

# On Normal Subgroups of Coxeter Groups Generated by Standard Parabolic Subgroups

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**Abstract :** We discuss one construction of nonstandard subgroups in the category of Coxeter groups.

Two formulae for the growth series of such a subgroups are given.

As an application we construct a flag simple convex polytope, whose f-polynomial has non-real roots.

The central object of this paper is the growth series of a Coxeter group. Many geometric features of such a group (or any group in general) are reflected in properties of its growth series.

We describe in detail the normal closure of a standard parabolic subgroup in a Coxeter group  $W$ . It is again a Coxeter group and its Coxeter presentation is given explicitly.

We give two formulae for the growth series of such a normal subgroup. The first one is given as a specialization of a multi-variable version of the growth series of  $W$ .

The second formula works when the normal subgroup is right-angled (this can be easily checked by analysis of the Dynkin diagram of  $W$ ). The formula is based on counting special subdiagrams of the Dynkin diagram of  $W$ .

As an application we construct a flag simple convex polytope, whose f-polynomial has non-real roots.

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## 1. PRELIMINARIES ON COXETER GROUPS

**Definition.** A Coxeter system  $(W, S)$  is a group  $W$  together with a set of generators  $S$ , and presentation

$$W = \langle S \mid (st)^{m_{st}} = 1 \text{ for all } s, t \in S \rangle,$$

where  $m_{ss} = 1$  (i.e. all the generators have order two) and  $m_{st} = m_{ts} \in \{2, 3, \dots, \infty\}$  if  $s \neq t$ . One reads  $(st)^\infty = 1$  as no relation between  $s$  and  $t$  is imposed. The matrix  $m$  is called the Coxeter matrix of  $W$ . We usually ignore  $S$  and call  $W$  a Coxeter group. The subgroup of  $W$  generated by  $T \subset S$  is denoted  $W_T$ . Such a subgroup is called a standard parabolic or parabolic for short.

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Traditionally a Coxeter matrix is depicted in the following decorated graph called *Dynkin diagram*. Its vertex set is  $S$  and any two distinct vertices are joined by an edge with label  $m_{st}$ . By convention

- we omit an edge if  $m_{st} = 2$ ,
- we omit a label if  $m_{st} = 3$ ,
- we draw double edge instead of an edge labeled by 4, and
- we draw dashed edge instead of an edge labeled by  $\infty$ .

We will use the following convention for the Cayley graph: the group acts on the left, therefore  $\{g_1, g_2\}$  is an edge provided  $g_1^{-1}g_2 \in S$ . We say that an edge  $\{g_1, g_2\}$  is *marked* by the generator  $g_1^{-1}g_2$ .

**Theorem 1.1 [D2].** *Let  $W$  be a group generated by a set of involutions  $S$ . Let  $C_W$  be the Cayley graph of  $(W, S)$ . Then  $(W, S)$  is a Coxeter system if and only if for any element  $s \in S$  the fixpoint set  $(C_W)^s$  separates  $C_W$  (i.e.  $C_W - (C_W)^s$  has two components being interchanged by  $s$ ).*

*Proof:* It is a part of Theorem 2.3.3 in [D2] (cf. [D2, Definitions 2.2.1 and 2.2.10]).  $\square$

**Definition.** *A conjugate of a generator from  $S$  is called a reflection. The set of all reflections is denoted by  $R(W)$ .*

**Corollary 1.2.** *Reflections are intrinsically defined by a Coxeter system. Precisely, if  $T \subset S$  then  $R(W_S) \cap W_T = R(W_T)$ .*

*Proof:* Assume that  $r \in R(W_S) \cap W_T$ . Since the action of  $r$  is conjugate to that of some generator, the fixpoint set of  $r$  also separates  $C_W$ . Then  $r$  and the neutral element  $e$  are in different components of  $C_{W_S} - (C_{W_S})^r$ , and thus in different components of  $C_{W_T} - (C_{W_T})^r$ . Take any path joining  $e$  and  $r$  in  $C_{W_T}$ . Such a path has to intersect the separating fixpoint set  $(C_{W_T})^r$  in a midpoint of some edge  $\{w, wt\}$ . Thus  $rw = wt$  and, what follows,  $r = wt w^{-1} \in R(W_T)$ .  $\square$

**Definition.** *The length of an element  $w$  of any group with respect to a given generating set  $S$  is denoted by  $\ell(w)$  and is equal to the minimal length of a word in  $S$  representing  $w$ . A word is called reduced if its length is equal to the length of the element it represents.*

**Lemma 1.3 [T].** *If  $(W, S)$  is a Coxeter system than any word can be reduced (into any reduced word representing the same element of  $W$ ) in a sequence of moves of the following form:*

- (a) removing a subword of a form  $ss$ , or
- (b) replacing an alternating subword  $st \dots$  of length  $m_{st}$  by an alternating subword  $ts \dots$  of the same length.

