

# SYMPLECTIC CONFIGURATIONS

ŚWIAT GAL AND JAREK KĘDRA

ABSTRACT. We define a class of symplectic fibrations called symplectic configurations. They are a natural generalization of Hamiltonian fibrations in the sense that they admit a closed symplectic connection two-form. Their geometric and topological properties are investigated. We are mainly concentrated on integral symplectic manifolds.

We construct the classifying space  $\mathbb{B}$  of symplectic integral configurations. The properties of the classifying map  $\mathbb{B} \rightarrow \mathbb{B}\text{Symp}(M, \omega)$  are examined. The universal symplectic bundle over  $\mathbb{B}$  has a natural connection whose holonomy group is the enlarged Hamiltonian group recently defined by McDuff. The space  $\mathbb{B}$  is identified with the classifying space of a certain extension of the symplectomorphism group.

## CONTENTS

1. Motivation	1
2. Preliminaries and statements of results	2
3. The topology of symplectic configurations	7
4. The geometry of symplectic configurations	12
5. Group cohomology and the group of McDuff	16
6. The cohomology ring of symplectic configurations	21
References	24

## 1. MOTIVATION

The paper introduces a new object, called a *symplectic configuration*, which is a symplectic fibration equipped with a map of the total space into some symplectic manifold so that the restrictions of the map to the fibers are symplectic embeddings. Symplectic configurations provide a generalization of Hamiltonian fibrations in the sense that they admit a closed symplectic connection two-form [MS98, Theorem 6.21].

We are partially motivated by the result of Narasimhan and Ramanan [NR61]. They showed that the natural connection  $\lambda$  on the universal principal bundle

$$U(n) \rightarrow E \rightarrow \text{Gr}(n, \infty)$$

has the following universal property. For any principal  $U(n)$ -bundle  $P \rightarrow B$  equipped with a connection  $\lambda_P$ , there exists a map  $f: B \rightarrow \text{Gr}(n, \infty)$  such that  $(P, \lambda_P) = f^*(E, \lambda)$ . Notice that the total space  $E$  is the space of linear embeddings

---

*Date:* August 30, 2008.

*2000 Mathematics Subject Classification.* Primary 53D, 55R; Secondary 53C.

*Key words and phrases.* symplectic fibration; h-principle; classifying space;

The first author is partially supported by the State Committee for Scientific Research (grant no. 2 P03A 017 25) and by the Foundation for Polish Science.

The second author is partially supported by the State Committee for Scientific Research (grant no. 1 P03A 023 27).

from  $\mathbb{C}^n$  into  $\mathbb{C}^\infty$  and  $\text{Gr}(n, \infty)$  is the Grassmannian of  $n$ -dimensional subspaces in  $\mathbb{C}^\infty$ .

This paper is devoted to proving a similar result about the group of symplectomorphisms  $\text{Symp}(M, \omega)$  in place of  $U(n)$ . Mostly, we deal with the case when  $\omega$  has integral periods. Assume for the moment that  $H^2(M, \mathbb{Z})$  is torsion free (the torsion issues are discussed in detail in Section 2).

The space  $\text{Symp}(M, \mathbb{C}\mathbb{P}^\infty)$  of symplectic embeddings of  $M$  into  $\mathbb{C}\mathbb{P}^\infty$  is not contractible (it has non-trivial  $\pi_2$ , see Corollary 3.7). Thus it is not the total space of the universal  $\text{Symp}(M, \omega)$ -bundle. However, we find a deep connection with the result of Narasimhan and Ramanan mentioned above. That is, we construct a natural connection on the principal bundle

$$\text{Symp}(M, \omega) \rightarrow \text{Symp}(M, \mathbb{C}\mathbb{P}^\infty) \rightarrow \text{Symp}(M, \mathbb{C}\mathbb{P}^\infty) / \text{Symp}(M, \omega) =: \mathbb{B}_M.$$

The connections on  $\text{Symp}(M, \omega)$ -bundles are in one-to-one correspondence with vertically closed extensions of  $\omega$  to the total space of the associated bundle with fiber  $M$  [MS98, Lemma 6.18]. Among them, the connections represented by closed two-forms are of special interest since they generalize the notion of a coupling form introduced by Guillemin, Lerman and Sternberg [GLS96]. A coupling form is a certain closed connection two-form satisfying a normalization condition. Furthermore, the fiber integrals of powers of the cohomology class of the coupling form give a sequence of symplectic characteristic classes [JK, KM05].

The connection two-form  $\Omega$  that we construct in the configuration  $(M, \omega) \rightarrow M_{\mathbb{B}} \rightarrow \mathbb{B}_M$  is universal in the following sense. For any symplectic bundle  $M \rightarrow E \rightarrow B$  endowed with a closed connection two-form  $\Omega_E$  there exists a map  $f: B \rightarrow \mathbb{B}_M$  such that  $(E, \Omega_E) = f^*(M_{\mathbb{B}}, \Omega)$ .

We show that  $\mathbb{B}_M$  is a classifying space of an extension of  $\text{Symp}(M, \omega)$  by the gauge group  $\text{Map}(M, U(1))$ . We determine the holonomy subgroup of this connection and prove that it is equal to the subgroup  $\text{Ham}^{s\mathbb{Z}}(M, \omega)$  of  $\text{Symp}(M, \omega)$  recently discovered by McDuff [McD]. Her paper deals with, among other things, a characterization of symplectic bundles with closed connection form. It was important to the development of the present paper that Dusa McDuff shared with us her ideas on a generalization of Hamiltonian fibrations.

**Organization of the paper.** In Section 2 we give the necessary definitions and state the main results. We give the direct parts of the proofs and postpone the more technical ones to later sections.

In Section 3 we prove the universality of the configuration  $(M, \omega) \rightarrow M_{\mathbb{B}} \rightarrow \mathbb{B}$  and prove the homotopy properties of the space  $\mathbb{B}$ . We also prove that  $\mathbb{B}$  is the classifying space of an extension  $\mathcal{G}$  of  $\text{Symp}(M, \widehat{\omega})$ .

Section 4 is devoted to the geometry of symplectic configurations. More precisely, we define a principal connection on a fibration  $\text{Symp}(M, W) \rightarrow \mathbb{B}_W$  and investigate its properties.

In Section 5 we investigate the group cohomology relations between groups  $\mathcal{D}$  and  $\mathcal{G}$ . We also compare these groups with the McDuff subgroup  $\text{Ham}^{s\mathbb{Z}}(M, \omega)$ . The section starts with some preparation on crossed homomorphisms and the necessary algebraic topology.

In Section 6 we investigate characteristic classes of integral symplectic configurations and relate them to other known characteristic classes.

**Acknowledgement.** The idea of configurations grew out of discussions with Tadeusz Januszkiewicz.

We warmly thank Dusa McDuff for showing us an early version of her paper [McD], discussions, comments and picking up mistakes and inaccuracies in earlier drafts of this paper. In particular, she drew our attention to the torsion issues.

We thank Agnieszka Bojanowska, Stefan Jackowski and Michael Weiss for homotopy discussions and hospitality.

## 2. PRELIMINARIES AND STATEMENTS OF RESULTS

**2.1. Symplectic configurations.** Let  $(M, \omega)$  be a closed symplectic manifold. A symplectic manifold is called *integral* if the symplectic form has integral periods.

**Definition 2.2.** Let  $(M, \omega)$  and  $(W, \omega_W)$  be symplectic manifolds. We say that a symplectic fiber bundle (a fiber bundle with a structure group  $\text{Symp}(M, \omega)$ )  $M \rightarrow E \rightarrow B$  is a *W-symplectic configuration* if there exists a map  $E \rightarrow W$  whose restriction to any fiber of  $E$  is a symplectic embedding. The fibration  $E$  is called an (integral) symplectic configuration if it is a  $W$ -configuration for some (integral)  $(W, \omega_W)$ .

Here is an alternative approach. Consider the space  $\text{Symp}(M, W)$  of symplectic embeddings  $f: (M, \omega) \rightarrow (W, \omega_W)$ . The group of all symplectomorphisms of the source acts freely on that space and the quotient is denoted by  $\mathbb{B}_W$ . We call it the *space of symplectic configurations of  $(M, \omega)$  in  $(W, \omega_W)$* . We get a principal fibration

$$\text{Symp}(M, \omega) \rightarrow \text{Symp}(M, W) \rightarrow \mathbb{B}_W$$

and the associated symplectic one

$$(M, \omega) \rightarrow M_W \rightarrow \mathbb{B}_W.$$

**Proposition 2.3.** *A symplectic fibration  $(M, \omega) \rightarrow E \rightarrow B$  is a  $W$ -symplectic configuration if and only if it is a pull-back of the bundle  $M_W$ .*

*Proof.* Let  $E \rightarrow W$  be a map such that it is a symplectic embedding on every fiber. Clearly, it defines a map  $B \rightarrow \mathbb{B}_W$  such that  $E$  is a pull-back of  $M_W$ .

To prove the converse choose a point  $p \in M$  and consider the following composition of two maps:

$$\text{Symp}(M, W) \xrightarrow{\mathcal{P}} M_W \xrightarrow{ev} W,$$

where

$$\mathcal{P}(f) := [f, p] \quad \text{and} \quad ev[f, q] = f(q).$$

The map  $ev$  is well defined. Indeed  $M_W$  is the quotient of  $\text{Symp}(M, W) \times M$  by the relation  $[f \circ \psi, q] \sim [f, \psi(q)]$  and  $ev([f \circ \psi, q]) = f(\psi(q)) = ev([f, \psi(q)])$ . The evaluation map is the required map for the fibration  $M_W$ . Since  $E$  is a pull-back of  $M_W$ , the proof is finished.  $\square$

**2.4. Closed connection two-forms.** Without loss of generality we will assume that the base of a fibration is a manifold, so we may speak of the differential forms on  $E$ .

Recall that if  $(M, \omega) \rightarrow E \rightarrow B$  is a symplectic fibration then a closed two-form  $\Omega \in \Omega^2(E)$  is called a *closed symplectic connection two-form* if it restricts to the symplectic form on any fiber (see Lemma 6.18 in [MS98] for a general definition). According to Thurston (Theorem 6.3 in [MS98]), the existence of a closed connection two-form is a purely cohomological condition, i.e. the form  $\Omega$  exists if and only if the class  $[\omega]$  belongs to the image of the map  $H^2(E; \mathbb{R}) \rightarrow H^2(M; \mathbb{R})$ .

**Lemma 2.5.** *A symplectic bundle  $(M, \omega) \rightarrow E \rightarrow B$  is a symplectic configuration if and only if it admits a closed connection two-form.*

*Proof.* If  $f: E \rightarrow W$  defines a configuration, then  $f^*(\omega_W)$  is a closed connection two-form on  $E$ .

To prove the converse construct a map  $f: (E, \Omega) \rightarrow (W, \omega_W) := \times_i (\mathbb{C}P^{n_i}, a_i \omega_{n_i})$  such that  $f^*[\omega_W] = [\Omega]$  for appropriate  $a_i \in \mathbb{R}$ , where  $\omega_n$  is the standard symplectic form on  $\mathbb{C}P^n$ . For each  $\eta_i \in H^2(E, \mathbb{Z})$  there exists a map  $f_i: E \rightarrow \mathbb{C}P^\infty$  such that  $f_i^*[\omega_\infty] = \eta_i$  (here  $[\omega_\infty]$  is the generator of  $H^2(\mathbb{C}P^\infty; \mathbb{Z})$ ). Hence  $[\Omega] = \sum a_i \eta_i = (\times f_i)^*(a_1[\omega_\infty] \times \cdots \times a_n[\omega_\infty])$  for the product map  $\times f_i: E \rightarrow \times_i \mathbb{C}P^\infty$ . The latter factors through a product of finite dimensional projective spaces, as required. For sufficiently large  $\sum n_i$  we can perturb  $f$  into an embedding preserving the two-form, according to the h-principle (see Section 3.4.2 in [Gro86]).  $\square$

**2.6. The universal integral configuration.** Due to Gromov, every integral symplectic manifold embeds into complex projective space  $(\mathbb{C}P^N, \omega_N)$  equipped with the standard symplectic form for  $N$  big enough (Exercise 3.4.2.(1) in [Gro86]). Spaces of symplectic embeddings satisfy the obvious functoriality properties and we define  $\text{Symp}(M, \mathbb{C}P^\infty) := \bigcup_{N \rightarrow \infty} \text{Symp}(M, \mathbb{C}P^N)$ . Its quotient by  $\text{Symp}(M, \omega)$  is denoted by  $\mathbb{B}_\infty$  and called *the universal symplectic configuration space*. As before we have the universal principal fibration

$$\text{Symp}(M, \omega) \rightarrow \text{Symp}(M, \mathbb{C}P^\infty) \rightarrow \mathbb{B}_\infty$$

and the associated one

$$(M, \omega) \rightarrow M_{\mathbb{B}_\infty} \rightarrow \mathbb{B}_\infty.$$

Let  $(M, \omega) \rightarrow E \rightarrow B$  be a symplectic fibration whose classifying map  $\xi: B \rightarrow \text{BSymp}(M, \omega)$  has a lift  $\widehat{\xi}: B \rightarrow \mathbb{B}_\infty$ . This lift defines a map  $E \rightarrow M_{\mathbb{B}_\infty} \xrightarrow{ev} \mathbb{C}P^\infty$  which factors through  $\mathbb{C}P^N$ , since  $E$  is finite dimensional. Thus  $E$  is an integral configuration (see Definition 2.2), and the pull-back of the standard symplectic form  $\omega_N$  on  $\mathbb{C}P^N$  is a closed integral connection form on  $E$ .

