

Euler characteristic of the configuration space of a complex[†]

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Abstract. A closed form formula (generating function) for the Euler characteristic of the configuration space of n particles in a simplicial complex is given.

Introduction.

The Euler characteristic, being independent of the structure of a simplicial complex on a topological space, may be computed in terms of zeroth order data, i.e. the number of cells of each dimension. We prove that the Euler characteristics of configuration spaces may be computed in terms of first order data of the underlying space, i.e. the Euler characteristics of links of cells.

When computing the Euler characteristic of the configuration space on n (ordered) particles in a simplicial complex X we mimic the inductive approach which gives the desired formula in the case of manifolds. Consider the projection $\mathcal{C}_n(X) \rightarrow X$ onto the first particle. Even though this map is not in general a fibration, we can relate the Euler characteristics of the base, the total space and the fibers of this map. A major feature here is a very interesting measure on X .

Next we restate the formula as a differential equation for the exponential generating function, which is then solved explicitly.

The result (and overall strategy) generalizes a similar result for the case of graphs, which was known to M. W. Davis, H. Glover, T. Januszkiewicz and J. Świątkowski around 1997.

o. Definitions and conventions.

For any topological space X define the *configuration space of n (ordered) particles in X* as $\mathcal{C}_n(X) := X^n - \{\xi : \xi_i = \xi_j \text{ for some } i \neq j\}$; ξ_i will be referred to as the i th *particle* of the *configuration* ξ . Define $\chi_n(X) := \chi\mathcal{C}_n(X)$ to be the Euler characteristic of $\mathcal{C}_n(X)$.

A subspace $S \subset X$ is *collared* if there is a tubular neighborhood $V \simeq S \times (-1, 1)$ of S in X . We say that a complex Y is obtained by *C(ut)&P(aste) surgery* from a complex X if one can find a collared subcomplex $S \subset X$ (one may think that X is a barycentric subdivision of some complex Z and S is transversal to Z , i.e. when restricted to any cell of Z it has codimension 1) such that Y can be obtained from X by cutting along S and then gluing again by some cellular automorphism of S . We define the *C&P equivalence relation* on the category of finite complexes as the equivalence relation generated by C&P surgery and subdivision. By \mathcal{CP} we denote the Grothendieck ring generated by C&P classes of finite complexes with disjoint union as sum and with Cartesian product as multiplication (the class of X will be denoted $[X]$).

Similarly, in the topological category we define *t(opological)C&P surgery*, and the *(topological) tCP ring*.

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Fact. The Euler characteristic is $t\mathcal{C}\&P$ invariant and may be regarded as a ring homomorphism $\chi: t\mathcal{C}\mathcal{P} \rightarrow \mathbb{Z}$.

Proof : From the exact sequence of a pair it follows that $\chi(X) = \chi(V) - \chi(X, V)$ and from the excision lemma $\chi(X, V) = \chi(X - S, V - S)$. Since the pairs $(X - S, V - S)$ and $(Y - S, V - S)$ are homeomorphic the proof is complete. \square

We will often use a special surgery (called an *amputation*): let $A \subset Z$ be an open subset with collared boundary; take $X = Z \sqcup (I \times \partial A)$, $S = \partial A \sqcup (\{1/2\} \times \partial A)$ and let the automorphism interchange both copies of ∂A ; then $Y = (Z - A) \sqcup \bar{A}I$, therefore $[Z] = [Z - A] + [\bar{A}] - [I \times \partial A]$.

1. The integral formula.

From now on, let X denote a finite simplicial complex. Let CX denote the cone over X . If σ is a cell of X , define:

- d_σ to be the dimension of σ ,
- L_σ to be the normal link of σ ,
- $\langle \sigma \rangle$ to be a sufficiently small contractible (open) neighborhood of the center of σ .

Theorem 1. Let μ be the measure¹⁾ on X given by $\mu(\sigma) = 1 - \chi L_\sigma$ for any cell σ . Then

$$(*) \quad \chi_n(X) = \int_X \chi_{n-1}(X - \langle \sigma \rangle) \mu(\sigma).$$

Remark : The above formula is obvious when X is a manifold, since then π is a bundle and the Euler characteristic of the total space is the product of the Euler characteristics of the fiber and the base. In this special case (*) reads: $\chi_n(X) = \chi_{n-1}(X - U)\chi(X)$, where U is a small open ball.