*Note:* The move (a) is not symmetric, i.e. we do not need creation of a pair  $ss$ . In particular any two reduced words representing the same element may be joined by moves of the form (b) only.

## 2. NORMAL CLOSURE OF A STANDARD PARABOLIC

If  $m_{st} = 2n + 1$  is odd then  $s$  and  $t$  are conjugates since the relation reads then

$$(st)^n s(st)^{-n} = t.$$

If one wants to find the normal closure of a subgroup  $W_T$  it is convenient to incorporate in  $T$  all the generators conjugate to those already in  $T$ . This clarifies the assumption of the following

**Proposition 2.1.** *Assume that  $T \subset S$  is such that for any  $t \in T$  and  $s \in S - T$  the exponent  $m_{st}$  is even (or  $\infty$ ). Then*

- (1) *there exists a well defined homomorphism  $\phi_T: W \rightarrow W_{S-T}$  that is the identity on  $W_{S-T}$  and sends  $W_T$  to the identity element,*
- (2) *the normal closure  $\overline{W_T}$  of  $W_T$  is the kernel of  $\phi_T$ ,*
- (3)  *$\overline{W_T}$  is generated by  $S_T$ , the set of all conjugates of  $T$  with respect to  $W_{S-T}$ ,*
- (4) *the Cayley graph  $C_{\overline{W_T}}$  of  $(\overline{W_T}, S_T)$  is made from the Cayley graph  $C_W$  of  $W$  by collapsing all the edges marked by the generators not in  $T$ ,*
- (5)  *$(\overline{W_T}, S_T)$  is a Coxeter system.*

*Proof:* Let  $t \in T$  and  $s \notin T$ , then the relation  $st \dots = ts \dots$ , after substituting the identity for  $s$ , becomes trivial exactly when  $m_{st}$  is even. Thus (1) follows. This also proves that the relation “ $s$  and  $t$  are conjugate” is the equivalence closure of the relation “ $m_{st}$  is odd”.

Any element  $w$  of  $W$  may be written as follows

$$w_0 t_1 w_1 \dots t_n w_n = (w_0 t_1 w_0^{-1}) (w_0 w_1 t_2 (w_0 w_1)^{-1}) \dots (w_0 \dots w_{n-1} t_n (w_0 \dots w_{n-1})^{-1}) (w_0 \dots w_n)$$

where  $t_i \in T$  and  $w_i \in W_{S-T}$ . If  $e = \phi_T(w) = w_0 \dots w_n$  then  $w$  is expressible in  $S_T$ , therefore (2) and (3).

Let  $g_1 u_1, g_2 u_2$ , where  $g_i \in \overline{W_T}$  and  $u_i \in W_{S-T}$ , be the endpoints of an edge corresponding to some generator in  $T$ , i. e.  $(g_1 u_1)^{-1} g_2 u_2 \in T$ . In particular  $e = \phi_T(u_1^{-1} g_1^{-1} g_2 u_2) = \phi_T(u_1^{-1} u_2)$ . Therefore  $u_1 = u_2$ . Furthermore

$$g_1^{-1} g_2 = u_1 ((g_1 u_1)^{-1} (g_2 u_1)) u_1^{-1} \in S_T.$$

On the contrary, if  $g_1^{-1} g_2 = u t u^{-1}$ , then the edge  $\{g_1, g_2\}$  in  $C_{\overline{W_T}}$  is covered by the edge  $\{g_1 u, g_2 u\}$  in  $C_W$ . This proves (4).

Let  $t \in S_T$ . By Theorem 1.1 we need to show that the fixpoint set of  $t$  separates  $C_{\overline{W_T}}$ .

Let  $W^+$  and  $W^-$  denote the two components of  $C_W - (C_W)^t$ . Two vertices, one from  $W^+$  and the other from  $W^-$  are joined by an edge precisely when they differ by  $t$  on the right. Since  $t$  is not a conjugate of any element of  $W_{S-T}$  all the cosets of  $W_{S-T}$  have to lie in  $W^+$  or  $W^-$ . Therefore the fixpoint set of  $t$  also separates  $C_{\overline{W_T}}$ . Thus (5).  $\square$

*Remark 2.2:* If  $W_{S-T}$  is not finite, then  $\overline{W_T}$  has in general an infinite number of generators.

