Dehn–Sommerville relations). *If X is a GHS $^{n-1}$ then $f_X(t-1) = (-1)^n f_X(-t)$.*

Proof [KL]: In fact, one needs a weaker assumption. Namely, that X is *Eulerian*, i.e. the Euler characteristic of the link of any simplex σ equals to that of the sphere of an appropriate dimension, or in other words:

$$\sum_{\tau \supset \sigma} (-1)^{\#\tau} = (-1)^n.$$

If τ is a simplex then $(1+s)^{\#\tau} = \sum_{\sigma \subset \tau} s^{\#\sigma}$. Therefore

$$f_X(t-1) = \sum_{\tau} (t-1)^{\#\tau} = \sum_{\sigma \subset \tau} (-t)^{\#\sigma} (-1)^{\#\tau} = \sum_{\sigma} (-t)^{\#\sigma} (-1)^n = (-1)^n f_X(-t).$$

□

If X is a GHS $^{n-1}$, then a more efficient invariant is the h-polynomial defined as

$$(1+t)^n h_X\left(\frac{1}{1+t}\right) := t^n f_X\left(\frac{1}{t}\right).$$

The Dehn–Sommerville relations written in terms of the h-polynomial say that h_X is *reciprocal* (i.e. $h_X(t) = t^n h_X(1/t)$). Moreover, if X is a convex sphere triangulation then h_X has a geometric interpretation in terms of a generic height function and $h_X(t^2)$ is the Poincaré series of the cohomology of the corresponding toric variety [St1].

Remark 1.2.5. The h-polynomial h_X is usually defined for any simplicial complex. Although in this exposition we will use it only when X is a GHS, the reader should be aware that some formulas (e.g. the formula for the h-polynomial of the subdivision along an edge) are incorrect when applied to the general complex.

2. THE γ -POLYNOMIAL

2.1 THE DEFINITION

Proposition 2.1.1. *Assume that h is a reciprocal polynomial of degree n . Then there exists a unique polynomial γ of degree at most $\lfloor n/2 \rfloor$ with the property*

$$(2.1.2) \quad h(t) = (1+t)^n \gamma \left(\frac{t}{(1+t)^2} \right).$$

Moreover if h has integral coefficients then so does γ .

Proof: Observe that the dimension of the space of reciprocal polynomials of degree n is $\lfloor n/2 \rfloor + 1$. The polynomials $t^i(1+t)^{n-2i}$ for $0 \leq i \leq n/2$ are reciprocal and linearly independent (being of different degree). Thus they constitute a basis of this space. Since the leading coefficients are equal to one, they also constitute a basis over the integers.

The coefficients of γ are the coefficients of h with respect to this basis. \square

Remark 2.1.3. If h is monic then the constant coefficient of γ equals one.

Definition 2.1.4. *Let X be a GHS (or at least Eulerian). The polynomial γ_X defined by*

$$(2.1.5) \quad h_X(t) = (1+t)^{\deg h_X} \gamma_X \left(\frac{t}{(1+t)^2} \right)$$

will be called the γ -polynomial of X .

Question 2.1.6. *What is the combinatorial/geometric interpretation of (the coefficients of) the γ -polynomial?*

Conjecture 2.1.7. *If X is a flag GHS then all the coefficients of the γ -polynomial γ_X are nonnegative.*

In the rest of this section we provide some evidence for the Conjecture 2.1.7.

Definition 2.1.8. *The join of two complexes X and Y on disjoint ground sets is defined as the complex $\{\sigma \cup \tau \mid \sigma \in X, \tau \in Y\}$.*

Remark 2.1.9. Any of f_\bullet , h_\bullet and γ_\bullet is multiplicative with respect to the joins.

Corollary 2.1.10. *If X and Y satisfy Conjecture 2.1.7 then so does their join $X * Y$.*

The Generalized Lower Bound Conjecture [St, Ch. II, Conj. 6.2]. *Let X be a GHS. Then h_X is unimodal, i.e., if $h_X(t) = \sum_{i=0}^n h_i t^i$ then $h_{\lfloor n/2 \rfloor} \geq \dots \geq h_2 \geq h_1 \geq h_0 = 1$.*

Remark 2.1.11. The above conjecture is true if X is a convex triangulation of a sphere [St1].