**2.7. Torsion.** The space  $\text{Symp}(M, \mathbb{C}P^\infty)$  has the same number of connected components as the set of maps from  $M$  to  $\mathbb{C}P^\infty$  with the property that the generator of  $H^2(\mathbb{C}P^\infty, \mathbb{R})$  pulls back to  $[\omega]$ . The homotopy classes of maps from  $M$  to  $\mathbb{C}P^\infty$  are in one-to-one correspondence with  $H^2(M, \mathbb{Z})$ . The kernel of the map  $H^2(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{R})$  consists of the torsion of  $H^2(M, \mathbb{Z})$ . Thus if this torsion is nontrivial then the space  $\text{Symp}(M, \mathbb{C}P^\infty)$  is disconnected. Since  $M$  is closed,  $\text{Symp}(M, \mathbb{C}P^\infty)$  has a finite number of connected components. Moreover, if  $\text{Symp}(M, \omega)$  acts non-transitively on the preimage of  $\omega$  in  $H^2(M, \mathbb{Z})$  then  $\mathbb{B}$  is disconnected.

Let us observe that any (finite) Abelian group  $T$  can be the torsion of  $H^2(M; \mathbb{Z})$ . First notice that the torsion of  $H^2(M; \mathbb{Z})$  is isomorphic to the torsion of  $H_1(M; \mathbb{Z})$  (see Corollary 5.5.4 in [Spa66]). Take a finitely presented group  $G$  such that its abelianization has torsion equal to  $T$  (e.g.  $G = T$ ). Due to Gompf [Gom95], there exists a closed symplectic 4-manifold  $(M, \omega)$  with fundamental group  $G$ . Hence the torsion of  $H^2(M; \mathbb{Z})$  is isomorphic to  $T$ .

Let  $(M, \omega)$  be an integral symplectic manifold and let  $\widehat{\omega}$  denote a pair  $(\omega, \tau)$  of the symplectic form and a lift of its class to  $H^2(M, \mathbb{Z})$ . Define

$$\text{Symp}(M, \widehat{\omega}) := \{\psi \in \text{Symp}(M, \omega) : \psi^*(\tau) = \tau\}$$

to be a finite index open-closed subgroup of  $\text{Symp}(M, \omega)$  consisting of the symplectomorphisms which preserve the class  $\tau$ . Notice that if  $H^2(M, \mathbb{Z})$  is torsion-free then  $\text{Symp}(M, \widehat{\omega}) = \text{Symp}(M, \omega)$ .

**Lemma 2.8.** *If  $(M, \omega) \rightarrow E \rightarrow B$  is an integral symplectic configuration, then there exists a lift  $\tau \in H^2(M, \mathbb{Z})$  of  $[\omega]$  such that the structure group of  $E$  reduces to  $\text{Symp}(M, \widehat{\omega})$ .*

*Proof.* If  $f: E \rightarrow \mathbb{C}P^\infty$  defines a configuration structure on  $E$  then the pull-back of the generator of  $H^2(\mathbb{C}P^\infty, \mathbb{Z})$  has the above property.  $\square$

Let  $\text{Symp}_\tau(M, \mathbb{C}P^\infty)$  denote the connected component of  $\text{Symp}(M, \mathbb{C}P^\infty)$  containing symplectic embeddings corresponding to  $\tau \in H^2(M, \mathbb{Z})$ . The quotient  $\mathbb{B} = \mathbb{B}_\tau = \text{Symp}_\tau(M, \mathbb{C}P^\infty) / \text{Symp}(M, \widehat{\omega})$  is the component of  $\mathbb{B}_\infty$  corresponding to  $\tau$ . In the sequel, we restrict ourselves to the investigation of  $\mathbb{B}$ . We will also call it a universal configuration space. Again we have the universal fibration:

$$\text{Symp}(M, \widehat{\omega}) \rightarrow \text{Symp}_\tau(M, \mathbb{C}P^\infty) \rightarrow \mathbb{B}$$

and the associated one

$$(M, \omega) \rightarrow M_{\mathbb{B}} \rightarrow \mathbb{B}.$$

A symplectic fibration whose classifying map lifts to  $\mathbb{B}_\tau$  will be called an integral symplectic configuration compatible with  $\tau$ . Notice that a symplectic integral configuration might be compatible with several lifts.

Since  $\text{Symp}(M, \widehat{\omega})$  is open-closed in  $\text{Symp}(M, \omega)$ , their connected components of the identity are equal. Therefore the map of classifying spaces induced by the inclusion is a covering. In particular, the higher homotopy groups of these classifying spaces are equal, as well as rational (co-) homologies.

**2.9. Homotopy properties of the universal configuration space.** Integral symplectic configurations constitute a big class of symplectic fibrations. To see this consider the classifying map

$$\mathcal{F}: \mathbb{B} \rightarrow \text{BSymp}(M, \omega)$$

and the maps  $\mathcal{F}_k: \pi_k(\mathbb{B}) \rightarrow \pi_k(\text{BSymp}(M, \omega))$  induced on homotopy groups. The following theorem is proved in Section 3.5.

**Theorem 2.10.** *Let  $(M, \omega)$  be a compact integral symplectic manifold. Then*

- (1)  $\pi_k(\mathbb{B}) \stackrel{\mathcal{F}_k}{\cong} \pi_k(\text{BSymp}(M, \omega))$  for  $k \neq 1, 2$ .
- (2) *The following sequences are exact.*

$$\begin{aligned} 0 \rightarrow H^0(M; \mathbb{Z}) \rightarrow \pi_2(\mathbb{B}) \xrightarrow{\mathcal{F}_2} \pi_2(\text{BSymp}(M, \omega)) \rightarrow \Gamma_\omega \rightarrow 0 \\ 0 \rightarrow \Gamma_\omega \rightarrow H^1(M; \mathbb{Z}) \rightarrow \pi_1(\mathbb{B}) \xrightarrow{\mathcal{F}_1} \pi_1(\text{BSymp}(M, \widehat{\omega})) \rightarrow 0, \end{aligned}$$

where  $\Gamma_\omega$  is the flux group (see [MS98, p. 321]).

**2.11. An extension of  $\text{Symp}(M, \widehat{\omega})$ .** We identify the universal symplectic configuration space  $\mathbb{B}$  as the classifying space  $B\mathcal{G}_\tau$  of a certain extension  $\mathcal{G}_\tau$  of  $\text{Symp}(M, \widehat{\omega})$ . More precisely this extension is of the form

$$0 \rightarrow \text{Map}(M, U(1)) \rightarrow \mathcal{G}_\tau \rightarrow \text{Symp}(M, \widehat{\omega}) \rightarrow 1,$$

where  $\text{Map}(M, U(1))$  is the (Abelian) gauge group of continuous maps from  $M$  to  $U(1)$  (see Section 3.11 for the precise definition). We shall usually omit the subscript  $\tau$  if it does not lead to confusion. The proof of the next theorem is in Section 3.11.

**Theorem 2.12.** *The classifying space of  $\mathcal{G}_\tau$  is homotopy equivalent to the universal symplectic configuration space:*

$$B\mathcal{G}_\tau = \mathbb{B}_\tau.$$

Moreover, the classifying map  $\mathcal{F}: \mathbb{B}_\tau = B\mathcal{G}_\tau \rightarrow \text{BSymp}(M, \widehat{\omega})$  is induced by the projection  $\mathcal{G}_\tau \rightarrow \text{Symp}(M, \widehat{\omega})$ .

**2.13. The holonomy group.** If  $\Omega$  is the symplectic form on  $\mathbb{C}P^\infty$  and  $ev: M_{\mathbb{B}} \rightarrow \mathbb{C}P^\infty$  is the evaluation map, then  $ev^*\Omega$  is the closed connection form on the universal symplectic configuration  $(M, \omega) \rightarrow M_{\mathbb{B}} \rightarrow \mathbb{B}$ . It defines a symplectic connection whose holonomy group is denoted by  $\mathcal{D} = \mathcal{D}_\tau \subset \text{Symp}(M, \widehat{\omega})$ .

Note that  $\mathcal{D}$  depends on the lift  $\tau$  and is defined only up to conjugacy. In general we will omit the subscript  $\tau$  except in necessary cases.

**Theorem 2.14.** (1)  $\mathcal{D}$  intersects every connected component of  $\text{Symp}(M, \widehat{\omega})$ ;  
 (2) The identity component  $\mathcal{D}_0$  of  $\mathcal{D}$  is equal to  $\text{Ham}(M, \omega)$ ;  
 (3)  $\mathcal{D} \cap \text{Symp}_0(M, \omega) = \text{Flux}^{-1}(H^1(M; \mathbb{Z})/\Gamma_\omega)$ , that is, the intersection consists of symplectomorphisms with an integral flux. In particular,  $\mathcal{D}$  is a closed subgroup of  $\text{Symp}(M, \widehat{\omega})$ .

*Proof.* (1) Since  $\pi_1(\mathbb{B}) \rightarrow \pi_1(B \text{Symp}(M, \widehat{\omega}))$  is a surjection (Theorem 2.10),  $\mathcal{D}$  intersects every connected component of  $\text{Symp}(M, \widehat{\omega})$ .

(2) Let  $(M, \omega) \rightarrow M_W \rightarrow \mathbb{B}_W$  be a symplectic configuration fibration. Since  $\Omega_W$  is a closed connection two-form, the holonomy about any contractible loop is Hamiltonian due to Theorem 6.21 in [MS98]. This proves that  $\mathcal{D}_0 \subset \text{Ham}(M, \omega)$ . The converse inclusion follows from Corollary 4.6.

(3) This part will be proved in Proposition 4.10(2).  $\square$

Finally we find a connection between the holonomy subgroup  $\mathcal{D}$  and the extension  $\mathcal{G}$ .

**Theorem 2.15.** Let  $\text{Map}_0(M, U(1))$  denote the identity component of  $\text{Map}(M, U(1))$ . There exists a commutative triangle of continuous homomorphisms

$$\begin{array}{ccc} & \mathcal{G}/\text{Map}_0(M, U(1)) & \\ & \nearrow & \downarrow \\ \mathcal{D} & \longrightarrow & \text{Symp}(M, \widehat{\omega}) \end{array}$$

in which the diagonal arrow is a homotopy equivalence. In particular, the structure group of  $E$  lifts to  $\mathcal{G}/\text{Map}_0(M, U(1))$  if and only if the structure group of  $E$  reduces to  $\mathcal{D}$ .

*Proof.* This follows straightforward from Theorem 5.17 and Corollary 5.14.  $\square$

The next two theorems summarize all the main results of the paper.

**Theorem 2.16.** Let  $(M, \omega)$  be a closed integral symplectic manifold and let  $\tau$  be an integral lift of  $[\omega]$ . Let  $(M, \omega) \rightarrow E \rightarrow B$  be a symplectic fibration. The following conditions are equivalent:

- (1)  $E$  admits a closed integral connection two-form  $\Omega$  compatible with  $\tau$ , i.e. there exists  $\mathcal{T} \in H^2(E, \mathbb{Z})$  such that it lifts  $\Omega$  and extends  $\tau$ ;
- (2)  $E$  is an integral symplectic configuration compatible with  $\tau$ ;
- (3) the structure group of  $E$  lifts to  $\mathcal{G}_\tau$ .

*Proof.* (1)  $\Rightarrow$  (2) is proved in Section 3.15,

(2)  $\Rightarrow$  (1) follows from Lemma 2.5,

(1)  $\Rightarrow$  (3) is proved in Section 3.16,

(2)  $\Leftrightarrow$  (3) follows from Theorem 2.12.  $\square$

Clearly, if  $E$  is an integral symplectic configuration with respect to the lift  $\tau$  then its structure group reduces to the holonomy group  $\mathcal{D}_\tau$ . In other words, if the structure group lifts to  $\mathcal{G}_\tau$  then it lifts to  $\mathcal{G}_\tau/\text{Map}_0(M, U(1))$ . The following example due to McDuff shows that the converse is not true.

**Example 2.17.** Let  $S^2 \rightarrow BSO(2) \xrightarrow{\pi} BSO(3)$  be the universal fibration associated with the action of  $SO(3)$  on  $S^2$ . Let  $\tau$  be the generator of  $H^2(S^2, \mathbb{Z})$ . We claim that  $\tau$  does not admit an integral extension although the fibration is Hamiltonian (i.e. its structure group  $SO(3)$  is a subgroup of  $\text{Ham}(S^2, \text{area})$ ). Indeed,  $H^3(BSO(2); \mathbb{Z}) = 0$  and therefore  $H^3(BSO(3), \mathbb{Z}) = \mathbb{Z}/2$  is in the kernel of  $\pi^*$ . Hence the generator of the latter group equals  $d_2(\tau)$ , where  $d_2: H^2(S^2) \rightarrow H^3(BSO(3))$  is the differential in the spectral sequence. Thus  $\tau$  is not in the image of  $\pi^*: H^2(BSO(2)) \rightarrow H^2(S^2, \mathbb{Z})$  i.e.  $\tau$  does not admit an integral extension.