### 3. COXETER MATRIX OF $\overline{W_T}$

**Definition.** Define the support  $\text{supp } w$  of an element  $w$  of a Coxeter group to be the set of generators that appear in its reduced presentation. By Lemma 1.3 this does not depend on the choice of a reduced presentation.

**Definition.** Let  $T_1, T_2 \subset S$ . We write  $T_1 \perp T_2$  if  $m_{t_1 t_2} = 2$  for any  $t_1 \in T_1$  and  $t_2 \in T_2$ . If  $T \subset S$  we define  $T^\perp = \{s \in S : m_{st} = 2 \text{ for all } t \in T\}$ .

Fix  $t_1, t_2 \in T$ . Let  $w^*$  be the shortest element in the double coset  $W_{t_1^\perp - T} w W_{t_2^\perp - T}$ . Such element is unique [B, Ch. IV, §1, Ex. 3].

**Corollary 3.1.** Let  $\tau_1 = w_1 t_1 w_1^{-1}$  and  $\tau_2 = w_2 t_2 w_2^{-1}$  be two generators of  $\overline{W_T}$ . Then  $\tau_1 = \tau_2$  if and only if

$$t_1 = t_2 \quad \text{and} \quad (w_1^{-1} w_2)^* = 1.$$

If  $\tau_1 \neq \tau_2$  then we have

$$m_{\tau_1 \tau_2} = \begin{cases} m_{t_1 t_2} & \text{if } t_1 \neq t_2 \text{ and } (w_1^{-1} w_2)^* = 1, \\ \frac{m_{s t_1}}{2} & \text{if } t_1 = t_2 \text{ and } (w_1^{-1} w_2)^* = s \in S - T, \\ \infty & \text{otherwise.} \end{cases}$$

*Proof:* Set  $w = (w_1^{-1} w_2)^*$ , so

$$w_1^{-1} w_2 = \eta_1 w \eta_2$$

for some  $\eta_i \in W_{t_i^\perp - T}$ . In particular  $t_i$  commutes with  $\eta_i$

Let  $N$  be a natural number such that

$$(3.2) \quad (\tau_1 \tau_2)^N = 1.$$

Then (3.2) reads

$$\begin{aligned} 1 &= \eta_1^{-1} w_1^{-1} w_1 \eta_1 = \eta_1^{-1} w_1^{-1} \left( w_1 t_1 (w_1^{-1} w_2) t_2 (w_1^{-1} w_2)^{-1} \dots \right) w_1 \eta_1 \\ &= \eta_1^{-1} \left( t_1 (\eta_1 w \eta_2) t_2 (\eta_1 w \eta_2)^{-1} \dots \right) \eta_1 \\ &= (\eta_1^{-1} t_1 \eta_1) w (\eta_2 t_2 \eta_2^{-1}) w^{-1} \dots \\ &= t_1 w t_2 w^{-1} \dots \end{aligned}$$

Lemma 1.3 implies that  $w$  has length at most one, otherwise no cancellation is possible. Indeed, any reduced form of  $w$  does not start with a letter that commutes with  $t_1$  nor ends with a letter that commutes with  $t_2$ .

The case  $\ell(w) = 0$  is obvious. If  $w$  is a generator, then it commutes neither with  $t_1$  nor with  $t_2$ . The only possibility to use Lemma 1.3 is described by the claim.  $\square$

## 4. MULTI-VARIABLE GROWTH SERIES

The length function on a Coxeter group may be refined in the following way. Take a set of generators  $S_0 \subset S$  with the property that whenever  $s$  belongs to  $S_0$ , then all the generators conjugate to  $s$  belong to  $S_0$ .

Lemma 1.3 implies the following

**Corollary.** *The number (counted with multiplicities) of generators from  $S_0$  appearing in a reduced presentation of  $w$ , and denoted  $\ell_{S_0}(w)$ , does not depend on the presentation.*

Note that  $\ell_{S_0}$  satisfies the triangle inequality, i.e.  $\ell_{S_0}(w_1 w_2) \leq \ell_{S_0}(w_1) + \ell_{S_0}(w_2)$ .

Let  $\varphi: S \rightarrow I$  be any partition of the generating set into sets with the above property. We call such a partition *allowable*. For each  $w \in W$  define a monomial in indeterminates  $I$ :  $w_\varphi(\mathbf{x}) = \prod_{x \in I} x^{\ell_{\varphi^{-1}(x)}(w)}$ , where  $\mathbf{x} = (x)_{x \in I}$ .

For any subset  $A$  of  $W$ , the formal power series  $A_\varphi(\mathbf{x}) = \sum_{w \in A} w_\varphi(\mathbf{x})$  is called the *multi-variable growth series* of  $A \subset W$ .