In the general case the fiber of π changes when ξ_1 approaches faces of non-zero codimension. The idea of the proof is to write a deformation retraction that corrects the picture at the vertices, use amputation surgery, and continue inductively.

Proof of Theorem 1. : Consider on X an auxiliary metric d compatible with the simplicial structure, such that each cell is a regular simplex with a side of length δ . Let $B_\delta(x)$ and $B_\delta(A)$ denote the open metric balls of radius δ around $x \in X$ and $A \subset X$ respectively.

We will denote by $\pi: \mathcal{C}_n(X) \rightarrow X$ the projection onto the first factor.

Let us define a *projection* p_k onto the k skeleton $X^{(k)}$ (it is defined only for some x from X , namely for those which are close to $X^{(k)}$, and far from $X^{(k-1)}$). We set $p_k(x)$ to be the point of $X^{(k)}$ nearest to x .

¹⁾ One should take the σ -algebra generated by the cells of X ; on the other hand μ may be understood as a functional on the space of functions which are constant on each open cell.

Put $\varepsilon := 1/8$; define inductively sequences of spaces:

(1) $A_k \subset \mathcal{C}_n(X)$ consists of the configurations such that $\text{dist}(\xi_{\mathcal{I}}, X^{(l)}) \geq \varepsilon^l$ for $l < k$ (roughly speaking $\xi_{\mathcal{I}}$ is far from $X^{(k-1)}$).

(2) $D_k \subset A_k$ consists of the configurations such that if $\text{dist}(\xi_{\mathcal{I}}, X^{(k)}) < \varepsilon^k$ then $\xi_{\mathcal{I}}$ is the only particle in $B_{\varepsilon^k}(p_k(\xi_{\mathcal{I}}))$.

For any σ such that $d_\sigma = k$ put $N_\sigma = B_{\varepsilon^{k/2}}(\sigma) \cap \pi(D_k)$ (N_σ is homeomorphic to $\sigma \times CL_\sigma$, its boundary in $\pi(D_k)$ is homeomorphic to $\sigma \times L_\sigma$). Then $\pi^{-1}(N_\sigma) \cap D_k$ is homeomorphic to the product $N_\sigma \times \mathcal{C}_{n-1}(X - \langle \sigma \rangle)$. Amputating all the N_σ 's we write

$$[D_k] = [E_k] + \sum_{\sigma: d_\sigma=k} [\sigma]([CL_\sigma] - [L_\sigma \times I])[C_{n-1}(X - \langle \sigma \rangle)],$$

and thus

$$\chi D_k = \chi E_k + \sum_{\sigma: d_\sigma=k} (\mathbf{1} - \chi L_\sigma) \chi_{n-1}(X - \langle \sigma \rangle),$$

where $E_k = \pi^{-1}(X - \bigcup N_\sigma) \cap D_k$.

We will complete the proof when we show that $\chi D_k = \chi E_{k-1}$. This will be done in two steps.

By *polar coordinates* on $CY = [0, \delta] \times Y / \{o\} \times Y$ (we will also denote the class of $\{o\} \times Y$ by o) we mean a pair of functions: the *modulus* $|\cdot| : CY \rightarrow [0, t]$, and the *argument* $\text{Arg} : (CY - o) \rightarrow Y$ which are the projections onto factors.

Step 1. D_k is a deformation retract of A_k (thus $\chi A_k = \chi D_k$).

We can assume that $\xi_{\mathcal{I}} \in B_{\varepsilon^k}(\sigma)$. Note that $B_{\varepsilon^k}(p_k(\xi_{\mathcal{I}}))$ is a cone over its boundary (in X). Consider polar coordinates on it.

For configuration ξ define $\rho := \min\{d(\xi_i, p_k(\xi_{\mathcal{I}})) : i > 1\}$. Both $\xi_{\mathcal{I}}$ and some other particle are in $B_{\varepsilon^k}(p_k(\xi_{\mathcal{I}}))$ iff $\rho < \varepsilon^k$. If this is the case (if not, then $\xi \in D_k$) define $r := \max(\rho, d(\xi_{\mathcal{I}}, p_k(\xi_{\mathcal{I}})))$.