The proof of the following theorem will be given in Section 5.18.

**Theorem 2.18.** *Suppose that the structure group of a symplectic fibration  $M \rightarrow E \rightarrow B$  reduces to  $\mathcal{D}_\tau \subset \text{Symp}(M, \bar{\omega})$ . If  $H_2(B, \mathbb{Z})$  is torsion-free then  $E$  is an integral configuration.*

*Remark 2.19.* We do not prove that  $E$  is a configuration compatible with  $\tau$ . We show that  $d_2(\tau) = 0$  and  $d_3(\tau) \in E_3^{3,0}$  is torsion in the spectral sequence associated with  $M \rightarrow E \rightarrow B$ . Thus, *a priori*,  $\tau$  might not extend. Notice that, although  $E_2^{3,0} = H^3(B, \mathbb{Z})$  is (by the assumption) torsion free,  $E_3^{3,0} = E_2^{3,0}/d_2(E_2^{1,1})$  might have nontrivial torsion. However, we do not know any example in which  $E$  is a configuration not compatible with  $\tau$ .

### 3. THE TOPOLOGY OF SYMPLECTIC CONFIGURATIONS

**3.1. Gromov's h-principle.** Let  $\text{Map}(M, \mathbb{C}P^\infty)$  (respectively  $\text{Map}^\infty(M, \mathbb{C}P^\infty)$ ) denotes the space of all continuous (respectively smooth) maps from  $M$  to  $\mathbb{C}P^\infty$  equipped with compact-open (respectively  $C^\infty$ ) topology. The space of smooth maps is defined as  $\bigcup_n C^\infty(M, \mathbb{C}P^n)$ . It is well known that the inclusion  $\text{Map}^\infty(M, \mathbb{C}P^\infty) \rightarrow \text{Map}(M, \mathbb{C}P^\infty)$  induces a (weak) homotopy equivalence. Due to a big codimension the first space is homotopy equivalent to the space of embeddings of  $M$  into  $\mathbb{C}P^\infty$ .

**Theorem 3.2.** *Let  $\tau \in H^2(M, \mathbb{Z})$  be a lift of  $[\omega] \in H^2(M, \mathbb{R})$ . Let  $\text{Symp}_\tau(M, \mathbb{C}P^\infty)$  and  $\text{Map}_\tau(M, \mathbb{C}P^\infty)$  denote the connected components corresponding to the class  $\tau$ . The inclusion  $i: \text{Symp}_\tau(M, \mathbb{C}P^\infty) \rightarrow \text{Map}_\tau(M, \mathbb{C}P^\infty)$  induces an isomorphism on homotopy groups.*

*Remark 3.3.* The above theorem easily follows from the parametric version of *h-principle for symplectic embeddings*. Unfortunately, it seems that there is no proof of it in the literature yet. That is why we prove the above theorem appealing only to the well known Gromov's *h-principle* for symplectic immersions and its parametric version (Theorem (A) and Exercise (2) in Section 3.4.2 of [Gro86]).

*Proof of Theorem 3.2. Injectivity:* Suppose that  $f \in \ker i_*$ , that is there exist a commutative diagram

$$\begin{array}{ccc} S^k \times M & \xrightarrow{f} & \mathbb{C}P^m \\ \downarrow & \nearrow F & \\ D^{k+1} \times M & & \end{array}$$

where  $f$  is a symplectic embedding on each fiber  $\{s\} \times M$  and  $F$  is a smooth embedding on fibers and equals  $f$  over the boundary. The argument consists of several steps.

We first deform  $F$  to a fiberwise symplectic immersion  $F'$  so that it equals  $f$  over the boundary. This can be done according to the parametric h-principle for symplectic immersions (Theorem 16.4.3 in [EM02]). We want to improve it so that it will be fiberwise symplectic embedding.

We will need an isotropic embedding  $j: M \rightarrow (\mathbb{C}^n, \omega_n)$ . For example one can compose an embedding of  $M$  into  $\mathbb{R}^N$  with the standard inclusion of Lagrangian  $\mathbb{R}^n$  into  $\mathbb{C}^n$

Secondly, we need a symplectic embedding  $\varphi$  of  $\mathbb{C}\mathbb{P}^m \times D^{2n}$  with the product form into  $\mathbb{C}\mathbb{P}^N$  such that  $\varphi$  is linear on  $\mathbb{C}\mathbb{P}^m \times 0$ . This can be achieved by realizing  $\varphi$  of  $\mathbb{C}\mathbb{P}^m \times D^{2n}$  in  $\mathbb{C}\mathbb{P}^m \times \mathbb{C}\mathbb{P}^n$  and embedding the latter in  $\mathbb{C}\mathbb{P}^{m+m+n}$  via the Segre embedding.

Take

$$D^{k+1} \times M \rightarrow \mathbb{C}\mathbb{P}^m \times D^{2n} \rightarrow \mathbb{C}\mathbb{P}^N$$

defined by

$$(d, m) \mapsto \varphi(F'(d, m), \alpha(d)j(m)),$$

where  $\alpha: D^{k+1} \rightarrow \mathbb{R}$  is a sufficiently small scaling function such that it equals 0 exactly on the boundary sphere. Clearly, it is an embedding because  $j$  is. It is fiberwise symplectic because  $F'$  is fiberwise symplectic and  $j$  is isotropic.

**Surjectivity:** We shall show that every fiberwise smooth embedding  $f: S^k \times M \rightarrow \mathbb{C}\mathbb{P}^m$  (fiberwise in  $\text{Map}_\tau(M, \mathbb{C}\mathbb{P}^m)$ ) is homotopic to a fiberwise symplectic embedding. First take the composition  $F: T^*S^k \times M \rightarrow \mathbb{C}\mathbb{P}^m$  of  $f$  and the projection  $T^*S^k \times M \rightarrow S^k \times M$ . According to the standard h-principle there exist an arbitrarily  $C^0$ -small deformation of  $F$  that is a symplectic embedding (with respect to the product symplectic form). The restriction of it to the image of the “zero section”  $S^k \times M \subset T^*S^k \times M$  gives a symplectic embedding.

The above homotopy might not preserve the base-point but this is not a problem due to the injectivity of  $i_*$  on  $\pi_0$ . Let  $s_0 \in S^k$  and  $m \in \text{Symp}_\tau(M, \mathbb{C}\mathbb{P}^\infty)$  be base-points. Let  $F: [0, 1] \times S^k \rightarrow \text{Map}_\tau(M, \mathbb{C}\mathbb{P}^\infty)$  be the free homotopy ( $F_0(s_0) = f(s_0) = m$ ) such that  $F_1(S^k) \subset \text{Symp}_\tau(M, \mathbb{C}\mathbb{P}^\infty)$ . Let  $q: [0, 1] \rightarrow \text{Map}_\tau(M, \mathbb{C}\mathbb{P}^\infty)$  be defined by  $q(t) := F_t(s_0)$ .

If  $l: [0, 1] \rightarrow X$  is a path in a topological space then  $l_*: \pi_k(X, l(0)) \rightarrow \pi_k(X, l(1))$  denotes the induced isomorphism. In our situation,  $F$  is a based homotopy from  $F_0$  to  $q_*F_1$ . Due to the injectivity of  $\pi_0(i)$ , there exists a path  $p: [0, 1] \rightarrow \text{Symp}_\tau(M, \mathbb{C}\mathbb{P}^\infty)$  from  $F_0(s_0)$  to  $F_1(s_0)$ . Further, we have that

$$p * F_1 \sim (p * q^{-1}) * q * F_1 \sim (p * q^{-1}) * F_0 \sim F_0.$$

The last equivalence follows from the fact that  $\text{Map}_\tau(M, \mathbb{C}\mathbb{P}^\infty)$  is an H-space, so the action of the fundamental group on all homotopy groups is trivial. This finishes the proof.  $\square$

*Remark 3.4.* Note that the result we mention at the beginning of the Section 1 implies injectivity on  $\pi_0$ . Indeed, an integral two-form is a curvature form of some  $U(1)$ -connection (on the pull-back of the universal  $U(1)$ -bundle  $U(1) \rightarrow S^\infty \rightarrow \mathbb{C}\mathbb{P}^\infty$  by the map  $f: M \rightarrow \mathbb{C}\mathbb{P}^\infty$ ) thus, following Narasimhan and Ramanan [NR61] one can approximate  $f$  (by a map classifying the same bundle, thus homotopic) to the map preserving not only curvature  $\omega$  but also the connection.

### 3.5. Proof of Theorem 2.10.

**Lemma 3.6.**  $\pi_k \text{Map}(M, \mathbb{C}\mathbb{P}^\infty) = H^{2-k}(M, \mathbb{Z})$ , where we set  $H^m(M, \mathbb{Z}) = 0$  for  $m < 0$ .

*Proof.* It follows from the basic algebraic topology (e.g. [Spa66]). Let  $s \in S^k$  and  $f \in \text{Map}(M, X)$  be a base point, the constant map  $m \mapsto x \in X$ . Moreover let  $M_+$

denote  $M$  with an artificially added base point. Then we calculate

$$\begin{aligned}\pi_k(\text{Map}(M, X)) &= [(S^k, s), (\text{Map}(M, X), f)] \\ &= [M_+ \wedge S^k, (X, x)] \\ &= [M_+, \Omega^k(X, x)]\end{aligned}$$

Since  $\mathbb{C}\mathbb{P}^\infty = K(\mathbb{Z}, 2)$ , we have that  $\pi_k(\text{Map}(M, \mathbb{C}\mathbb{P}^\infty)) = [M, K(\mathbb{Z}, 2-k)] = H^{2-k}(M, \mathbb{Z})$ .

It is easy to see a composition of this isomorphism with the projection  $H^{2-k}(M, \mathbb{Z}) \rightarrow \text{Hom}(H_{2-k}(M, \mathbb{Z}), \mathbb{Z})$  (which is an isomorphism for  $k \in \{1, 2\}$ ) as follows. Let  $[\xi] \in \pi_k(\text{Map}(M, \mathbb{C}\mathbb{P}^\infty))$  be represented by a map  $\xi: S^k \times M \rightarrow \mathbb{C}\mathbb{P}^\infty$ . The value of the resulting cohomology class on  $(2-k)$ -cycle  $a$  is equal to  $\langle [S^k \times a], \xi^* \omega_\infty \rangle$ , where  $\omega_\infty \in H^2(\mathbb{C}\mathbb{P}^\infty; \mathbb{Z})$  is the generator.  $\square$

**Corollary 3.7.**  $\pi_k \text{Symp}_\tau(M, \mathbb{C}\mathbb{P}^\infty) = \begin{cases} H^{2-k}(M; \mathbb{Z}) & \text{for } k \in \{1, 2\} \\ 0 & \text{otherwise} \end{cases}$  and hence  $\pi_k(\mathbb{B}) \cong \pi_k(\text{BSymp}(M, \widehat{\omega}))$  for  $k > 3$ .

*Proof.* Since  $\text{Map}(M, \mathbb{C}\mathbb{P}^\infty)$  is an H-space, all its connected components are homotopy equivalent. The first statement follows immediately from Theorem 3.2 and Lemma 3.6. The second is the direct application of the long exact sequence of homotopy group for the fibration  $\text{Symp}(M, \widehat{\omega}) \rightarrow \text{Symp}_\tau(M, \mathbb{C}\mathbb{P}^\infty) \rightarrow \mathbb{B}$ .  $\square$

**Example 3.8.** Let  $\Sigma_g$  be a surface of genus  $g > 0$ . Let  $i_0, i_1 \in \text{Symp}(\Sigma_g, \mathbb{C}\mathbb{P}^\infty)$  be two embeddings such that  $i_0 = i_1 \circ f$ , where  $f: \Sigma_g \rightarrow \Sigma_g$  is a Dehn twist. One can construct a symplectic Lefschetz fibration  $(M, \omega) \rightarrow S^2$  with the generic fiber  $\Sigma_g$  such that the monodromy about some critical value is exactly  $f$  and  $(M, \omega)$  is an integral symplectic manifold [ABKP00]. Embedding  $(M, \omega)$  symplectically into  $\mathbb{C}\mathbb{P}^N$  we also get a family of symplectic embeddings  $i_t: \Sigma_g \rightarrow \mathbb{C}\mathbb{P}^N$  for  $t \in [0, 1]$ . Since  $\pi_0(\text{Symp}(\Sigma_g, \text{area}))$  is generated by Dehn twists, we obtain that the map  $\pi_0(\text{Symp}(\Sigma_g, \text{area})) \rightarrow \pi_0(\text{Symp}(\Sigma_g, \mathbb{C}\mathbb{P}^\infty))$  is trivial.