The above definition needs some finiteness assumption on  $(W, S)$ . It is enough to assume that  $S$  is finite (if not, the coefficients of the growth series may be infinite). Since we also want to study the growth series of  $\overline{W_T}$ , we want to know that  $S_T$  is finite. Therefore, due to Remark 2.2, we will assume that  $W_{S-T}$  is finite.

The notation may look a little ambiguous. However, note that  $W_\varphi$  does not depend on whether  $W$  is a standard parabolic subgroup of some bigger Coxeter group or not. Similarly  $(\overline{W_T})_\varphi$  may denote two things: either the growth series of the group  $\overline{W_T}$  or the subset  $\overline{W_T} \subset W$ . In fact these two series coincide, as shown by the following

**Proposition 4.1.** *Assume that  $T \subset S$  is as in Proposition 2.1 and  $\varphi^{-1}(x_0) = S - T$ . There is a well defined prolongation  $\varphi_T$  of  $\varphi$  to  $S_T$  by requiring that  $\varphi(w t w^{-1}) = \varphi(t)$  for any  $t \in T$  and  $w \in W_{S-T}$ . It is obviously allowable. Then*

$$(\overline{W_T})_{\varphi_T}(\mathbf{x}) = (\overline{W_T})_\varphi(\mathbf{x})|_{x_0=1},$$

where the left series is the growth series of  $\overline{W_T}$  and on the right — of  $\overline{W_T} \subset W$ .

*Note:* The right hand side is well defined, i.e. there is only a finite number of nonzero coefficients at  $J x_0^m$  for given monomial  $J$  in variables different from  $x_0$ .

If  $w_\varphi(\mathbf{x}) = J x_0^m$  then  $w = w_0 t_{j_1} w_1 \dots w_{n-1} t_{j_n} w_n$ , where  $\prod \varphi(t_{j_k}) = J$  and  $w_k \in W_{S-T}$ . There is only a finite number of possibilities for  $t_{j_k}$  and  $w_k$  (notice that we have assumed that  $W_{S_T}$  is finite).

*Proof of Proposition 4.1:* Take a reduced word in  $S$  representing an element from  $\overline{W_T}$  and write it as

$$(4.2) \quad w_0 t_1 w_1 \dots t_n w_n$$

where  $t_i \in T$  and  $w_j$  are subwords in  $S - T$ .

If the following word in  $S_T$  (representing the same element)

$$(4.3) \quad (w_0 t_1 w_0^{-1}) (w_0 w_1 t_2 (w_0 w_1)^{-1}) \dots (w_0 \dots w_{n-1} t_n (w_0 \dots w_{n-1})^{-1})$$

were not reduced, one would do some cancellations in (4.3), expand the result to obtain the word in  $S$  with smaller  $\ell_{\varphi^{-1}(x)}$  for some  $x \neq x_0$  than that of (4.2). This would show that (4.2) was not reduced.

Finally (4.2) and (4.3) define the same monomial. Thus the claim.  $\square$

**Theorem 4.4.** *Under the assumptions of the preceding proposition*

$$W_{\varphi}(\mathbf{x})|_{x_0=1} = (\#W_{S-T}) \cdot (\overline{W_T})_{\varphi_T}(\mathbf{x}).$$

*Proof:* Any element  $w$  of  $W$  decomposes uniquely as  $w = w_T w_0$  where,  $w_T \in \overline{W_T}$  and  $w_0 \in W_{S-T}$ . We need to show that  $\ell_{\varphi^{-1}(x)}(w) = \ell_{\varphi^{-1}(x)}(w_T)$  for any  $x \neq x_0$ .

Since  $\ell_{\varphi^{-1}(x)}(w_0) = 0$ , by the triangle inequality we have

$$\ell_{\varphi^{-1}(x)}(w_T) \geq \ell_{\varphi^{-1}(x)}(w_T w_0) \geq \ell_{\varphi^{-1}(x)}(w_T w_0 w_0^{-1}) = \ell_{\varphi^{-1}(x)}(w_T).$$

$\square$

*Note:* Two generators of  $\overline{W_T}$  conjugate in  $W$  may not be conjugate in  $\overline{W_T}$ , thus not every multi-variable growth series of  $\overline{W_T}$  is specialization of that of  $W$ .