Corollary 2.1.12. *If Conjecture 2.1.7 holds for X then h_X is unimodal.*

Proof: Each of the polynomials $t^i(1+t)^{n-2i}$ is unimodal. A sum of reciprocal unimodal polynomials is unimodal. \square

Definition 2.1.13. *The cross polytope O^n is the n -fold join of the zero-dimensional sphere.*

Lemma 2.1.14. *Let X be a flag GHS $^{n-1}$ and $\gamma_X(t) = \sum_{i=0}^n \gamma_i t^i$. Then*

- (1) $\gamma_1 \geq 0$,
- (2) if $\gamma_1 = 0$ then X is a cross-polytope.

Proof: Part (1) is equivalent to the condition that X has at least $2n$ vertices. We prove this by induction on the dimension of X . Take two vertices v and w not joined by an edge. The vertices adjacent to v are vertices of the link of v , and, by induction, there is at least $2(n-1)$ of them. Together with v and w there is $2n$ of them, as desired. This also proves (2). \square

2.2 CHARNEY–DAVIS CONJECTURE

Conjecture 2.1.7 is a strengthening of the well known

Charney–Davis Conjecture [CD]. *If X is a flag GHS $^{2n-1}$ then $(-1)^n h_X(-1) \geq 0$.*

Precisely, $(-1)^n h_X(-1)$ equals to the highest coefficient of γ_X . Indeed (2.1.5) can be rewritten as

$$\left(\frac{1}{t}\right)^n h_X(t) = \left(\frac{(1+t)^2}{t}\right)^n \gamma_X\left(\frac{t}{(1+t)^2}\right).$$

Passing to the limit $t \rightarrow -1$ we obtain the claim.

The Charney–Davis Conjecture, obvious for $n = 1$, is proven for $n = 2$ [DO]. It is motivated by its being a consequence of the Euler Characteristic Conjecture which says that the sign of the Euler characteristic of a $2n$ -dimensional closed aspherical manifold is $(-1)^n$.

Lemma 2.2.1. *The sum of the f -polynomials of the links of all simplices with k vertices of any simplicial complex X is equal to $f_X^{(k)}/k!$.*

Proof: For any $\tau \in \text{Lk}_\sigma$ define $\tau^* = \tau \cup \sigma$. We have

$$\sum_{\substack{\sigma \in X \\ \#\sigma=k}} f_{\text{Lk}_\sigma}(t) = \sum_{\substack{\sigma \subset \tau^* \in X \\ \#\sigma=k}} t^{\#\tau^*-k} = \sum_{\tau^* \in X} \binom{\#\tau^*}{k} t^{\#\tau^*-k} = \frac{f_X^{(k)}(t)}{k!}.$$

\square

Corollary 2.2.2. *If X is even-dimensional GHS then the highest coefficient of γ_X is non-negative provided the Charney–Davis Conjecture is true for links of vertices of X .*

Proof: Let $\dim X = 2n$. By definition

$$\sum_{i=0}^n \gamma_i t^i (t+1)^{2n-2i+1} = (t-1)^{2n+1} f_X \left(\frac{1}{t-1} \right).$$

Differentiate and substitute $t = -1$. Since $f_X(-1/2) = 0$ (by the Dehn–Sommerville relations) one observes that

$$\gamma_n = (-1)^n (1/2)^{2n-1} f'_X(-1/2) \geq 0,$$

where the last inequality follows from the fact that $2^{2n} f_Y(-1/2) = h_Y(-1)$ for any $(2n-1)$ dimensional link Y of any vertex in X . \square

Corollary 2.2.3. *If X is flag GHS of dimension less than five then the Conjecture 2.1.7 holds.*

Proof: Let $(n-1)$ be the dimension of the GHS. Observe that if $n \leq 5$ then γ_X is at most quadratic. Write $\gamma_X(t) = 1 + \gamma_1 t + \gamma_2 t^2$. The linear term γ_1 is nonnegative by Lemma 2.1.14 (1). Thus the claim follows for $n \leq 3$. If $n = 4$ then γ_2 is nonnegative by the Davis–Okun Theorem. If $n = 5$ then γ_2 is nonnegative by the Davis–Okun Theorem and Corollary 2.2.2. \square

2.3 THE CD-INDEX

The reader may find the combinatorial background of this Section in [St1 Ch. III.4, St2].