**Lemma 3.9.** *The map  $\pi_2(\text{Symp}(M, \omega)) \rightarrow \pi_2(\text{Symp}(M, \mathbb{C}\mathbb{P}^\infty))$  is trivial. In particular,  $\pi_3(\mathbb{B}) \cong \pi_3(\text{BSymp}(M, \omega))$ .*

*Proof.* Let  $\xi \in \pi_2(\text{Symp}(M, \omega))$ . Its image in  $\pi_2(\text{Map}(M, \mathbb{C}\mathbb{P}^\infty)) = \mathbb{Z}$  is equal to  $\langle [\omega], ev_*(\xi) \rangle$ , where  $ev: \text{Symp}(M, \omega) \rightarrow M$  is the evaluation at the base point  $pt \in M$  (cf. proof of Lemma 3.6). More precisely, consider the following commutative diagram.

$$\begin{array}{ccc}\text{Symp}(M, \omega) & \xrightarrow{i} & \text{Symp}(M, \mathbb{C}\mathbb{P}^\infty) \\ \downarrow ev & & \downarrow ev \\ M & \xrightarrow{f} & \mathbb{C}\mathbb{P}^\infty\end{array}$$

The isomorphism  $\pi_2(\text{Map}(M, \mathbb{C}\mathbb{P}^\infty)) = \mathbb{Z}$  is given by the integration of the generator  $[\omega_\infty] \in H^2(\mathbb{C}\mathbb{P}^\infty)$  over the sphere  $S^2 \rightarrow \text{Map}(M, \mathbb{C}\mathbb{P}^\infty) \xrightarrow{ev} \mathbb{C}\mathbb{P}^\infty$ . Hence we calculate that  $\langle [\omega_\infty], ev_* i_*(\xi) \rangle = \langle [\omega_\infty], f_* ev_*(\xi) \rangle = \langle f^* [\omega_\infty], ev_*(\xi) \rangle = \langle [\omega], ev_*(\xi) \rangle$  as required.

We claim that this number is equal to zero. Indeed, if  $ev^*[\omega]$  is a sum of products of 1-dimensional classes then  $\langle ev^*[\omega], \xi \rangle = \langle \xi^*(ev^*[\omega]), [S^2] \rangle = 0$ . Suppose, by contradiction, that  $ev^*[\omega]$  is not a sum of products. Recall that, according to Milnor and Moore [MM65], the cohomology ring of  $\text{Symp}(M, \omega)$  (which is an H-space) is a free graded commutative algebra. If  $ev^*[\omega]$  were nonzero then it would be of infinite order, which is impossible since  $[\omega]^{n+1} = 0$ .  $\square$

Let  $\text{Flux}: \pi_1(\text{Symp}(M, \omega)) \rightarrow H^1(M; \mathbb{R})$  be the *flux homomorphism* defined as follows.  $\langle \text{Flux } \xi, \gamma \rangle = \langle [\omega], \xi_*(\gamma) \rangle$ , where  $\xi_*(\gamma): \mathbb{T}^2 \rightarrow M$  sends  $(t, s) \rightarrow \xi_t(\gamma(s))$  (see Chapter 9 in [MS98] for details). Its image is called the flux group and is denoted by  $\Gamma_\omega$ . Notice that in the case of an integral symplectic structure  $\Gamma_\omega$  is a subgroup of the integral cohomology  $H^1(M, \mathbb{Z})$ . The following lemma is an immediate consequence of the above definition.

**Lemma 3.10.** *The following diagram is commutative.*

$$\begin{array}{ccc} \pi_1(\text{Symp}(M, \omega)) & \longrightarrow & \pi_1(\text{Symp}(M, \mathbb{C}P^\infty)) \\ \text{Flux} \downarrow & & \cong \downarrow \\ \Gamma_\omega & \longrightarrow & H^1(M, \mathbb{Z}) = \pi_1(\text{Map}(M, \mathbb{C}P^\infty)) \end{array}$$

□

*Proof of Theorem 2.10.*

- (1) This part follows from Corollary 3.7 and Lemma 3.9.
- (2) To prove that the first sequence is exact one applies Lemma 3.9 and 3.10 to the exact sequence associated with the fibration  $\text{Symp}(M, \widehat{\omega}) \rightarrow \text{Symp}_\tau(M, \mathbb{C}P^\infty) \rightarrow \mathbb{B}$ . Similarly, Corollary 3.7 and Lemma 3.10 applied to the long exact sequence of the the same fibration proves the exactness of the second sequence.

□

**3.11. Proof of Theorem 2.12.** Recall that principal  $G$ -bundles over  $M$  are classified by  $H^1(M, G)$  and  $H^1(M, U(1))$  is canonically isomorphic to  $H^2(M, \mathbb{Z})$ . The bundle corresponding to  $\tau \in H^2(M, \mathbb{Z})$  will be denoted  $L_\tau$ , and the class of a bundle  $L$  in  $H^2(M, \mathbb{Z})$  will be denoted by  $[L]$ .

Assume that  $(M, \omega)$  and  $(W, \omega_W)$  are integral symplectic manifolds. Choose  $\tau \in H^2(M, \mathbb{Z})$  and  $\tau_W \in H^2(W, \mathbb{Z})$  to be some lifts of  $\omega$  and  $\omega_W$  respectively. Let

$$\text{Map}_{\text{Symp}}^{U(1)}(L_\tau, L_{\tau_W})$$

be the space of maps from  $L_\tau$  to  $L_{\tau_W}$  which are  $U(1)$ -equivariant and cover a symplectomorphism  $(M, \omega) \rightarrow (W, \omega_W)$ :

$$\begin{array}{ccc} L_\tau & \xrightarrow{\widehat{\psi}} & L_{\tau_W} \\ \downarrow & & \downarrow \\ M & \xrightarrow{\psi} & W. \end{array}$$

Observe that if  $\psi: M \rightarrow W$  is a continuous map, then  $\psi^*(L_\tau) = L_{\psi^*\tau}$ , thus there exists a covering  $U(1)$ -equivariant map  $\widehat{f}: L_{\psi^*\tau} \rightarrow L_\tau$ . In particular, for  $(W, \omega_W) = (M, \omega)$  we get a group

$$\mathcal{G} := \text{Map}_{\text{Symp}}^{U(1)}(L_\tau, L_\tau).$$

**Lemma 3.12.** *There is an extension*

$$1 \rightarrow \text{Map}(M, U(1)) \rightarrow \mathcal{G} \rightarrow \text{Symp}(M, \widehat{\omega}) \rightarrow 1$$

*Proof.* The kernel of the map that assigns to  $\widehat{\psi}$  the underlying symplectomorphism  $\psi$  consists of the automorphisms which cover the identity. Over each point of  $M$  such automorphism is given by an element of  $U(1)$ . □

**Proposition 3.13.** *Let  $\omega_n$  denote the standard symplectic form on  $\mathbb{C}\mathbb{P}^n$ , and let  $[L_n] = [\omega_n] \in H^2(\mathbb{C}\mathbb{P}^n, \mathbb{Z})$ . Then the space*

$$\text{Map}_{\text{Symp}}^{U(1)}(L_\tau, L_\infty) := \bigcup_n \text{Map}_{\text{Symp}}^{U(1)}(L_\tau, L_n)$$

is contractible.

*Proof.* Let  $\text{Map}^{U(1)}(L_\tau, L_\infty)$  be a space of all  $U(1)$ -equivariant maps  $L_\tau \rightarrow L_\infty$ . Then we have a bundle:

$$\text{Map}(M, U(1)) \rightarrow \text{Map}^{U(1)}(L_\tau, L_\infty) \rightarrow \text{Map}_\tau(M, \mathbb{C}\mathbb{P}^\infty),$$

Consider the following two exact sequences of homotopy groups

$$\begin{array}{ccccc} \dots & \pi_k(\text{Map}(M, U(1))) & \longrightarrow & \pi_k(\text{Map}^{U(1)}(L_\tau, L_\infty)) & \longrightarrow & \pi_k(\text{Map}_\tau(M, \mathbb{C}\mathbb{P}^\infty)) \dots \\ & \uparrow & & \uparrow & & \uparrow \\ & = & & & & \\ \dots & \pi_k(\text{Map}(M, U(1))) & \longrightarrow & \pi_k(\text{Map}_{\text{Symp}}^{U(1)}(L_\tau, L_\infty)) & \longrightarrow & \pi_k(\text{Symp}_\tau(M, \mathbb{C}\mathbb{P}^\infty)) \dots \end{array}$$

where the vertical arrows are induced by inclusions. According to Theorem 3.2, the map

$$\pi_k(\text{Symp}_\tau(M, \mathbb{C}\mathbb{P}^\infty)) \rightarrow \pi_k(\text{Map}_\tau(M, \mathbb{C}\mathbb{P}^\infty))$$

is an isomorphism. By the five lemma the same is true for

$$\pi_k(\text{Map}_{\text{Symp}}^{U(1)}(L_\tau, L_\infty)) \rightarrow \pi_k(\text{Map}^{U(1)}(L_\tau, L_\infty)).$$

We are left to show that  $\text{Map}^{U(1)}(L_\tau, L_\infty)$  is contractible. Since  $L_\infty = EU(1)$ , the following lemma finishes the proof.  $\square$

**Lemma 3.14.** *Let  $E$  be a free  $G$ -space. Then  $\text{Map}^G(E, EG)$  is contractible.*

*Proof.* It is a standard fact (about universal principal bundles) that  $\text{Map}^G(E, EG)$  is connected (see Theorem 8.12 on page 58 in [tD87]). We need to show that for all  $k$  the space  $\text{Map}(S^k, \text{Map}^G(E, EG)) = \text{Map}^G(E \times S^k, EG)$  is connected. But  $S^k \times E$  is again a free  $G$ -space and we apply the observation we began with.  $\square$

*Proof of Theorem 2.12.* Consider the following two fibrations

$$\text{Map}(M, U(1)) \rightarrow \text{Map}_{\text{Symp}}^{U(1)}(L_\tau, L_\infty) \xrightarrow{p_1} \text{Symp}_\tau(M, \mathbb{C}\mathbb{P}^\infty)$$

and

$$\text{Symp}(M, \widehat{\omega}) \rightarrow \text{Symp}_\tau(M, \mathbb{C}\mathbb{P}^\infty) \xrightarrow{p_2} \mathbb{B}_\tau,$$

where the base  $\mathbb{B}_\tau$  is defined to be this quotient. Observe that the composition  $p_2 \circ p_1$  defines a fibration

$$\mathcal{G} \rightarrow \text{Map}_{\text{Symp}}^{U(1)}(L_\tau, L_\infty) \xrightarrow{p_2 \circ p_1} \mathbb{B}_\tau$$

with a contractible total space.  $\square$

**3.15. Proof of Theorem 2.16 (1  $\Rightarrow$  2).** By assumption the connection form  $\Omega$  has a lift  $\mathcal{T} \in H^2(E, \mathbb{Z}) = [E, K(\mathbb{Z}, 2)]$ . Since  $K(\mathbb{Z}, 2) = \mathbb{C}\mathbb{P}^\infty$ , we get an embedding  $f : E \rightarrow \mathbb{C}\mathbb{P}^N$  such that  $\mathcal{T}$  is the pull-back of the generator. According to the original Gromov's h-principle (Theorem 3.4.2 A in [Gro86]) we can deform  $f$  to an immersion preserving forms. That is an immersion  $f' : E \rightarrow \mathbb{C}\mathbb{P}^N$  so that  $f'^*(\omega_N) = \Omega$ . At the price of increasing  $N$  we can deform it further to get a embedding as we did in the proof of Theorem 3.2.  $\square$

3.16. **Proof of Theorem 2.16** (1  $\Rightarrow$  3). A class  $\mathcal{J}$  defines a line bundle  $L_{\mathcal{J}} \rightarrow E$ . Notice that the pull-back bundle under the inclusion of the fiber  $i: M \rightarrow E$  is the line bundle  $L_{i^*\mathcal{J}} \rightarrow M$ . Thus  $i^*\mathcal{J}$  is a lift of  $[\omega]$  to  $H^2(M, \mathbb{Z})$ .

Consider the composition  $L_{\mathcal{J}} \rightarrow E \rightarrow B$  as a bundle over  $B$  with fiber  $U(1)$ . The structure group of this fibration is contained in  $\mathcal{G}$ . Thus we get a classifying map  $\widehat{f}: B \rightarrow B\mathcal{G}$  such that the following diagram commutes up to homotopy (here  $f$  classifies  $E$ ):

$$\begin{array}{ccc} & & B\mathcal{G} \\ & \nearrow \widehat{f} & \downarrow \\ B & \xrightarrow{f} & \text{BSymp}(M, \omega) \end{array}$$

Recall that, according to Theorem 2.12,  $B\mathcal{G} \simeq \mathbb{B}$ . The classifying map  $\widehat{f}: B \rightarrow B\mathcal{G} = \mathbb{B}$  is the one constructed in the previous section.  $\square$

#### 4. THE GEOMETRY OF SYMPLECTIC CONFIGURATIONS

4.1. **A principal connection on  $\text{Symp}(M, W)$ .** The main reference for connections is Kobayashi-Nomizu [KN96a]. The tangent space to  $\text{Symp}(M, W)$  at point  $f$  is equal to

$$\{\mathcal{X} \in \Gamma(f^*TW): \mathcal{X}(p) = \frac{d}{dt}f_t(p) \quad f_t \in \text{Symp}(M, W)\}.$$

We define the horizontal space  $\mathcal{H}_f$  to be the space of the sections  $f^*\omega_W$ -orthogonal to  $M$ :

$$\mathcal{H}_f := \{\mathcal{X} \in T_f \text{Symp}(M, W): f^*\omega_W(\mathcal{X}, Y) = 0 \quad \forall Y \in \Gamma(TM)\}.$$

Consider the following one-form  $\theta$  on  $\text{Symp}(M, W)$  with values in closed one-forms on  $M$  which is identified with the Lie algebra of  $\text{Symp}(M, \omega)$

$$\theta_f(\mathcal{X}) := f^*(\iota_{\mathcal{X}_f}\omega_W)$$

for  $\mathcal{X} \in T_f \text{Symp}(M, W)$ . More precisely

$$(\theta_f(\mathcal{X}))(p)(Y) = \omega_W(\mathcal{X}_f(p), f_*Y(p)),$$

where  $Y$  is a vector field on  $M$  and  $\mathcal{X}_f$  is a vector field on  $W$  defined on  $f(M)$ .