**Corollary.** *Assume that  $T \subset S$  is as in Proposition 2.1. Let  $\varphi: S \rightarrow \{x_0, x\}$  be such that  $\varphi(t) = x$  if and only if  $t \in T$ . Then the ordinary growth series of  $W_T$  and  $\overline{W_T}$  are computed as follows:*

$$\begin{aligned} W_T(x) &= W_{\varphi}(0, x), \\ \overline{W_T}(x) &= \frac{W_{\varphi}(1, x)}{W_{\varphi}(1, 0)}, \end{aligned}$$

where  $W_{\varphi}(x_0, x)$  is a multi-variable growth series of  $W$  associated to the allowable partition  $\varphi$ .

*Proof:*  $W_{\varphi}(1, 0) = W_{S-T}(1) = \#W_{S-T}$ .  $\square$

## 5. RIGHT-ANGLED COXETER GROUPS AND FLAG COMPLEXES

**Definition.** *The nerve  $N_W$  of a Coxeter system  $(W, S)$  is a simplicial complex consisting of  $T \subset S$  such that the subgroup  $W_T$  is finite.*

*Remark:* Some authors define the nerve as the barycentric subdivision of the above.

**Definition 5.1.** A Coxeter group is said to be right-angled if  $m_{st} \in \{1, 2, \infty\}$  for all  $s, t \in S$ .

If  $W$  is a right-angled Coxeter group, then  $W_T$  is finite if and only if  $W_{T'}$  is finite for any two-element subset  $T'$  of  $T$ , therefore  $N_W$  is the *flag completion* of its one-skeleton:  $T$  is a face of  $N_W$  if and only if its one skeleton is contained in  $N_W$ . Such a simplicial complex is called a *flag complex*.

On the other hand, let  $\Gamma$  be any graph. Let  $S$  denote the set of vertices of  $\Gamma$ . Declare

$$m_{st} = \begin{cases} 1 & \text{if } s = t, \\ 2 & \text{if there is an edge joining } s \text{ and } t, \\ \infty & \text{otherwise.} \end{cases}$$

Then  $N_W$  is the flag completion of  $\Gamma$ .

Let  $X$  be any simplicial complex. For a given function  $\varphi: S \rightarrow I$  on the set of vertices one defines the  $f$ -polynomial of  $X$  by the formula

$$f_{X,\varphi}(\mathbf{x}) := \sum_{\sigma \in X} \prod_{s \in \sigma} \varphi(s).$$

**Proposition 5.2 [S, Prop. 26].** Let  $W$  be an arbitrary Coxeter group and  $\varphi$  an allowable partition. Then  $W_\varphi(\mathbf{x})$  is a series of a rational function. Moreover, if  $W$  is infinite, then

$$\frac{1}{W_\varphi(\mathbf{x}^{-1})} = \sum_{T \subset S} \frac{(-1)^{\#T}}{(W_T)_{\varphi|_T}(\mathbf{x})},$$

where  $T$  runs over all subsets of  $S$  such that  $W_T$  is finite, and  $\mathbf{x}^{-1} = (x^{-1})_{x \in I}$ .

**Corollary 5.3.** Assume that  $W$  is a right angled Coxeter group. Since no two generators are conjugated, any function  $\varphi: S \rightarrow I$  is an allowable partition and

$$(5.4) \quad f_{N_W,\varphi} \left( \frac{-1}{1+\mathbf{x}} \right) = \frac{1}{W_\varphi(\mathbf{x}^{-1})}.$$

*Proof:* If  $W_T$  is finite, then  $(W_T)_\varphi(\mathbf{x}) = \prod_{t \in T} (1 + \varphi(t))$ , thus (5.4) follows from Proposition 5.2.  $\square$

## 6. COMPUTATION OF F-POLYNOMIAL

Corollary 3.1 implies the following

**Corollary 6.1.**  $\overline{W_T}$  is a right-angled Coxeter group if and only if

- (1) if  $s \notin T$  and  $t \in T$  then  $m_{st} \in \{2, 4, \infty\}$ , and
- (2) if  $t, t' \in T$  then  $m_{tt'} \in \{1, 2, \infty\}$ .

Let us describe the basic example which will serve as a model for the general case.

**Definition.** The Coxeter system  $(W, S)$  is of type  $B_k$  if its Dynkin diagram is the following:

$$s_k \quad s_{k-1} \quad s_{k-2} \quad \dots \quad s_4 \quad s_3 \quad s_2 \quad s_1$$

Coxeter group of type  $B_k$  is the group of symmetries of a regular  $k$ -dimensional cube. Assume that the cube has vertices with each coordinate equal to  $\pm 1$ . In this case the generator  $s_1$  corresponds to the reflection in the hyperplane defined by vanishing of the first coordinate ( $s_1$  changes the sign of the first coordinate). The group  $\overline{W_{\{s_1\}}}$  is right-angled Coxeter group generated by reflections in the hyperplanes defined by vanishing of some coordinate. We will call such a hyperplane a *coordinate hyperplane*.