We need a continuous function $n : [0, 2\varepsilon^k] \times (0, \varepsilon^k] \rightarrow [0, 2\varepsilon^k]$ such that $n(\cdot, R)$ is a homeomorphism (preserving endpoints) of $[0, 2\varepsilon^k]$ for any R , $n(R, R) = \varepsilon^k$, and $n(\cdot, \varepsilon^k) = \text{id}$. An example of such a function is

$$n(t, R) := \frac{2\varepsilon^{2k} - R^2}{R(2\varepsilon^k - R)} t + \frac{R - \varepsilon^k}{R(2\varepsilon^k - R)} t^2.$$

Define

$$\lambda_{\xi, \theta}(y) := \begin{cases} n(|y|, (\mathbf{1} - \theta)r + \theta\varepsilon^k) \text{Arg}(y) & \text{if } y \in B_{\varepsilon^k}(p_k(\xi_{\mathcal{I}})), \\ y & \text{otherwise.} \end{cases}$$

For each θ this is a homeomorphism of X continuously depending on ξ .

The desired deformation retraction is now given by the formula

$$\Lambda_\theta(\xi_1, \dots, \xi_n) := (\lambda_{\xi, \theta}(\xi_1), \dots, \lambda_{\xi, \theta}(\xi_n)).$$

Step 2. A_{k+1} is a deformation retract of E_k (thus $\chi E_k = \chi A_{k+1}$).

For simplicity of notation we will assume that $B_{2\varepsilon^k}(\sigma)$ is a metric product $(\sigma \cap \pi(D_k)) \times CL_\sigma$ (to do this one has to change the metric a bit in the neighborhood of σ). Consider polar coordinates on CL_σ .

We need a continuous function $n : [\varepsilon^k, 2\varepsilon^k] \times [0, \varepsilon^k/2) \rightarrow [\varepsilon^k, 2\varepsilon^k]$ such that $n(\cdot, R)$ is an embedding of $[\varepsilon^k, 2\varepsilon^k]$ into itself, $n(2\varepsilon^k, \cdot) = 2\varepsilon^k$, if $R > 0$ then $n(\cdot, R) > \varepsilon^k$, and $n(\cdot, 0) = id$. An example of such a function is

$$n(t, R) := \frac{\varepsilon^k - R}{\varepsilon^k} t + \frac{2R}{\varepsilon^k}.$$

Define

$$\lambda_{\xi, \theta}(y) := \begin{cases} (p_k(y), n(|y|, \theta(\varepsilon^k - |\xi_I|))\text{Arg}(y)) & \text{if } y \in B_{2\varepsilon^k}(\sigma), \\ y & \text{otherwise,} \end{cases}$$

and

$$\zeta_\theta := (p_k(\xi_I), (|\xi_I| + \theta(\varepsilon^k - |\xi_I|))\text{Arg}(\xi_I)).$$

The desired deformation retraction is now given by the formula

$$\Lambda_\theta(\xi_1, \dots, \xi_n) := (\zeta_\theta, \lambda_{\xi, \theta}(\xi_2), \dots, \lambda_{\xi, \theta}(\xi_n)).$$

□

2. Generating function

Definition. Let $\varepsilon u_X(t) := \sum \chi_n(X) t^n / n!$.

We use this particular generating function since it satisfies an interesting differential equation. Namely, it is clear that Theorem 1 can be restated as the differential equation

$$(\star) \quad \varepsilon u'_X = \int_X \varepsilon u_{X - \langle \sigma \rangle} \mu(\sigma).$$

A preliminary step to solve (\star) is

Proposition 1. εu is a group homomorphism of the additive group of \mathcal{CP} to the group of units $\mathbb{Z}[[t]]^\times$.

Proof: First we prove that this is a homomorphism. For any configuration of n particles in $X \sqcup Y$ there is a partition $I \sqcup J = \{1, \dots, n\}$ such that $x_i \in X$ for $i \in I$ and $x_j \in Y$ for $j \in J$, i.e.

$$\mathcal{C}_n(X \sqcup Y) \simeq \bigsqcup_{I \sqcup J = \{1, \dots, n\}} \mathcal{C}_{\#I}(X) \mathcal{C}_{\#J}(Y).$$

Therefore

$$\frac{\chi_n(X \sqcup Y)}{n!} = \frac{1}{n!} \sum_k \binom{n}{k} \chi_k(X) \chi_{n-k}(Y) = \sum_k \frac{\chi_k(X)}{k!} \cdot \frac{\chi_{n-k}(Y)}{(n-k)!}.$$