**Lemma 4.2.** *The one-form  $\theta$  is a connection form induced by  $\mathcal{H}$ , that is, it satisfies*

- (1)  $\theta_{f \circ \psi} = Ad_{\psi}\theta = \psi^* \circ \theta$ ,
- (2)  $\theta_f(\underline{X}) = X$ , where  $X \in \Omega^1(M)$  is a closed one-form and  $\underline{X}$  denotes the fundamental vector field (i.e. the vector field generated by the infinitesimal action of a Lie algebra),
- (3)  $\mathcal{H}$  is the kernel of  $\theta$ .

*Proof.* (1) This is an immediate calculation:

$$\begin{aligned} (\theta_{f \circ \psi}(\mathcal{X}))(p)(Y) &= \omega_W(\mathcal{X}_{f \circ \psi}(p), (f \circ \psi)_*Y(p)) \\ &= \omega_W(\mathcal{X}_f(\psi(p)), f_*(\psi_*Y(p))) \\ &= (\theta_f(\mathcal{X}_f))(\psi(p))(\psi_*Y(p)) \\ &= (\psi^* \circ \theta_f)(\mathcal{X}_f)(p)(Y). \end{aligned}$$

(2) If  $X$  is a closed one-form on  $M$  then we denote by  $X^\sharp$  the  $\omega$ -corresponding vector field. That is  $\iota_{X^\sharp}\omega = X$ . Thus  $\underline{X}_f = X^\sharp$  under the identification between  $M$  and  $f(M) \in W$  given by  $f$ . Hence we get

$$\theta_f(X^\sharp) = f^*(\iota_{X^\sharp}^{\omega_W}) = X.$$

(3) Obvious. □

**Proposition 4.3.** *The connection associated to  $\theta$  on  $M_W$  and connection defined by the two-form  $\Omega_W := ev^*(\omega_W)$  coincide.*

*Proof.* Both distributions consist of vectors  $X \in T_{[f,p]}M_W$  satisfying the condition  $ev_*(X) \in T_{f(p)}W$  is  $\omega_W$ -orthogonal to  $f_*(T_pM)$ . □

**4.4. The curvature of  $\theta$ .** Let's calculate the curvature two-form of the connection  $\theta$ . By definition it is

$$\Theta(X, Y) := d\theta(X^h, Y^h) = -\theta([X^h, Y^h]),$$

where  $X^h, Y^h \in \mathcal{H}$  are horizontal parts of tangent vectors. Calculating it further we get

$$\begin{aligned} -\Theta_f(X^h, Y^h) &= \theta_f[X^h, Y^h] \\ &= f^*(\iota_{[X^h, Y^h]}\omega_W) \\ &= f^*(\{\alpha_X, \alpha_Y\}) \end{aligned}$$

Here  $\alpha_X, \alpha_Y$  are one-forms associated with extensions of  $X^h, Y^h \in \Gamma(f^*TW)$  to vector field on some neighbourhood of  $f(M) \subset W$  and  $\{\alpha_X, \alpha_Y\}$  is the Poisson bracket. Since  $X^h$  and  $Y^h$  are horizontal, the one-forms  $\alpha_X, \alpha_Y$  vanish on  $M$ . Therefore their local extensions might be chosen to be exact and we get

$$-\Theta_f(X^h, Y^h) = d[f^*\{H_X, H_Y\}],$$

where  $H_X, H_Y: W \rightarrow \mathbb{R}$  are Hamiltonians, i.e.  $dH_X = \alpha_X$  and  $dH_Y = \alpha_Y$ .

In other words the curvature of the connection  $\theta$  at the embedding  $f: (M, \omega) \rightarrow (W, \omega_W)$  is measured by the subspace of functions on  $M$  consisting of restrictions of Poisson brackets of functions on  $W$  which are constant on  $f(M)$ . We will use this curvature form to construct characteristic classes in Section 6.9.

**Lemma 4.5.** *Let  $f: (M, \omega) \rightarrow (M, \omega) \times (\Sigma, \omega_\Sigma)$  be a symplectic embedding given by  $f(p) = (p, s_0)$ , where  $s_0 \in \Sigma$  is a base point on the surface. The image of the curvature two-form*

$$\Theta_f: \mathcal{H}_f \times \mathcal{H}_f \rightarrow \text{Lie}(\text{Symp}(M, \omega))$$

*contains the subspace of all exact one-forms.*

*Proof.* What in fact we need to show is that given a function  $H: M \rightarrow \mathbb{R}$  there are functions  $F, G: M \times \Sigma \rightarrow \mathbb{R}$  constant on  $f(M)$  such that  $\{F, G\} \circ f = H$ .

Let  $F', G': \Sigma \rightarrow \mathbb{R}$  be smooth functions such that  $F'(s_0) = 0$  and  $\{F', G'\}(s_0) = 1$ . Define  $F(p, s) := F'(s)$ ,  $G(p, s) = G'(s)$  and  $\widehat{H}(p, s) = H(p)$ . The following calculation follows from the Leibniz property of the Poisson bracket.

$$\{\widehat{H}F, G\} = \widehat{H}\{F, G\} + F\{\widehat{H}, G\} = \widehat{H}\{F, G\}$$

Hence,  $\{\widehat{H}F, G\} \circ f = \widehat{H} \circ f = H$ . Note that the functions  $\widehat{H}F$  and  $G$  are constant on  $f(M)$  as required. □

**Corollary 4.6.** *The image of the curvature two-form*

$$\Theta_{iof}: \mathcal{H}_{iof} \times \mathcal{H}_{iof} \rightarrow \text{Lie}(\text{Symp}(M, \omega))$$

*of the connection in the universal configuration fibration contains the subspace of all exact one-forms for every  $f: (M, \omega) \rightarrow \mathbb{C}\mathbb{P}^\infty$ .* □

**4.7. The holonomy group.** Let  $(M, \omega) \xrightarrow{i} E \xrightarrow{\pi} B$  be a symplectic fibration. Let  $K := \ker [\pi_* : H_2(E, \mathbb{Z}) \rightarrow H_2(B, \mathbb{Z})]$ . The following exact sequence:

$$0 \rightarrow H_2(M, \mathbb{Z}) \rightarrow K \rightarrow H_1(B; H_1(F, \mathbb{Z})) \rightarrow 0$$

is derived from the Leray-Serre spectral sequence associated with  $E$ . An element of  $H_1(B, H_1(M, \mathbb{Z}))$  may be described as a (class of a) loop  $\gamma$  in  $B$  and a  $\gamma$ -invariant cycle  $[\ell] \in H_1(M, \mathbb{Z})^\gamma$  in the fiber  $F$  over a base point of  $\gamma$ .

Let  $\Omega$  be a closed extension of  $\omega$ . We construct a certain lift  $\Sigma_{\gamma, \ell} \in H_2(E, \mathbb{Z})$  of  $(\gamma, \ell) \in H_1(B; H_1(M, \mathbb{Z}))$  to  $K$ . It is a union of two chains  $f(S^1 \times I)$  and  $C$ . The map  $f$  is such that  $f(\{s\} \times I)$  is a horizontal lift of  $\gamma$  for each  $s$  and  $f(S^1 \times \{0\}) = \ell$ . The chain  $C$  is a chain in  $F$  such that  $\partial C = \gamma_* \ell - \ell$ , where  $\gamma_*$  is the holonomy of  $\gamma$  with respect to  $\Omega$ . Observe that  $\Omega(f_*(\frac{d}{ds}), f_*(\frac{d}{dt})) = 0$  because the kernel of  $\Omega$  is the horizontal subspace. Thus  $\int_{\Sigma_{\gamma, \ell}} \Omega = \int_C \omega$ .

Let  $\psi \in \text{Symp}(M, \omega)$  and  $H_1(M, \mathbb{Z})^\psi = \{\ell \in H_1(M, \mathbb{Z}) : \psi_*(\ell) = \ell\}$  be the  $\psi$ -invariant part of the first cohomology. Define a flux-like homomorphism

$$\text{Flux}^\psi : H_1(M, \mathbb{Z})^\psi \rightarrow \mathbb{R}/\omega$$

as follows (here  $\mathbb{R}/\omega$  denotes the quotient of  $\mathbb{R}$  by the group of periods of the symplectic form). If  $[\ell] \in H_1(M, \mathbb{Z})^\psi$  then there exists a two-chain  $C$  such that  $\partial C = \ell - \psi_* \ell$ . Then

$$\text{Flux}^\psi(\ell) := \int_C \omega \quad \text{mod (periods of } \omega).$$

The integral is defined up to the period of  $\omega$  and does not depend on the choice of representative  $\ell$  in the homology class.

Let  $\overline{\mathcal{D}}$  be a subset of  $\text{Symp}(M, \widehat{\omega})$  consisting of those  $\psi \in \text{Symp}(M, \widehat{\omega})$  for which  $\text{Flux}^\psi = 0$ . Note that  $\overline{\mathcal{D}}$  is conjugacy invariant, but it is not a subgroup of  $\text{Symp}(M, \widehat{\omega})$ .

**Proposition 4.8.** *Let  $M \rightarrow E \rightarrow B$  be a symplectic fibration. Let  $\Omega$  be an extension of the symplectic form  $\omega$  and let  $K := \ker H_2(E, \mathbb{Z}) \rightarrow H_2(B, \mathbb{Z})$ . Then  $\langle \Omega, K \rangle \subset \mathbb{Z}$  if and only if the holonomy  $\pi_1(B) \rightarrow \text{Symp}(M, \widehat{\omega})$  is contained in  $\overline{\mathcal{D}}$ .*

*Proof.* If  $C' \in H_2(M, \mathbb{Z})$  then  $\int_{i_*(C')} \Omega = \int_{C'} \omega$  since  $\Omega$  extends  $\omega$ . If  $(\gamma, \ell) \in H_1(B; H_1(M, \mathbb{Z}))$  then

$$\int_{\Sigma_{\gamma, \ell}} \Omega = \int_C \omega = \text{Flux}^\psi(\ell) \quad \text{mod (periods of } \omega),$$

where  $\psi$  is the monodromy around the loop  $\gamma$ . □

Recall that we define  $\mathcal{D}$  to be the holonomy group of the universal connection  $\theta$ . Notice that  $\mathcal{D}$  is not well defined as a subgroup of  $\text{Symp}(M, \omega)$  since it depends of the reference point  $f : (M, \omega) \rightarrow \mathbb{C}\mathbb{P}^\infty$ . However, according to standard theory of connections, different holonomy subgroups are conjugate.

**Corollary 4.9.** *The set  $\overline{\mathcal{D}}$  is the sum of all subgroups conjugate to  $\mathcal{D}$ .*

*Proof.* Let  $E = M \times \mathbb{R} / \sim$ , where  $(m, t + k) \sim (\psi^k(m), t)$  for  $k \in \mathbb{Z}$ . The form  $\Omega = \pi_M^* \omega$  on  $M \times \mathbb{R}$  descends to a closed connection form on  $E$  with a holonomy  $\psi$ . Since  $H_2(S^1, \mathbb{Z}) = 0$ , we deduce from Proposition 4.8 that if  $\psi \in \overline{\mathcal{D}}$  then  $\Omega$  has integral periods.

To prove the converse inclusion notice that it is enough to show that  $\mathcal{D} \subset \overline{\mathcal{D}}$ , according to the conjugacy invariance. Since  $\mathcal{D}$  is the universal holonomy group, for any  $\psi \in \mathcal{D}$  there exists a bundle  $M \rightarrow E \rightarrow S^1$  with a closed integral connection

form  $\Omega$  such that  $\psi$  is a holonomy of that form. Thus  $\psi \in \overline{\mathcal{D}}$  again by Proposition 4.8.  $\square$

**Proposition 4.10.**

- (1) Let  $H \subset \text{Symp}(M, \widehat{\omega})$  be any subgroup. If  $H \subset \overline{\mathcal{D}}$  then  $H < \mathcal{D}$  (up to conjugacy).
- (2)  $\mathcal{D} \cap \text{Symp}_0(M, \widehat{\omega}) = \overline{\mathcal{D}} \cap \text{Symp}_0(M, \widehat{\omega})$ ,
- (3)  $\mathcal{D}$  is a closed subgroup of  $\text{Symp}(M, \widehat{\omega})$

*Proof.* First we will prove (1) with the additional assumption that  $H$  is countable. Then we will use it to prove (2) and (3). Finally we will conclude general case of (1) with the use of (3).