The parabolic subgroup  $W_{\{s_2, \dots, s_n\}}$  is the symmetric group of  $n$  letters. It acts by permuting coordinates. Precisely,  $s_k$  transposes  $(k-1)$ -st and  $k$ -th coordinates.

**Proposition 6.2.** Let  $(W, \{s_1, \dots, s_m\})$  be of type  $B_m$ . If  $\Theta$  is a family of  $n$  commuting conjugates of  $s_1$  then  $\Theta$  may be conjugated by an element from  $W_{\{s_2, \dots, s_m\}}$  in such a way that  $\Theta$  consists of all conjugates of  $s_1$  in  $W_{\{s_1, \dots, s_n\}}$ .

*Proof:* Family of reflections in coordinate hyperplanes is defined by the intersection of these hyperplanes. Since all such intersections can be conjugated by permuting the coordinates, provided they have the same dimension, the claim follows.  $\square$

Now return to the general question. Let  $\Theta \subset S_T$  be a family of commuting generators of a right-angled group  $\overline{W_T}$ . The aim of the rest of the present section is to compute the f-polynomial of  $N_{\overline{W_T}}$ , in case when  $\overline{W_T}$  is right angled, in terms of the Coxeter matrix of  $(W, S)$  (in contrast with Theorem 4.4, which gives the f-polynomial in terms of the growth series of  $W$ ).

In order to compute the f-polynomial of the nerve we need to determine maximal families of commuting conjugates of generators.

**Theorem 6.3.** Assume that  $T$  is as in Corollary 6.1. For any commuting family  $\Theta$  of conjugates of generators there is a unique subset  $\Sigma \subset S$  of the same cardinality as  $\Theta$ , such that

- (1)  $W_\Sigma$  is finite, and if for any  $t \in T' = T \cap \Sigma$  the set of vertices of the connected component of the Dynkin diagram of  $W_\Sigma$  containing  $t$  is denoted by  $\Sigma_t$ , then:
  - $\Sigma = \bigcup_{t \in T'} \Sigma_t$ ,
  - $\Sigma_t$  is of type  $B_{k(t)}$ ,

- $t$  is the unique element of  $\Sigma_t \cap T$  and it is the distinguished generator  $s_t$ ;
- (2) there exists  $w_\Theta \in W_{S-T}$  such that

$$w_\Theta \Theta w_\Theta = \bigcup_{t \in T'} \{wtw^{-1} | w \in W_\Sigma\} = \bigcup_{t \in T'} \{wtw^{-1} | w \in W_{\Sigma_t}\}.$$

**Lemma 6.4.** *If  $G$  is a finite subgroup of a Coxeter group, then it is conjugate to a subgroup of some finite standard parabolic.*

*Proof:* Any Coxeter group acts on its Davis complex [D1] which is CAT(0) (nonpositively curved) by the theorem of Moussong [M]. The only stabilizers of that action are the conjugates of finite standard parabolics.

The claim follows, since any action of a finite group on a CAT(0) complex has a fixed point [BH].  $\square$

Obviously, since the set of parabolic subgroups is closed under the intersection, there exist unique smallest parabolic containing  $G$ .

*Proof of Theorem 6.3:* Since a family  $\Theta$  of commuting involutions generates a finite subgroup, by Lemma 6.4 we may assume that  $\Theta$  lies in a finite parabolic  $W_{S'}$ . Let  $T'$  denote a set of those elements of  $T$  whose conjugates appear in  $\Theta$ .

We claim that  $T' \subset S' \cap T$  and moreover that each element of  $\Theta$  is conjugated to some element of  $T'$  by an element of  $W_{S'}$ . Indeed, by Corollary 1.2 each element of  $\Theta$  (being a reflection) is conjugated inside  $W_{S'}$  to some generator from  $S'$ . By Corollary 6.1 no element of  $T$  is a conjugate of any other generator from  $S$ , thus the claim.

By Corollary 6.1 any generator from  $T'$  may be connected in the Dynkin diagram to any other only by an edge labeled by 4. Thus by the classification of finite Coxeter groups ([B]), the connected component  $S'_t$  of  $t \in T'$  (within the Dynkin diagram of  $W_{S'}$ ) is a graph of type  $B_k$  for some  $k$ .