Let P be a finite graded poset of rank $n+1$ with $\hat{0}$ and $\hat{1}$. Let ρ be the rank function. For any chain C of the form $\hat{0} < x_1 < \cdots < x_d < \hat{1}$ define a noncommutative monomial $u_C = \prod_{i=1}^n u_i$ in the variables a and b putting

$$u_i = \begin{cases} a & \text{if } i \neq \rho(x_k) \text{ for any } k, \\ b & \text{if } i = \rho(x_k) \text{ for some } k. \end{cases}$$

Finally let

$$\gamma_P(a, b) := \sum_C u_C$$

and

$$\Psi_P(a, b) := \gamma_P(a-b, b).$$

One may think that γ_P is a generalization of f-polynomial, while Ψ_P is that of an h-polynomial in the following sense. Define the nerve or the order complex $N(P)$ of P to be the simplicial complex with the set of vertices $P - \{\hat{0}, \hat{1}\}$ such that $C \subset P - \{\hat{0}, \hat{1}\}$ is in $N(P)$ if and only if C is a chain. Then

$$f_{N(P)}(t) = \gamma_P(1, t),$$

$$h_{N(P)}(t) = \Psi_P(1, t).$$

Note that $N(P)$ is always a flag complex.

Definition 2.3.1. A graded poset P as above is called Eulerian if for any $x < z$ one has

$$\sum_{x \leq y \leq z} (-1)^{\rho(y)} = 0.$$

Proposition [St1,St2] (Bayer–Billera, Fine, Stanley). If P is an Eulerian poset then $\Psi_P(a, b)$ may be written as $\Phi_P(c, d)$ in $c = a + b$ and $d = ab + ba$.

The noncommutative polynomial Φ_P is called the cd-index of P . It follows directly from the the definition that

$$\gamma_{N(P)}(t) = \Phi_P(1, 2t).$$

Remark 2.3.2. E. Babson was the first to notice that $h_{N(P)}(-1) = \Psi_P(0, -2)$, i.e. if $n = 2m$ then $(-1)^m h_{N(P)}(-1)$ equals 2^m times the coefficient of d^m in Ψ_P .

The main conjecture on the cd-index is the following

Conjecture 2.3.3 [St2, Conj. 2.1]. The coefficients of the cd-index of any Gorenstein* poset are nonnegative.

As proven by Stanley [St2], if P is a face poset of an S-shellable cell (e.g. a convex cell), then Conjecture 2.3.3 holds for P . In this case $N(P)$ is the barycentric subdivision of that cell.

Corollary 2.3.4. If X is the barycentric subdivision of an S-shellable cell then Conjecture 2.1.7 holds.

Recently Karu [Kar] has announced the proof of Conjecture 2.3.3. As a corrolary we obtain

Corollary 2.3.5. If X is the barycentric subdivision of a regular CW-sphere then Conjecture 2.1.7 holds.

2.4 EDGE SUBDIVISION.

Definition 2.4.1. Let X be a simplicial complex. Let $\eta = \{s, t\}$ be an edge. Define $\text{Sub}_\eta(X)$ to be a simplicial complex constructed from X by bisection of all simplices containing η . In other words let e be any letter not in the vertex set S of X . Then $S \cup \{e\}$ is the vertex set of $\text{Sub}_\eta(X)$ and

$$\text{Sub}_\eta(X) = \{\sigma | \eta \not\subset \sigma \in X\} \cup \{\sigma \cup \{e\}, \sigma \cup \{s, e\}, \sigma \cup \{t, e\} | \sigma \in \text{Lk}_\eta\}.$$

We say that $\text{Sub}_\eta(X)$ is the subdivision of X along η .

The geometric realizations of X and $\text{Sub}_\eta(X)$ are homeomorphic. In particular if X is a sphere triangulation then so is $\text{Sub}_\eta(X)$.

Proposition 2.4.2. *Assume that X is a convex sphere triangulation. Then $\text{Sub}_\eta(X)$ may be realized as a convex sphere triangulation.*

Proof: Take any vector v starting at the midpoint of η and pointing inside X . Taking e to be a sufficiently small translation of the midpoint of η in the direction of $-v$ we obtain that the boundary of the convex hull of the vertices of X and e is a (convex) realization of $\text{Sub}_\eta(X)$. \square

Proposition 2.4.3. *Subdividing a GHS X along an edge η affects h and γ as follows*

$$(2.4.4) \quad \begin{aligned} h_{\text{Sub}_\eta(X)}(t) &= h_X(t) + t h_{\text{Lk}_\eta}(t), \\ \gamma_{\text{Sub}_\eta(X)}(t) &= \gamma_X(t) + t \gamma_{\text{Lk}_\eta}(t). \end{aligned}$$