- (1) We shall construct a fibration over a noncompact surface (possibly of infinite genus) with closed connection form  $\Omega$  and holonomy equal to  $H$ . Since  $H < \overline{\mathcal{D}}$  then, according to Proposition 4.8  $\langle \Omega, K \rangle \subset \mathbb{Z}$ . Because the base is a noncompact surface then the latter is equivalent to the integrality of  $\Omega$ . To see this notice that  $K = H_2(E, \mathbb{Z})$  as  $H_2(B, \mathbb{Z})$  is trivial. By the universality of  $\mathcal{D}$  we will conclude that, up to a conjugation,  $H < \mathcal{D}$ .

Take  $M_{\psi_i} \times S^1 \rightarrow T^2$  for each  $\psi_i \in H$ ,  $i = 1, 2, \dots$  and form a countable fiber connected sum  $(M_{\psi_1} \times S^1) \#_M (M_{\psi_2} \times S^1) \#_M \dots$ . It is a bundle over  $\Sigma_\infty$  the surface of infinite genus and admits a closed connection form. Indeed, every  $M_\psi \times S^1$  admits a closed connection form as in the proof of Corollary 4.9. One can easily extend it over the fiber connected sum.

Every loop in the connected sum is a composition of loops in the summands, thus the holonomy of the bundle is contained in  $H$ . Since all the generators of  $H$  are contained in the holonomy,  $H$  equals the holonomy group of that bundle.

- (2) Let  $\psi \in \overline{\mathcal{D}} \cap \text{Symp}_0(M, \omega)$ . Observe that  $\text{Flux}^\psi : H^1(M, \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z}$  is equal to the value of the usual flux on  $\psi$ . This shows that  $\overline{\mathcal{D}} \cap \text{Symp}_0(M, \omega) \subset \text{Flux}^{-1}(H^1(M, \mathbb{Z}))$ . In fact, the equality holds, because  $\mathcal{D}_0 = \text{Ham}(M, \omega)$ , according to Theorem 2.14 (2).

Now take one element  $\psi$  from each component of  $\overline{\mathcal{D}} \cap \text{Symp}_0(M, \omega)$  and generate subgroup  $H \subset \text{Symp}_0(M, \omega)$ . It is clearly countable, hence it is contained in  $\mathcal{D} \cap \text{Symp}_0(M, \omega)$ , due to the part (1). Since  $\mathcal{D}_0 = \text{Ham}(M, \omega)$ , we have  $\overline{\mathcal{D}} \cap \text{Symp}_0(M, \omega) = H \cdot \text{Ham}(M, \omega) \subset \mathcal{D} \cap \text{Symp}_0(M, \omega)$ .

- (3) Since  $\overline{\mathcal{D}} \cap \text{Symp}_0(M, \widehat{\omega}) = \text{Flux}^{-1}(H^1(M, \mathbb{Z}))$  is a closed subgroup of  $\text{Symp}_0(M, \omega)$ , (3) follows.

To finish the proof we need to show that (1) holds in full generality. Let  $H'$  be a countable dense subgroup of  $H$ . Such a group exists because  $\text{Symp}(M, \omega)$  is a group modelled on a separable and metrizable space of closed one-forms. Thus it is second countable. The subgroup  $H$  is second countable as well, according to the hereditary properties of the second countability, and hence it is separable. Take  $H'$  to be a countable subgroup of  $H$  generated by a countable dense subset. Then  $gH'g^{-1} \subset \mathcal{D}$  by already proved part of (1) and  $gHg^{-1} \subset \mathcal{D}$  by (3).  $\square$

**4.11. Other results.** We conclude this section with several geometric results. The proofs are straightforward and are left to the reader.

**Proposition 4.12.**

- (1) The action of  $\text{Symp}(W, \omega_W)$  on  $\text{Symp}(M, W)$  given by the composition  $\varphi \cdot f := \varphi \circ f$  preserves the connection  $\theta$ .

- (2) *The induced action on the associated bundle  $(M, \omega) \rightarrow M_W \rightarrow \mathbb{B}_W$  is given by  $\varphi \cdot [f, p] := [\varphi \circ f, p]$ . The evaluation map  $ev: M_W \rightarrow W$  is  $\text{Symp}(W, \omega_W)$ -equivariant.*

The general idea of this paper is to investigate spaces  $\mathbb{B}_W = \text{Symp}(M, W)/\text{Symp}(M, \omega)$ , where  $(M, \omega)$  is fixed and  $(W, \omega_W)$  varies. One can do the other way around and fix the target manifold  $(W, \omega_W)$ . This approach was taken by Haller-Vizman in [HV04]. They write explicitly a moment map for the action and express  $\mathbb{B}_W$  as a coadjoint orbit in the dual Lie algebra of the group  $\text{Ham}(W, \omega_W)$ .

**Proposition 4.13** ([HV04]).

- (1) *The fiber integral  $p_!(ev^*(\omega_W^{n+1})) \in \Omega^2(\mathbb{B}_W)$  is a symplectic form on  $\mathbb{B}_W$ .*  
(2) *The action induced on  $\mathbb{B}_W$  preserves the symplectic form*

$$p_!(ev^*(\omega_W^{n+1})).$$

## 5. GROUP COHOMOLOGY AND THE GROUP OF McDUFF

**5.1. Preparation on crossed homomorphisms.** Let  $S$  be a topological group and  $V$  be a topological  $S$ -module. All the modules we consider are right modules. A continuous map  $\phi: S \rightarrow V$  is called a *continuous crossed homomorphism* or *continuous 1-cocycle (with values in  $V$ )* if  $\phi(gh) = \phi(h) + \phi(g) \cdot h$ .

A crossed homomorphism  $S \rightarrow V$  may be understood as a deformation of the map  $S \rightarrow V \rtimes_\rho S$ . Precisely, the map

$$(\psi, \rho): S \rightarrow V \rtimes S$$

is a homomorphism if and only if  $\psi$  is a crossed homomorphism.

**5.2. An obstruction associated to a crossed homomorphism.** The kernel  $H$  of a 1-cocycle  $\psi$  is a pull-back in the following diagram:

$$\begin{array}{ccc} H & \longrightarrow & S \\ \downarrow & & \downarrow s \\ S & \xrightarrow{h} & V \rtimes S \end{array}$$

Where  $h = (\psi, \rho)$ , and  $s: S \rightarrow V \times S$  is the inclusion onto the second factor. Thus  $BH$  is a homotopy pull-back in the following diagram

$$\begin{array}{ccc} BH & \longrightarrow & BS \\ \downarrow & & \downarrow B_s \\ BS & \xrightarrow{Bh} & B(V \rtimes S) \end{array}$$

The homotopy fiber of the map  $B_s: BS \rightarrow B(V \rtimes S)$  equals  $V$ , moreover the fibration

$$V \rightarrow BA = V \times_{V \rtimes S} E(V \rtimes S) \rightarrow B(V \rtimes S)$$

is the universal  $V$ -bundle over  $B(V \rtimes S)$ . Note that  $V$  is an affine  $V \rtimes S$ -space. In particular the obstruction to the existence of a section  $BS \rightarrow BH$  is a pull-back of the obstruction to the existence of a section of  $B_s: BS \rightarrow B(V \rtimes S)$ .

**Theorem 5.3** (McDuff). *There exists a class  $\mathcal{O}_M \in H^2(B \text{Symp}(M, \widehat{\omega}), H^1(M, \mathbb{Z}))$  with the following property. A  $\text{Symp}(M, \widehat{\omega})$ -bundle  $M \rightarrow P \rightarrow B$  admits a reduction to  $\mathcal{D}_\tau$  if and only if  $f^*\mathcal{O} = 0 \in H^2(B, H^1(M, \mathbb{Z})/\text{torsion})$ , where  $f: B \rightarrow B \text{Symp}(M, \widehat{\omega})$  is the classifying map for the bundle  $P$ .*

*Proof.* Let  $S = \text{Symp}(M, \widehat{\omega})$  and  $V = H^1(M, \mathbb{R}/\mathbb{Z})$ . Due to Theorem 5.17 (which we shall prove later),  $\mathcal{D}_\tau$  is the kernel of the cocycle of McDuff  $F^s: \text{Symp}(M, \widehat{\omega}) \rightarrow H^1(M, \mathbb{R}/\mathbb{Z})$ . Since  $V = K(H^1(M, \mathbb{Z}), 1)$ , the total obstruction class  $\mathcal{O}_V$  to the existence of the lift  $B(V \rtimes S) \rightarrow BS$  is the element of  $H^2(BS, \pi_1(V)) = H^2(BS, H^1(M, \mathbb{Z}))$ . Thus we set  $\mathcal{O}_M = (Bh)^* \mathcal{O}_V$  where  $h = (\rho, F^s)$  and  $\rho$  is an action of  $\text{Symp}(M, \widehat{\omega})$  on  $H^1(M, \mathbb{Z})$ .  $\square$

In Section 5.5 we describe the Bockstein map  $\beta$  which, for given short exact sequence of  $S$ -modules

$$0 \rightarrow Z \rightarrow R \rightarrow R/Z \rightarrow 0,$$

to every (cohomology class of a) crossed homomorphism  $S \rightarrow R/Z$  associates an extension  $0 \rightarrow Z \rightarrow G \rightarrow S \rightarrow 0$ . If  $Z$  is discrete, then such an extension is classified by an element  $\sigma_G \in H^2(BS, Z)$  (cf. [Seg70]).

**Corollary 5.4.**  $\mathcal{O}_M = \sigma_{\mathcal{G}/\text{Map}_0(M, U(1))} = \beta([F^s])$ .

*Proof.* By the result of 5.5,  $\mathcal{D}_\tau$  and  $\mathcal{G}/\text{Map}_0(M, U(1))$  are homotopy equivariant, thus the obstruction to the existence of a section  $\mathcal{D}_\tau \rightarrow \text{Symp}(M, \widehat{\omega})$  equals the the obstruction to the existence of a section  $\mathcal{G}/\text{Map}_0(M, U(1)) \rightarrow \text{Symp}(M, \widehat{\omega})$ .

The second equality follows directly from the definition of  $\beta$ .  $\square$

**5.5. The homomorphism of Bockstein.** Let  $S$  be a topological group,  $R$  be a topological  $S$ -module, and  $Z$  its submodule. Let  $\phi: S \rightarrow R/Z$  be a crossed homomorphism. Define  $G$  to be the subgroup of the semidirect product  $R \rtimes S$  consisting of the pairs  $(s, r)$  such that  $\phi(s) = r + Z$ . We call  $G$  a pull-back of the following diagram:

$$\begin{array}{ccc} G & \xrightarrow{\Phi} & R \\ p \downarrow & & \downarrow \\ S & \xrightarrow{\phi} & R/Z \end{array}$$

where  $p$  and  $\Phi$  are the projections on the factors. The vertical arrows are homomorphisms and  $\Phi$  is a crossed homomorphism with respect to the action factored through  $S$ .

*Remark 5.6.*  $G$  is a pull-back in the category theory sense [GM03, p. 81] of the following diagram of groups

$$\begin{array}{ccc} G & \xrightarrow{(\Phi, \rho \circ p)} & R \rtimes S \\ p \downarrow & & \downarrow \\ S & \xrightarrow{(\phi, \rho)} & (R/Z) \rtimes S \end{array}$$

**Lemma 5.7.** *Let  $H$  be a kernel of  $\phi$ . Then there exist a section  $\sigma: H \rightarrow G$  (i.e. continuous homomorphism, such that  $p \circ \sigma = id_H$ ).*

*Proof.* Consider a pull-back:

$$\begin{array}{ccc} G' & \xrightarrow{\Phi'} & R \\ p_H \downarrow & & \downarrow \\ H & \xrightarrow{\phi|_H} & R/Z \end{array}$$

One observes that since  $\phi|_H$  is trivial then  $G'$  is a semidirect product  $Z \rtimes H$  and, in particular, there is a splitting  $H \rightarrow G'$ . Clearly,  $G'$  is a subgroup of  $G$ . Hence composing the splitting  $H \rightarrow G'$  with the inclusion  $G' < G$  we get the required section.  $\square$

Note that  $\ker p = Z$  and consider the two (topological) fibrations:

$$Z \rightarrow G \xrightarrow{p} S,$$

and

$$H \rightarrow S \xrightarrow{\phi} R/Z.$$

**Lemma 5.8.** *The corresponding long exact sequences of homotopy groups are matched up as in the following commutative diagram:*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \pi_{k+1}(S) & \longrightarrow & \pi_{k+1}(R/Z) & \longrightarrow & \pi_k(H) \longrightarrow \cdots \\ & & \parallel & & \partial_R \downarrow & & \pi_k(\sigma) \downarrow \\ \cdots & \longrightarrow & \pi_{k+1}(S) & \longrightarrow & \pi_k(Z) & \longrightarrow & \pi_k(G) \longrightarrow \cdots \end{array}$$

where  $\partial_R$  is the connecting homomorphism in the long exact sequence of the homotopy groups of the fibration  $Z \rightarrow R \rightarrow R/Z$ .