We prove inductively that χ_n is C&P invariant. Let Y be obtained from X by C&P surgery along S . To show that $\chi_n(X) = \chi_n(Y)$ using (\star) , it is sufficient to check that for any simplex σ its measure in X and Y is the same, and $X - \langle \sigma \rangle$ and $Y - \langle \sigma \rangle$ are C&P related. This is obvious when σ is not in S . If σ does belong to S then its link in X is the same as a link in Y (it is the suspension of a link in S), the second statement follows from the fact that a neighborhood of S is isomorphic to $S \times (0, 1)$, and $\langle \sigma \rangle$ can be pushed away from S by some automorphism of $S \times (0, 1)$ which is the identity near the ends of the interval. Then we can apply C&P surgery along S and eventually move $\langle \sigma \rangle$ back. \square

3. Explicit formula for ϵu .

Proposition 2.

$$\frac{\epsilon u_{CX}(t)}{\epsilon u_{X \times I}(t)} = 1 + (1 - \chi X)t.$$

Remark : The results of the previous sections are still valid if we assume that cells of X are products of simplices. Therefore we can put a product structure on $X \times I$. The cells of CX are cones of cells of X and their sides.

Proof of Proposition 2. : Let ρ be a cell of CX different from the vertex of the cone. Then

$$[CX - \langle \rho \rangle] = [CX] + [X \times I - \langle \rho \rangle] - [X \times I],$$

i.e.

$$\epsilon u_{CX - \langle \rho \rangle} = \frac{\epsilon u_{CX}}{\epsilon u_{X \times I}} \epsilon u_{X \times I - \langle \rho \rangle}.$$

If ρ is a vertex of a cone then $CX - \langle \rho \rangle = X \times I$. Combining the above equalities with (\star) we obtain

$$\begin{aligned} \epsilon u'_{CX} &= (1 - \chi X) \epsilon u_{X \times I} + \int_{X \times [0,1]} \frac{\epsilon u_{CX}}{\epsilon u_{X \times I}} \epsilon u_{X \times I - \langle \rho \rangle} \mu(\rho) \\ &= (1 - \chi X) \epsilon u_{X \times I} + \frac{\epsilon u_{CX} \epsilon u'_{X \times I}}{\epsilon u_{X \times I}}. \end{aligned}$$

The last equality follows since $\mu(\sigma \times \{1\}) = 0$ (the link is a cone) and the integral may be taken over $X \times I$. Eventually we obtain

$$\left(\frac{\epsilon u_{CX}}{\epsilon u_{X \times I}} \right)' = 1 - \chi X.$$

\square

Theorem 2. Let X be a complex. For any cell σ let d_σ and v_σ denote respectively the dimension of σ and the Euler characteristic of the normal link L_σ of σ . Then

$$(1) \quad \epsilon u_X(t) = \prod_{\sigma} (1 + (-1)^{d_\sigma} (1 - v_\sigma) t)^{(-1)^{d_\sigma}}.$$

Proof: Apply ϵu to the equality $[X - \langle \sigma \rangle] = [X] + [\partial \langle \sigma \rangle \times I] - [\overline{\langle \sigma \rangle}]$, and integrate it over X to obtain

$$\epsilon u'_X = \epsilon u_X \int_X \frac{\epsilon u_{\partial \langle \sigma \rangle \times I}}{\epsilon u_{\overline{\langle \sigma \rangle}}} \mu(\sigma).$$

Since $\overline{\langle \sigma \rangle} = C \partial \langle \sigma \rangle$, we apply Proposition 2:

$$(2) \quad \begin{aligned} (\log \epsilon u_X)'(t) &= \int_X \frac{1}{1 + (1 - \chi \partial \langle \sigma \rangle) t} \mu(\sigma) \\ &= \int_X \frac{\frac{d}{dt} \log(1 + (1 - \chi \partial \langle \sigma \rangle) t)}{1 - \chi \partial \langle \sigma \rangle} \mu(\sigma). \end{aligned}$$

Also $\partial \langle \sigma \rangle$ is the d_σ -fold suspension of L_σ , so $1 - \chi \partial \langle \sigma \rangle = (-1)^{d_\sigma} (1 - v_\sigma)$. Therefore, rewriting the right hand side we have

$$(\log \epsilon u_X)'(t) = \sum_{\sigma} (-1)^{d_\sigma} \frac{d}{dt} \log(1 + (-1)^{d_\sigma} (1 - v_\sigma) t),$$

which proves the theorem. □

Note : When $\chi \partial \langle \sigma \rangle = 1$, the corresponding summand in (2) is 0 and the factor in (1) is 1.

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