Proof: Clearly $h_{\text{Sub}_\eta(X)} - h_X$ does not depend on X but only on the link of η . Therefore it suffices to check (2.4.4) in one particular case. Define X to be the join of Y and the k -gon. Let η be an edge of a the k -gon. Then $\text{Lk}_\eta = Y$ and $\text{Sub}_\eta(X)$ is the join of Y and the $(k + 1)$ -gon. Thus the claim follows from Remark 2.1.9 and simple calculation that shows that if Δ_m is an m -gon, then

$$(2.4.5) \quad \begin{aligned} f_{\Delta_m}(t) &= 1 + m(t + t^2), \\ h_{\Delta_m}(t) &= 1 + (m - 2)t + t^2, \\ \gamma_{\Delta_m}(t) &= 1 + (m - 4)t. \end{aligned}$$

\square

Proposition 2.4.6. *If X is flag then so is $\text{Sub}_\eta(X)$.*

Proof: Let σ be a clique in $\text{Sub}_\eta(X)$. If $e \notin \sigma$, then σ is a clique in X . Thus $\sigma \in X$, but then, by construction, $\sigma \in \text{Sub}_\eta(X)$.

Now assume that $e \in \sigma$. Define $\sigma_* := \sigma - \{e, s, t\}$. Each of the vertices in σ_* being connected by an edge to e has to be connected to s and t . Thus σ_* is a clique in Lk_η , and, in particular, a simplex in X .

On the other hand $\eta \not\subset \sigma$, since, by definition, η is not an edge of $\text{Sub}_\eta(X)$. Thus σ is of the form $\sigma_* \cup \{t, e\}$, $\sigma_* \cup \{e\}$ or $\sigma_* \cup \{e, s\}$. Therefore $\sigma \in \text{Sub}_\eta(X)$. \square

Corollary 2.4.7. *If X and Lk_η satisfy Conjecture 2.1.7, then so does $\text{Sub}_\eta(X)$.*

3. REAL ROOTS

3.1 THE CONJECTURE

Stating Conjecture 2.1.7 we were motivated by

The Real Root Conjecture. *The zeroes of the h -polynomial of a flag GHS are all real.*

Remark 3.1.1. The Real Root Conjecture implies Conjecture 2.1.7 by the following argument. Assume that a reciprocal polynomial h and a polynomial γ are related by (2.1.2). Then h has only real negative roots if and only if the same holds for γ because $t/(1+t)^2$ is real negative or infinite if and only if t is real negative. The “if” part is obvious. Conversely, if $t/(1+t)^2 = 1/(\sqrt{t} + (\sqrt{t})^{-1})^2$ is real negative then $\sqrt{t} + (\sqrt{t})^{-1}$ is purely imaginary. Since $\operatorname{Re}(z)$ and $\operatorname{Re}(z^{-1}) = \operatorname{Re}(z)/|z|^2$ have the same sign, \sqrt{t} has to be purely imaginary. Thus t is real negative.

The Real Root Conjecture was stated by T. Januszkiewicz in a series of questions concerning L^2 -cohomology of buildings (the details may be found in [DDJO]).

Independently, V. Reiner and V. Welker observed that the positive answer would prove the Neggers-Stanley Conjecture for graded naturally labeled posets of width 2 (details in [RW]).

Remark 3.1.2. As pointed out by the editor there have been recently announced counterexamples to the Stanley–Neggers Conjecture for the general case [Br] and for the naturally labeled one [Ste].

Theorem 3.1.3. *The Real Root Conjecture is true if the dimension of the GHS is less than five.*

Proposition 3.1.4. *Let $h_X = \sum_{i=0}^n h_i t^i$ be the h -polynomial of a flag GHS $^{n-1}$. Then*
 (1) *among the roots of h_X with the smallest modulus there is a real negative one, and*
 (2) *if this root is equal to -1 , then X is a cross-polytope.*

Proof: By the definition of the h -polynomial

$$(3.1.5) \quad \frac{(1+t)^n}{h_X(-t)} = \left(f_X \left(\frac{-t}{1+t} \right) \right)^{-1}.$$

In Section 3.3, for the sake of the self consistency of the exposition, we will recall a proof of the following well known

Proposition 3.1.6. *All coefficients in the power series expansion of (3.1.5) are positive.*

As a corollary we obtain that there is a pole of (3.1.5) at its convergence radius. Since the modulus of any pole is not smaller than the convergence radius, among the poles of (3.1.5) with the smallest modulus there is a real one. As the poles of (3.1.5) have opposite values to the zeroes of h_X , this proves (1).