*Proof.* This is straightforward to check using standard diagram-chasing.  $\square$

**Corollary 5.9.** *If  $R$  is contractible then  $\sigma$  constructed above is a homotopy equivalence.*

*Proof.* If  $R$  is contractible then  $\partial_R$  is an isomorphism. Hence, by the five lemma,  $\pi_k(\sigma)$  is also an isomorphism and the statement follows.  $\square$

**5.10. The cocycles and group of McDuff.** Let  $\omega$  be a closed (possibly singular) two-form on a manifold  $W$ . Let  $SH_1(W; \mathbb{Z})$  be space of integral 1-cycles quotient out by boundaries of chains of zero  $\omega$ -area. There is a split extension

$$0 \rightarrow \mathbb{R}/\omega \rightarrow SH_1(W, \mathbb{Z}) \xrightarrow{p} H_1(W, \mathbb{Z}) \rightarrow 0.$$

The existence of a section  $s: H_1(W, \mathbb{Z}) \rightarrow SH_1(W, \mathbb{Z})$  follows since  $\mathbb{R}/\omega$  is divisible. Precisely such a section may be constructed inductively, as follows. Assume that the section is defined on a submodule  $V \subset H_1(W, \mathbb{Z})$ ,  $v \in H_1(W, \mathbb{Z})$  and  $n$  is the least integer such that  $nv \in V$ . Let  $\tilde{v}$  be any lift of  $v$  in  $SH_1(W, \mathbb{Z})$ . If  $n = \infty$  we set  $s(v) = \tilde{v}$ . In the other case we see that  $s(nv) - n\tilde{v} = [r] \in \mathbb{R}/\omega$ . If  $\gamma_{r/n}$  is a contractible loop bounding a disk of area  $r/n$  we set  $s(v) = \tilde{v} + \gamma_{r/n}$ .

Clearly, any group  $G$  that fixes  $\omega$  acts on  $SH_1(W, \mathbb{Z})$ . In [McD] McDuff defines a continuous crossed homomorphism  $F_s: G \rightarrow H^1(M; \mathbb{R}/\omega)$  as follows

$$F_s(g)(\gamma) := s(g_*(\gamma)) - g_*(s(\gamma)) \in \mathbb{R}/\omega = \ker(SH_1(W, \mathbb{Z}) \rightarrow H_1(W, \mathbb{Z})).$$

**Lemma 5.11.**  *$F_s$  satisfies*

$$F_s(gh)(\gamma) = g_*(F_s(h)(\gamma)) + F_s(g)(h_*(\gamma)).$$

*Proof.* Write for short  $F_s(g) = [s, g_*]$ . Then

$$F_s(gh) = [s, g_* h_*] = g_* [s, h_*] + [s, g_*] h_* = g_* F_s(h) + F_s(g) h_*$$

by the Leibniz rule.  $\square$

Since  $G$  acts trivially on  $\mathbb{R}/\omega$ ,  $g_*(F_s(h)(\gamma)) = F_s(h)(\gamma)$ . Hence

$$F_s(gh) = F_s(g) + (F_s(g)) \cdot h,$$

that is  $F_s$  is indeed a crossed-homomorphism.

We shall apply this construction in the following situation:

- (1)  $(W_1, \omega_1) = (L_\tau, \pi^* \omega)$  where  $L_\tau$  is a prequantum  $U(1)$ -bundle with  $[L] = \tau \in H^2(M, \mathbb{Z})$  and  $G_1 = \mathcal{G}$ .
- (2)  $(W_2, \omega_2) = (M, \omega)$  and  $G_2 = \text{Symp}(M, \widehat{\omega})$ .

**Lemma 5.12.**

- (1) Since  $\pi^*\omega$  is exact on  $L_\tau$ , the group of periods is trivial i.e.  $H^1(L_\tau, \mathbb{R}/\pi^*\omega) = H^1(M, \mathbb{R})$ .
- (2) One can choose sections  $\tilde{s}: H_1(L_\tau, \mathbb{Z}) \rightarrow SH_1(L_\tau, \mathbb{Z})$  and  $s: H_1(M, \mathbb{Z}) \rightarrow SH_1(M, \mathbb{Z})$  such that the following diagram commutes.

$$\begin{array}{ccccc} \mathbb{R} & \longrightarrow & SH_1(L_\tau, \mathbb{Z}) & \xleftarrow{\tilde{s}} & H_1(L_\tau, \mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow = \\ \mathbb{R}/\omega & \longrightarrow & SH_1(M, \mathbb{Z}) & \xleftarrow{s} & H_1(M, \mathbb{Z}) \end{array}$$

- (3) If  $\phi: \mathcal{G} \rightarrow \text{Symp}(M, \widehat{\omega})$  assigns to a bundle map its underlying symplectomorphism then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{F_{\tilde{s}}} & H^1(M; \mathbb{R}) \\ \phi \downarrow & & \downarrow \\ \text{Symp}(M, \widehat{\omega}) & \xrightarrow{F_s} & H^1(M; \mathbb{R}/\omega) \end{array}$$

- (4) The restriction of  $F_{\tilde{s}}$  to  $\text{Map}_0(M, U(1))$  is trivial, and thus descends to a crossed homomorphism  $\widehat{F}_s: \mathcal{G}/\text{Map}_0(M, U(1)) \rightarrow H^1(M; \mathbb{R})$ .
- (5) The following diagram

$$\begin{array}{ccc} \mathcal{G}/\text{Map}_0(M, U(1)) & \xrightarrow{\widehat{F}_s} & H^1(M; \mathbb{R}) \\ \downarrow & & \downarrow \\ \text{Symp}(M, \omega) & \xrightarrow{F_s} & H^1(M; \mathbb{R}/\mathbb{Z}) \end{array}$$

is a pull-back.

*Proof.* Most of the statements are trivial.

(2) First we have to define  $\tilde{s}$  on  $K = \ker H_1(L_\tau, \mathbb{Z}) \rightarrow H_1(M, \mathbb{Z})$  in such a way that its image lives in  $\ker SH_1(L_\tau, \mathbb{Z}) \rightarrow SH_1(M, \mathbb{Z})$ . The generator of  $K$  is represented by a loop in a fiber, and the value of  $\tilde{s}$  may also be taken to be a loop in the fiber. Then we extend  $\tilde{s}$  to whole  $H_1(L_\tau, \mathbb{Z})$  as explained above. It clearly descends to the map  $s: H_1(M, \mathbb{Z}) \rightarrow SH_1(M, \mathbb{Z})$ .

(4) This is obvious since  $\text{Map}_0(M, U(1))$  is connected and the fiber of the map  $H^1(M, \mathbb{R}) \rightarrow H^1(M, \mathbb{R}/\mathbb{Z})$  is discrete.  $\square$

*Remark 5.13* (about Lemma 5.12 (2)). McDuff [McD] shows that for each  $s$  one can choose a lift  $\tau$  of  $[\omega]$  and a section  $\tilde{s}$  such that  $s$  is constructed as above.

McDuff proves that the restriction of  $F_s$  to  $\text{Symp}_0(M, \omega)$  is equal to the flux homomorphism. She defines  $\text{Ham}^{s\mathbb{Z}}(M, \omega) := \ker F_s$ . Clearly, the component of the identity of  $\text{Ham}^{s\mathbb{Z}}(M, \omega)$  is equal to  $\text{Ham}(M, \omega)$ .

**Corollary 5.14.** *There exists a continuous group homomorphism*

$$\text{Ham}^{s\mathbb{Z}}(M, \omega) \rightarrow \mathcal{G}/\text{Map}_0(M, U(1))$$

*that is a homotopy equivalence.*

*Proof.* Since  $H^1(M; \mathbb{R})$  is contractible, the statement follows from Corollary 5.9.  $\square$

**5.15. The groups  $\text{Ham}^{\text{SZ}}$  and  $\mathcal{D}$  are equal.** Let  $G$  be a topological group and  $F: G_0 \rightarrow A$  a  $G$ -equivariant homomorphism to a right  $G$ -module  $A$ . That is it satisfies  $F(f^{-1}gf) = F(g) \cdot f$ . Let  $H$  be a subgroup of  $G$  intersecting every connected component of  $G$  and such that  $H \cap G_0 = \ker F$ . Let  $\sigma: G/G_0 = \pi_0(G) \rightarrow H$  be a set theoretic section of the natural projection  $p: G \rightarrow G/G_0$ . Define 1-cocycle  $F^\sigma: G \rightarrow A$  by

$$F^\sigma(g) := F(\sigma(p(g))^{-1}g).$$

The following lemma that is a reformulation of Proposition 1.8(ii) from [McD].

**Lemma 5.16.** *The 1-cocycle  $F^\sigma$  extends  $F$  and  $H = \ker F^\sigma$ .*

*Proof.* It is straightforward that  $F^\sigma$  extends  $F$ . To prove that  $F^\sigma$  is a 1-cocycle choose  $f, g \in G$ . Since  $\sigma(G) \subset H$ ,  $h = \delta(\sigma \circ p)(f, g) = \sigma(p(fg))\sigma(p(g))^{-1}\sigma(p(f))^{-1} \in H$ . We calculate

$$\begin{aligned} G_0 \ni \sigma(fg)^{-1}fg &= h\sigma(p(g))^{-1}\sigma(p(f))^{-1}fg \\ &= h\sigma(p(g))^{-1}gg^{-1}\sigma(p(f))^{-1}fg \\ &= h(\sigma(p(g))^{-1}g)(g^{-1}\sigma(p(f))^{-1}fg) \in hG_0. \end{aligned}$$

Thus  $h \in G_0$  and finally  $h \in \ker F = G_0 \cap H$ . Then

$$\begin{aligned} F^\sigma(fg) &= F(\sigma(p(fg))^{-1}fg) \\ &= F(h) + F(\sigma(p(g))^{-1}g) + F(g^{-1}\sigma(p(f))^{-1}fg) \\ &= F^\sigma(g) + F^\sigma(f) \cdot g \end{aligned}$$

The proof that  $H = \ker F^\sigma$  is also immediate and is left to the reader.  $\square$

**Theorem 5.17.** *The group  $\text{Ham}^{\text{SZ}}(M, \omega)$  and the holonomy group  $\mathcal{D}$  are isomorphic.*

*Proof. Step 1.* We will show that  $\text{Ham}^{\text{SZ}}(M, \omega) \subset \overline{\mathcal{D}}$ . We have to observe that if  $[l] \in H_1(M, \mathbb{Z})^\psi$  then  $F_s(\psi)(l) = \text{Flux}^\psi(l)$ .

*Step 2.* By Proposition 4.10 (1) we conclude that  $\text{Ham}^{\text{SZ}}(M, \omega) \subset \mathcal{D}$  (up to conjugacy).

*Step 3.* Let  $F$  be the following composition

$$\text{Symp}_0(M, \widehat{\omega}) \xrightarrow{\text{Flux}} H^1(M, \mathbb{R})/\Gamma_\omega \rightarrow H^1(M, \mathbb{R}/\mathbb{Z}).$$

It follows from the definition that  $\text{Ham}^{\text{SZ}}(M, \omega) \cap \text{Symp}_0(M, \widehat{\omega}) = \ker F$ . Let  $\sigma$  be a section  $\pi_0(\text{Symp}(M, \widehat{\omega})) \rightarrow \text{Ham}^{\text{SZ}}(M, \omega)$ . Then by Lemma 5.16  $\text{Ham}^{\text{SZ}}(M, \omega) = \ker F^\sigma$ . By Theorem 2.14(3)  $\mathcal{D} = \text{Ham}(M, \omega) \cap \text{Symp}_0(M, \widehat{\omega}) = \ker F$ . Since  $\sigma(\pi_0(\text{Symp}(M, \widehat{\omega}))) \subset \text{Ham}^{\text{SZ}}(M, \omega) \subset \mathcal{D}$ , by Lemma 5.16, we conclude that  $\mathcal{D} = \ker F^\sigma$ .  $\square$

**5.18. Proof of Theorem 2.18.** Let  $\tau \in H^2(M, \mathbb{Z})$  be a preimage of the class  $[\omega] \in H^2(M, \mathbb{R})$  of the symplectic form. Let  $(M, \omega) \rightarrow E \rightarrow B$  be a symplectic fibration with the classifying map  $f: B \rightarrow B\text{Symp}(M, \widehat{\omega})$ .

**Lemma 5.19.** *There exists a lift  $\widehat{f}: B \rightarrow B(\mathcal{G}/\text{Map}_0(M, U(1)))$  if and only if  $d_2(\tau) = 0$ . Here  $d_2$  is the differential in the spectral sequence associated to  $E$ .*

*Proof.* There exists a lift  $\widehat{f}: B \rightarrow B(\mathcal{G}/\text{Map}_0(M, U(1)))$  if and only there exist a lift  $\widetilde{f}: B_2 \rightarrow B\mathcal{G}$  over the two-skeleton of  $B$ . Indeed, we take a lift over the two skeleton by composing the lift  $\widetilde{f}$  with the projection  $B\mathcal{G} \rightarrow B(\mathcal{G}/\text{Map}_0(M, U(1)))$ . Since the fiber  $BH^1(M, \mathbb{Z})$  of the bundle  $B(\mathcal{G}/\text{Map}_0(M, U(1))) \rightarrow B\text{Symp}(M, \widehat{\omega})$  has trivial higher homotopy then all the further obstructions vanish and we extend the lift to whole  $B$ .