Let  $S'_0 = S' - \bigcup_{t \in T'} S'_t$ . From the definition it follows that any Coxeter group is the product of its finite parabolics corresponding to the connected components of its Dynkin diagram. Thus  $W_{S'} = W_{S'_0} \times \prod_{t \in T'} W_{S'_t}$ . Since each element of  $\Theta$  is a conjugate of some element of  $T'$ , we have  $\Theta \subset \prod_{t \in T'} W_{S'_t}$ . Moreover, if  $\Theta_t$  denotes the set of those elements from  $\Theta$  which are conjugated to  $t \in T'$  then  $\Theta_t \subset W_{S'_t}$ .

By Proposition 6.2 we may find elements  $w_t \in W_{S'_t}$  such that each  $w_t \Theta_t w_t^{-1}$  consists of all conjugates of  $t$  in  $W_{\Sigma_t}$  for a suitable subset  $\Sigma_t \subset S'_t$ . Since different  $W_{S'_t}$  commute, setting  $w_\Theta = \prod_{t \in T'} w_t$  completes the proof.  $\square$

Note that  $\Theta$  determines  $\Sigma$  but  $\Sigma$  determines  $\Theta$  only up to conjugacy. In order to count all commuting families conjugate to  $\Theta$  we need to understand the action of  $W_{S-T}$  on the set of families as in Theorem 6.3 (1).

Let  $\Sigma$  be such a family. The stabilizer of  $\Sigma$  is the centralizer of  $\Sigma$  in  $W_{S-T}$ , i.e.  $W_{\Sigma_0}$ , where  $\Sigma_0 = (\bigcup_{t \in T} \Sigma_t)^\perp - T$ .

Summarizing the above we have proved

**Theorem 6.5.** *The  $f$ -polynomial of  $\overline{W_T}$  reads*

$$f_{N_{\overline{W_T}}, \varphi} = \sum_{\Sigma \subset S} [W_{S-T} : W_{\Sigma_0}] \prod_{t \in T} \frac{\varphi(t)^{k(t)}}{k(t)!}$$

Where the sum runs over all subsets  $\Sigma$  as in Theorem 6.3 (1).

This formula allows the following refinement. Assume that  $\sigma \subset \tau \in N_{\overline{W}}$ . The corresponding families  $\Sigma^\sigma$  and  $\Sigma^\tau$  satisfy  $\Sigma_t^\sigma \subset \Sigma_t^\tau$  for all  $t \in T$ . However if we set  $\sigma$  and  $\Sigma$  such that  $\Sigma_t^\sigma \subset \Sigma_t$  for all  $t \in T$ , then there are several  $\tau$  with the property  $\tau \supset \sigma$  and  $\Sigma^\tau \subset \Sigma$ .

The centralizer of  $\Sigma^\sigma$  act on the set of such  $\tau$ . thus the number of such  $\tau$  equals

$$[W_{\Sigma_0} : W_{\Sigma_0^\sigma}].$$

**Definition.** *The link  $Lk_\sigma$  of  $\sigma$  a simplex in a simplicial complex  $X$  is a subcomplex consisting of all  $\tau \in X$  such that  $\tau^* = \sigma \cup \tau \in X$  and  $\sigma \cap \tau = \emptyset$ .*

**Corollary 6.6.** *Fix a simplex  $\sigma \in N_{\overline{W_T}}$ . It is the commuting family corresponding to  $\Sigma^\sigma \subset S$ . The  $f$ -polynomial of the link of  $\sigma$  reads*

$$f_{Lk_\sigma, \varphi}(t) = \sum_{\Sigma \supset \Sigma^\sigma} [W_{\Sigma^\sigma} : W_\Sigma] \prod_{t \in T} \frac{\varphi(t)^{k(t) - k^\sigma(t)}}{(k(t) - k^\sigma(t))!}.$$

## 7. AN EXAMPLE

In this section we present a specific triangulation of a sphere (which is the boundary complex of a convex polytope), such that its  $f$ -polynomial has non-real roots. Other such examples are constructed by a different method in [G], where the reader may also find a connection of the (former) conjecture on real-rootedness of  $f$ -polynomials of such triangulations to geometry and combinatorics with further references.

Let  $W$  be defined by the following Dynkin diagram:

$$t_1 \qquad \qquad \qquad t_2$$

(7.1)

Let  $T = \{t_1, t_2\}$  consist of white dots in the Diagram 7.1. Recall that  $\overline{W_T} = \ker(W_S \rightarrow W_{S-T})$ .

**Remark 7.2.**  $N_{\overline{W_T}}$  may be realized as a convex triangulation of a sphere, i.e. the boundary complex of a convex polytope.