If the root with the smallest modulus is -1 , then by reciprocity, it has the greatest modulus, and it follows that all roots lie on the unit circle. Thus the linear term, which is the negative of the sum of these roots, is less than or equal to the degree. However, Lemma 2.1.14(1) gives the opposite inequality. The equality is possible only when all roots are equal to -1 . Thus (2) follows from Lemma 2.1.14(2). \square

Corollary 3.1.7. *If X is not a cross-polytope, then γ_X has a negative real root.*

Proof of Theorem 3.1.3: By Remark 3.1.1 we need to check that γ_X has only real negative roots. If the dimension of the flag GHS X is less than five, then γ_X is at most quadratic. As γ_X has real coefficients and, by previous corollary, at least one real root (if γ_X is not constant), it cannot have a single non-real root. Since, by Corollary 2.2.3, all the coefficients of γ_X are nonnegative, it cannot have a real positive root. Thus the claim. \square

3.2 LOW DIMENSIONAL GEOGRAPHY

In this section we prove a partial converse of Theorem 3.1.3.

A quadratic polynomial $1 + h_1 t + t^2$ has only real negative roots if and only if $h_1 \geq 2$. Then it is the h-polynomial of an $(h_1 + 2)$ -gon.

A reciprocal polynomial H of odd degree is of the form $H(t) = (1 + t)h(t)$ for some reciprocal polynomial h . If H has only negative roots then so does h . If h is the h-polynomial of some complex X , then H is the h-polynomial of a suspension of X (X joined with S^0).

Therefore a reciprocal monic polynomial with integer coefficients of degree at most three is the h-polynomial of a flag sphere triangulation if and only if it has negative real roots. This is almost true if the degree is four (or five, by the previous remark). Namely, we have

Theorem 3.2.1. *Let γ be a quadratic polynomial with constant term 1 and integer coefficients. Assume that $\gamma(t) - t$ has only negative real roots. Then $h(t) = (1 + t)^4 \gamma(t / (1 + t)^2)$ is the h-polynomial of a flag sphere triangulation.*

Is the condition that $\gamma(t) - t$ has only negative real zeroes essential? In other words, is there a monic reciprocal polynomial of degree four with natural coefficients having only negative real roots such that h is not the h-polynomial of a flag sphere triangulation? The smallest example not covered by the above Theorem is $1 + 9t + 21t^2 + 9t^3 + t^4$. It is the average of the h-polynomials of a join of a 5-gon and an 8-gon and a join of a 6-gon and a 7-gon. However we state a

Conjecture 3.2.2. *Assume that X is a flag triangulation of S^3 such that $\gamma_X(t) - t$ has some non-real roots. Then X is a join of two polygons.*

For fixed h_2 there is at most one monic reciprocal polynomial with real negative roots such that $\gamma(t) - t$ has some non-real roots. It is not an h-polynomial of a join of polygons for $h_2 = 21, 25, 31, 35, 36, 41, 43, 48, 49, 54, \dots$

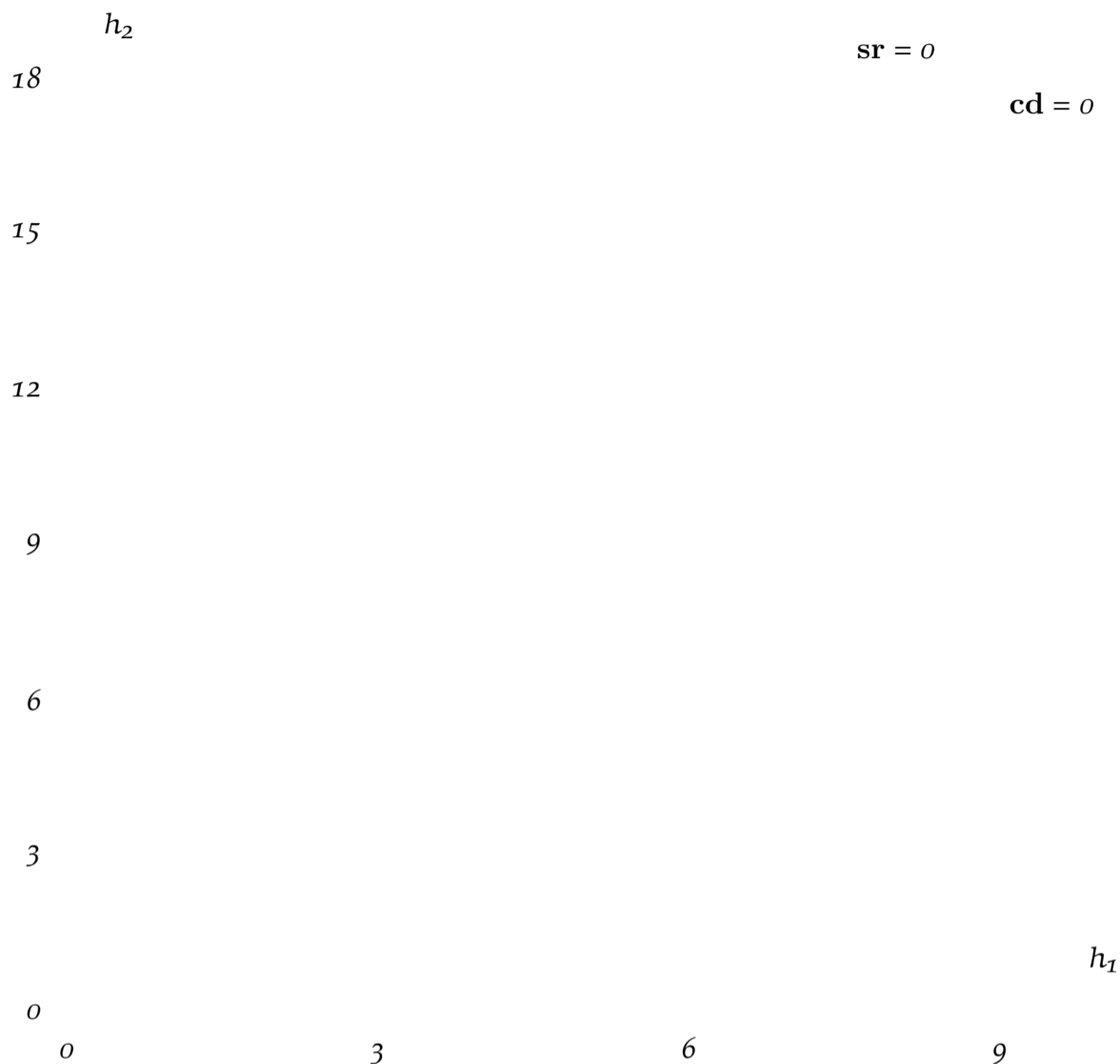
Lemma 3.2.3. *A polynomial $h(t) = 1 + h_1 t + h_2 t^2 + h_1 t^3 + t^4$ has only real negative roots if and only if*

$$\begin{aligned} \text{cd} &:= h_2 - 2h_1 + 2 \geq 0, \\ \text{sr} &:= h_1^2 - 4(h_2 - 2) \geq 0, \\ &h_1 \geq 4. \end{aligned}$$

Proof: First note that if $h(t) = (1 + t)^4 + \gamma_1 t(1 + t)^2 + \gamma_2 t^2$ then by Remark 3.1.1 we have to show that $\gamma(t) = 1 + \gamma_1 t + \gamma_2 t^2$ has only real negative roots.

γ has only real roots if and only if the discriminant $sr = \gamma_1^2 - 4\gamma_2$ is nonnegative. Then the roots are negative if and only if $\gamma_2 = cd$ and $\gamma_1 = h_1 - 4$ are nonnegative. \square

The plot below shows regions, in which various configurations of roots appear. White dots mark double roots.



The line $cd = 0$ describes the polynomials that have zero at -1 (of even multiplicity, since h is reciprocal). The other pair of zeroes has to be real or lie on the unit circle. In other words the curve separates the regions with positive and negative value of $h(-1)$ (as in the Charney–Davis Conjecture).

The curve $sr = 0$ describes the polynomials that have a pair of double roots. They have to be real or lie on the unit circle. In other words the curve is the border of the region where the smallest modulus root is real or all the roots have modulus one.

The above two curves in the (h_1, h_2) -plane are tangent at the point $(4, 6)$. The third inequality separates two domains. In the first all roots are real, while in the second they all lie on the unit circle.

Proof of Theorem 3.2.1: Recall that the h-polynomial is multiplicative with respect to the join and the h-polynomial of an m -gon equals $1 + (m - 2)t + t^2$.