*Proof:* Consider the group defined by the following Dynkin diagram:

It can be realized as a finite-covolume reflection group in the hyperbolic space with the fundamental domain  $D$ , a simplex with a unique ideal vertex [B, Ch. V, §4, Ex. 17]. Let  $\overline{D} = W_{S_0} \cdot D$ . It is the convex hyperbolic polytope with some ideal vertices. Finally let  $D^*$  be  $\overline{D}$  with ideal vertices truncated. It is the polytope dual to  $N_{\overline{W_T}}$ .

Since  $D^*$ , realized in the Klein model of the hyperbolic space, is a (Euklidian) convex polytope, the polar dual of  $D^*$  realizes  $N_{\overline{W_T}}$  as a convex triangulation of a sphere [Z, Cor. 2.14].  $\square$

According to Theorem 6.5 we have to find all suitable triples  $\{\Sigma_{t_1}, \Sigma_{t_2}, \Sigma_0\}$ . They are listed in the following picture. Black dots in the connected component of  $t_i$  are elements of  $\Sigma_{t_i}$ . The remaining black dots (in the middle) are elements of  $\Sigma_0$ .

$$\sum k(t)$$

8

7

6

5

4

3

2

1

0

The orders of finite Coxeter groups are well known ([B]) Substituting the above to The-

orem 6.5 one obtains

$$\begin{aligned}
f_{N_{\overline{W_T}}}(t_1, t_2) &= 696729600 \left( \left( \frac{2t_2^7 t_1}{5040} + \frac{t_2^8}{40320} \right) + \left( \frac{t_2^6 t_1}{720} + \frac{t_2^7}{5040} + \frac{t_2^7}{2 \cdot 5040} \right) \right. \\
&\quad + \left( \frac{t_2^5 t_1}{2 \cdot 2 \cdot 120} + \frac{t_2^6}{2 \cdot 720} \right) + \left( \frac{t_2^4 t_1}{24 \cdot 24} + \frac{t_2^5}{2 \cdot 6 \cdot 120} \right) \\
&\quad + \left( \frac{t_2^3 t_1}{6 \cdot 192} + \frac{t_2^4}{24 \cdot 120} \right) + \left( \frac{t_2^2 t_1}{2 \cdot 1920} + \frac{t_2^3}{6 \cdot 1920} \right) \\
&\quad + \left( \frac{t_2 t_1}{23040} + \frac{t_2^2}{2 \cdot 51840} \right) + \left( \frac{t_1}{322560} + \frac{t_2}{2903040} \right) \\
&\quad \left. + \frac{1}{696729600} \right) \\
&= t_1(276480 t_2^7 + 967680 t_2^6 + 1451520 t_2^5 + 1209600 t_2^4 + 604800 t_2^3 \\
&\quad + 181440 t_2^2 + 30240 t_2 + 2160) \\
&\quad + (17280 t_2^8 + 207360 t_2^7 + 483840 t_2^6 + 483840 t_2^5 + 241920 t_2^4 \\
&\quad + 60480 t_2^3 + 6720 t_2^2 + 240 t_2 + 1).
\end{aligned}$$

Therefore, in particular,

$$\begin{aligned}
f_{N_{\overline{W_T}}}(t, t) &= 293760 t^8 + 1175040 t^7 + 1935360 t^6 + 1693440 t^5 + 846720 t^4 \\
&\quad + 241920 t^3 + 36960 t^2 + 2400 t + 1.
\end{aligned}$$

and

$$\frac{1}{\overline{W_T}(t)} = \frac{t^8 - 2392 t^7 + 20188 t^6 - 70504 t^5 + 107590 t^4 - 70504 t^3 + 20188 t^2 - 2392 t + 1}{(1+t)^8}$$

Thus the poles of  $\overline{W_T}(\cdot)$  (with 2 digit precision) are:  $0.41 \cdot 10^{-3}$ ,  $0.24 \pm 0.16i$ ,  $0.63$ ,  $1.6$ ,  $2.9 \pm 1.9i$ ,  $2.4 \cdot 10^3$ . This provides the first known counterexample to the Real Roots Conjecture [G].

The link  $L$  (of codimension 2) of a face corresponding to

is a triangulation of a five dimensional flag sphere. Using Corollary 6.6 we check that  $f_L(-1/2) \neq 0$ . A link  $K$  (in  $L$ ) of an edge defined by

satisfies  $f_K(t) = (1 + 2t)^4$ , which is an  $f$ -polynomial of a cross polytope (and, as follows,  $K$  is a cross polytope).

The pair  $(K, L)$  allows us to construct a triangulation of the five dimensional sphere which is a counterexample to the Real Roots Conjecture [G, Th. 5.2]. Five is also the smallest dimension when this is possible [G, Th. 3.2].

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