Since $h_1 = f_1 - 4$ is a natural number, one has

$$\left\lfloor \frac{h_1^2}{4} \right\rfloor = \left\lfloor \frac{h_1}{2} \right\rfloor \left\lfloor \frac{h_1}{2} \right\rfloor = \left(h_1 - \left\lfloor \frac{h_1}{2} \right\rfloor \right) \left\lfloor \frac{h_1}{2} \right\rfloor = \max_{k \in \mathbb{N}} (h_1 - k) k.$$

Let $\alpha_k = h_2 - 2 - k(h_1 - k)$. In particular $\text{cd} = \alpha_2$. Define

$$C_k = \{(h_1, h_2) \in \mathbb{Z}^2 : \alpha_{k-1} \geq 0 \geq \alpha_k\},$$

for $k \geq 3$ and $C_2 = \{(h_1, h_2) \in \mathbb{Z}^2 : \alpha_2 = 0, h_1 \geq 4\}$. Then

$$\left\{ (h_1, h_2) \in \mathbb{Z}^2 : 6 \leq 2h_1 - 2 \leq h_2 \leq \frac{h_1^2}{4} + 2 \right\} = \bigcup_{k=2}^{\infty} C_k.$$

C_k is a cone with the vertex at the point $(2k - 1, k^2 - k + 2)$, which corresponds to the h-polynomial of the join of a $(k + 1)$ -gon and a $(k + 2)$ -gon and is generated over \mathbb{N} by the primitive vectors $(1, k - 1)$ and $(1, k)$.

Assume that $(h_1 - 1, h_2 - 2) = (2k - 1, k^2 - k + 2) + a(1, k - 1) + b(1, k) \in C_k$. Let X be the join of a $(k + 1)$ -gon and a $(k + 2)$ -gon subdivided along an edge whose link is a quadrilateral. X has two disjoint edges whose links are a k -gon and a $(k + 1)$ -gon. Subdividing the former a times and the latter b times we obtain a triangulation with the desired h-polynomial. \square

Theorem 3.2.1 should be compared to what is known about the h-polynomials of arbitrary (not necessarily flag) sphere triangulations.

Theorem 3.2.4. *A reciprocal polynomial $h(t) = \sum_{i=0}^n h_i t^i$ of degree $n = 4$ or 5 is the h-polynomial of a triangulation of a sphere if and only if*

$$h_1(h_1 + 1)/2 \geq h_2 \geq h_1 \geq h_0 = 1.$$

The “if” part is due to Billera and Lee [BL]. The meaning of the inequalities is the following:

- $h_1(h_1 + 1)/2 \geq h_2$ is equivalent to the fact that two vertices may be joined by at most one edge,
- $h_2 \geq h_1$ is a part of the Lower Bound Theorem [Ba,Kal],
- $h_1 \geq h_0$ is equivalent to the fact that a minimum of the number of vertices is achieved on a boundary of a simplex.

3.3 THE REAL ROOT CONJECTURE FOR S^5 .

Lemma 3.3.1. *If $1 + \gamma_1 t + \gamma_2 t^2 + \gamma_3 t^3 = (1 + xt)(1 + yt)(1 + zt)$ has only real roots then*

$$(3.3.2) \quad \gamma_2^2 \geq 3\gamma_3\gamma_1.$$

Proof:

$$2((xy + yz + zx)^2 - 3xyz(x + y + z)) = x^2(y - z)^2 + y^2(z - x)^2 + z^2(x - y)^2.$$

□

Note that if O is the cross-polytope then $\gamma_O(t) = 1$.

Theorem 3.3.3. *Assume that X is a flag triangulation of S^5 that has an edge η whose link is a cross-polytope and $h_X(-1) < 0$ (the cubical coefficient of γ_X is positive). Then for a sufficiently large natural number m , the m -fold subdivision $\text{Sub}_\eta^m(X)$ of X along η contradicts the Real Root Conjecture, i.e. $h_{\text{Sub}_\eta^m(X)}$ has a non-real root.*

Proof: By (2.4.4), subdivision along η increases the linear term of γ_X without changing other coefficients. However (3.3.2) for γ_X is false when $\gamma_3 = -h_X(-1)$ is positive and the linear coefficient γ_1 is sufficiently large. □

What is left to do is to exhibit a complex satisfying the hypothesis of Theorem 3.3.3.

Take the join X_1 of two pentagons. Subdivide an edge whose link is a quadrilateral to obtain X_2 . Finally let X be the join of X_2 and another pentagon. X_1 has a vertex (the new one) whose link is a cross polytope. Thus, X has an edge whose link is a cross polytope (join of the above with any edge of the pentagon).

By (2.4.4) and (2.4.5) we calculate:

$$\gamma_X(t) = (1 + t)((1 + t)^2 + t \cdot 1) = 1 + 4t + 4t^2 + t^3.$$

Therefore X satisfies the hypothesis of Theorem 3.3.3 and, twice subdivided, X becomes a counterexample to the Real Root Conjecture. In fact, it suffices to take a single subdivision, but this requires an extra check. This (smallest known) counterexample to the Real Root Conjecture has f-polynomial $f_{\text{Sub}_\eta X}(t) = 1 + 17t + 109t^2 + 345t^3 + 575t^4 + 483t^5 + 161t^6$.

The above example still satisfies Conjecture 2.1.7 and, in particular, the Charney–Davis Conjecture (cf. Corollary 2.4.7).

Corollary 3.3.4. *Taking the join with any flag sphere triangulations one finds that there are counterexamples to the Real Root Conjecture if the dimension of the sphere is greater than or equal to 5.*

Other counterexamples to the Real Root Conjecture are presented in the forthcoming paper [Ga].

3.4 THE SMALLEST ROOT.

We briefly present the classical proof of Proposition 3.1.6 that uses Coxeter groups. Another homological proof can be given by showing that Stanley–Reisner face ring $R(X)$ of a flag complex X is Koszul, thus the coefficient of (3.1.5) at t^j equals $\dim \operatorname{Tor}_j^{R(X)}(k, k)$ (see e.g. [RW, Prop. 4.13]).

With any flag complex X with vertex set S one associates a *right-angled Coxeter group* W with the following presentation

$$W = \langle S \mid s^2 = 1 \text{ for all } s \in S, st = ts \text{ for all } \{s, t\} \in X \rangle.$$

The definition of W uses only the one-skeleton of X , but the following observation links the whole X to W . Let the subgroup of W generated by $T \subset S$ be denoted by W_T . Then W_T is finite if and only if $T \in X$. In this case $W_T = (\mathbb{Z}/2)^{\#T}$.

Definition 3.4.1. Define a formal series $W(t) = \sum_{w \in W} t^{\ell(w)}$, where ℓ denotes the length function with respect to the generating set S . We call $W(\cdot)$ the growth series of W .

Proposition 3.4.2 [Se]. $W(t)$ represents a rational function. Moreover, if W is infinite, then

$$\frac{1}{W(t)} = \sum_{T \subset S} \frac{(-1)^{\#T}}{W_T(t^{-1})},$$

where T runs over subsets of S such that W_T is finite.

Corollary 3.4.3. Let W be the Coxeter group associated to a flag complex X . Then

$$(3.4.4) \quad f_X \left(\frac{-t}{1+t} \right) = \frac{1}{W(t)}.$$

Proof: If W_T is finite, then $W_T(t) = (1+t)^{\#T}$, thus Proposition 3.4.2 reduces to (3.4.4). \square This finishes the proof of Proposition 3.1.6.

Note that proving Proposition 3.1.4(1) we did not use the assumption that X is GHS. The second part of Proposition 3.1.4 allows the following generalization (one may consult [DDJO, Prop. 3.10] for ample discussion):

Proposition 3.4.5. If X is any flag complex and the radius of convergence of $W_X(\cdot)$ equals one (i.e. h_X has no zeroes in the interior of the unit disk), then X is a join of a cross-polytope and a simplex (i.e. a multiple suspension of a simplex).

Proof: Let T be any subset of S . The coefficients of $W_T(\cdot)$ are dominated by those of $W(\cdot)$. This is straightforward either by noticing that the length function on W_T is the restriction of the length function on W or by interpreting the coefficients as dimensions of Tor modules.

Thus, the convergence radius of the $W_T(\cdot)$ is greater or equal to that of $W(\cdot)$.

Let x and y be two vertices not joined by an edge (if there is no such pair then X is a simplex and we are done). We want to show that X is a suspension of its subcomplex spanned by $S - \{x, y\}$. To do this we need to show that if z is any vertex different from x and y then z is joined with both x and y .

If not, then straightforward computation, using Proposition 3.4.2, shows that the convergence radius of $W_{\{x,y,z\}}(\cdot)$ is strictly smaller than one. \square

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Wrocław, 31 December 2003/1 January 2004