Next, according to Theorem 2.16 (2)  $\iff$  (3), the existence of a lift  $\widehat{f}: B_2 \rightarrow B\mathcal{G}$  is equivalent to the fact that the class  $\tau = i^*(\widehat{\tau})$ , where  $i: M \rightarrow E_2$  is the inclusion of the fiber into the total space of the restriction of  $E \rightarrow B$  over the two-skeleton  $B_2$ . This in turn is equivalent to the vanishing of  $d_2(\tau)$  in the spectral sequence for  $E$ .  $\square$

**Lemma 5.20** ([KM05]). *If  $d_2(\tau) = 0$  then  $d_3(\tau) \in E_3^{3,0}$  is a torsion element.*

*Proof.* Recall that  $\dim M = 2n$  and that  $\pi_1(\text{BSymp}(M, \widehat{\omega}))$  acts trivially on  $H^{2n}(M, \mathbb{R})$ . Then we have that

$$\begin{aligned} 0 &= d_3[\tau^{n+1}] \\ &= (n+1)d_3[\tau] \otimes [\tau^n] \in H^3(\text{BSymp}(M, \omega); H^{2n}(M, \mathbb{Z})). \end{aligned}$$

It means that  $d_3[\tau]$  is a torsion element.  $\square$

*Proof of Theorem 2.18.* Suppose that there is a lift  $\widehat{f}: B \rightarrow B\mathcal{D}$ . It implies, due to Theorem 5.17, that there is a lift  $\widehat{f}: B \rightarrow B\mathcal{G}/\text{Map}_0(M, U(1))$ . Lemma 5.19 and Lemma 5.20 show that  $d_2(\tau) = 0$  and  $d_3(\tau)$  are torsion elements.

Since all differentials of  $[\omega]$  vanish, there exists a closed extension  $\Omega'$  of  $\omega$ . We shall define a new closed connection form  $\Omega := \Omega' + \pi^*\beta$  so that it is integral (cf. Section 3 in [McD]). Since  $H_2(B, \mathbb{Z})$  is torsion-free,  $H_2(E, \mathbb{Z}) = \pi_*(H_2(E, \mathbb{Z})) \oplus K$ , where  $K := \ker \pi_*$  as in Section 4.7. It follows from Proposition 4.8 that  $\Omega'$  has integral periods on the cycles from  $K$ . We then define  $\beta \in \Omega^2(B)$  to be any two-form satisfying  $\int_{\pi_*(C_i)} \beta = -\int_{C_i} \Omega'$ , where  $\{\pi_*(C_i)\}$  forms a basis of the image  $\pi_*(H_2(E, \mathbb{R}))$ .

Finally, since  $E$  admits closed integral connection form it is an integral symplectic configuration, due to Theorem 2.16. Notice that the class of the above integral connection form might not restrict to  $\tau$ , i.e.  $\tau \neq i^*[\Omega] \in H^2(M, \mathbb{Z})$ .  $\square$

## 6. THE COHOMOLOGY RING OF SYMPLECTIC CONFIGURATIONS

**6.1. Characteristic classes of configurations.** We define a characteristic class of configurations of  $(M, \omega)$  in  $(W, \omega_W)$  to be an element of the cohomology ring  $H^*(\mathbb{B}_W)$ . If  $M$  is compact then certain characteristic classes can be obtained as fiber integrals

$$\chi_k^W := p_!(e\nu^*([\omega_W^{n+k}])).$$

These classes are natural in the sense that if  $\psi: (W, \omega_W) \rightarrow (V, \omega_V)$  is a symplectic embedding then  $\chi_k^W = \Psi^*\chi_k^V$ , where  $\Psi: \mathbb{B}_W \rightarrow \mathbb{B}_V$  is the map induced by  $\psi$ .

Since any integral symplectic manifold embeds into  $\mathbb{C}\mathbb{P}^N$ , we define universal characteristic classes with respect to  $\mathbb{C}\mathbb{P}^\infty$ . We denote the above fiber integrals by  $\chi_k \in H^{2k}(\mathbb{B})$ . In this case those are usual characteristic classes of a fibration with the structure group  $\mathcal{G}$ .

A fundamental question is whether these classes are *symplectic* that is whether they come from  $H^*(B\text{Symp})$  via the canonical map  $\mathbb{B} \rightarrow B\text{Symp}$ . In general the answer is negative.

**Lemma 6.2.** *Let  $\sigma \in \pi_2(\mathbb{B})$  be an element coming from  $H^0(M; \mathbb{Z})$  (see Theorem 2.10). Then  $\langle \chi_1, \sigma \rangle \neq 0$ . In particular,  $\chi_1$  is not symplectic class. Moreover,  $\chi_1$  is of infinite order in  $H^*(\mathbb{B})$ .*

*Proof.* Recall that  $\sigma$  is represented by a configuration  $M \times S^2 \rightarrow \mathbb{C}\mathbb{P}^\infty$  such that the  $S^2$  summand represents the positive generator of  $\pi_2(\mathbb{C}\mathbb{P}^\infty)$ . Clearly,  $e\nu^*(\omega_\infty) = \omega \oplus \omega_{S^2}$  and hence  $\pi_!(e\nu^*(\omega_\infty^{n+1})) = (n+1)\omega_{S^2}$ . Since the configuration is a trivial fibration, no symplectic characteristic class can be nontrivial.

In order to prove the last statement, consider a configuration  $P := M \times (S^2 \times \cdots \times S^2) \rightarrow \mathbb{C}\mathbb{P}^\infty$  such that every  $S^2$  is as above. Clearly,  $\chi_1(P)$  is a sum of generators and its top power is nonzero. Since it works for any number of factors  $S^2$ , it follows that  $\chi_1$  is of infinite order.  $\square$

Let  $(M, \omega) \xrightarrow{i} M_{\text{Ham}(M, \omega)} \xrightarrow{p} \text{BHam}(M, \omega)$  be the universal Hamiltonian fibration. Recall that the *coupling class*  $\Omega \in H^2(M_{\text{Ham}(M, \omega)})$  is defined by the following two conditions:

- (1)  $i^*(\Omega) = [\omega]$
- (2)  $p_!(\Omega^{n+1}) = 0$

One can define characteristic classes by  $\mu_k := p_!(\Omega^{n+k}) \in H^{2k}(\text{BHam}(M, \omega))$  [JK, KM05].

Let  $\widetilde{\text{B}} := \text{Symp}(M, \mathbb{C}\mathbb{P}^\infty) / \text{Ham}(M, \omega)$  and let  $(M, \omega) \xrightarrow{i} M_{\widetilde{\text{B}}} \xrightarrow{p} \widetilde{\text{B}}$  be the associated bundle. Notice that  $\widetilde{\text{B}} \rightarrow \text{B}$  is the universal cover. We have the following commutative diagram of classifying spaces:

$$\begin{array}{ccc} \widetilde{\text{B}} & \longrightarrow & \text{B} \\ \downarrow & & \downarrow \\ \text{BHam}(M, \omega) & \longrightarrow & \text{BSymp}(M, \omega) \end{array}$$

We want to compare the pull-backs of characteristic classes  $\chi_k$  and  $\mu_k$  in  $H^*(\widetilde{\text{B}})$ . To avoid clumsy notation we will denote these pull-backs by the same symbols as the original classes. The guiding idea is that the  $\chi_k$  classes might be easier to calculate than  $\mu_k$ 's.

**Lemma 6.3.** *Let  $(M, \omega) \xrightarrow{i} M_{\widetilde{\text{B}}} \xrightarrow{p} \widetilde{\text{B}}$  be the above associated bundle. Then  $\chi_k \cong \mu_k$  modulo an ideal in  $H^*(\widetilde{\text{B}})$  generated by  $\chi_1$ .*

*Proof.* First observe that the coupling class  $\Omega = ev^*(\omega_0) - \frac{1}{n+1}p^*(\chi_1)$ . Calculating the appropriate fiber integral we get the statement.

$$\begin{aligned} \chi_k &= p_!(ev^*(\omega_0^{n+k})) \\ &= p_!\left(\left(\Omega + \frac{1}{n+1}p^*(\chi_1)\right)^{n+k}\right) \\ &= p_!\left(\sum_{i=0}^{n+k} \binom{n+k}{i} \Omega^{n+k-i} \left(\frac{1}{n+1}p^*(\chi_1)\right)^i\right) \\ &= \sum_{i=0}^{n+k} \binom{n+k}{i} \left(\frac{1}{n+1}\right)^i p_!(\Omega^{n+k-i}) (\chi_1)^i \\ &= \sum_{i=0}^k \binom{n+k}{i} \left(\frac{1}{n+1}\right)^i \mu_{k-i} (\chi_1)^i \end{aligned}$$

$\square$

**Corollary 6.4.** *Let  $(M, \omega) \rightarrow P \rightarrow B$  be a configuration whose classifying map admits a lift to  $\widetilde{\text{B}}$ . If  $H^2(B) = 0$  then  $\chi_k(P) = \mu_k(P)$ . In particular, if  $\chi_k(P)$  is nonzero then so does  $\mu_k(P)$ .*  $\square$

**6.5. Calculating  $\chi$ -classes.** It is important to know if there are relations between characteristic classes. If  $(M, \omega) \rightarrow P \rightarrow S^{2k}$  is a symplectic configuration with nontrivial  $\chi_k(P)$  then  $\chi_k$  is not a product in  $H^*(B)$ . Indeed, if it were a product then its pull-back  $\chi_k(P) \in H^{2k}(S^{2k})$  would vanish.

**Theorem 6.6.** *Let  $(M, \omega) \rightarrow P \rightarrow S^{2k}$  be a symplectic configuration,  $k > 1$ . The following conditions are equivalent.*

- (1)  $\chi_k(P) \neq 0$ ;
- (2)  $\mu_k(P) \neq 0$ ;
- (3)  $ev_*[P] \neq 0$  in  $H_{2n+2k}(\mathbb{C}P^\infty; \mathbb{Z})$ ;
- (4) *There exists  $a \in H^2(P)$  such that  $a^{n+k} \neq 0$  (i.e.  $P$  is c-symplectic), and  $a$  extends  $[\omega]$ .*

Moreover, any of these conditions implies that  $k \leq n + 1$ .

*Proof.* The first two conditions are equivalent, according to Corollary 6.4. Each of them is equivalent to the third by the basic properties of fiber integration:

$$\langle \chi_k(P), [S^{2k}] \rangle = \langle ev^*(\omega_\infty^{n+k}), [P] \rangle = \langle \omega_\infty^{n+k}, ev_*[P] \rangle.$$

The third condition implies the fourth one since the c-symplectic class is given by  $[ev^*(\omega_\infty)]$ . The converse implication follows from Theorem 2.16.

For the last statement, observe that  $H^{2n+2}(P) = 0$  if  $k > n + 1$ . Hence  $P$  cannot be c-symplectic.  $\square$

*Remark 6.7.* Consider the question of finding a large  $k \in \mathbb{Z}$  such that the Hurewicz map  $\pi_{2k}(\text{BSymp}(M, \omega)) \rightarrow H_{2k}(\text{BSymp}(M, \omega); \mathbb{Q})$  is nontrivial. The characteristic classes constructed as fiber integrals give quite restricted method for answering this question. As mentioned in the above theorem one can detect homologically nontrivial spheres up to dimension  $\dim M + 2$ .

### 6.8. Examples.

- (1) Consider the symplectic fibration  $\mathbb{C}P^1 \rightarrow \mathbb{C}P^3 \rightarrow S^4 = \mathbb{H}P^1$ . The total space is symplectic hence, due to Theorem 6.6, we get that  $\mu_2 = \chi_2 \neq 0$  in  $H^4(B(S^2))$ .
- (2) Moreover, take a symplectic fibration  $F \rightarrow P \rightarrow \mathbb{C}P^3$  such that  $P$  admits a compatible symplectic structure. Composing with the above we get another symplectic fibration

$$(M, \omega) \rightarrow P \rightarrow S^4,$$

whose fiber is the total space of  $F \rightarrow (M, \omega) \rightarrow \mathbb{C}P^1$ . Again, we obtain that  $\chi_2 = \mu_2 \neq 0$  in  $H^4(B(M, \omega))$ .

- (3) Similarly, consider the symplectic fibration

$$\mathbb{C}P^3 \rightarrow \mathbb{C}P^7 \rightarrow S^8 = \text{CaP}.$$

Here, we get that  $\chi_4$  and  $\mu_4$  are nonzero for  $\mathbb{C}P^3$ .

- (4) Taking a symplectic fibration over  $\mathbb{C}P^7$  admitting a compatible symplectic structure we get a result analogous to the one in (2).
- (5) Let  $(M, \omega) := U(m)/U(m_1) \times \cdots \times U(m_l)$ ,  $m = m_1 + \cdots + m_l$  and  $m_1 \geq m_2 \geq \cdots \geq m_l$  be a generalized flag manifold equipped with the homogeneous symplectic form. It is proved in Keřdra-McDuff [KM05] that the classes  $\mu_k \in H^{2k}(\text{BSymp}_0(M, \omega))$  are nonzero for  $1 < k \leq m$ . Moreover, it is shown that they are nontrivial for certain fibrations over spheres, say  $(M, \omega) \rightarrow P \rightarrow S^{2k}$ , with the structure group  $SU(m)$ . According to Theorem 6.6,  $P$  is c-symplectic, however, it is not clear if it admits any symplectic form in general.

As in the previous examples we consider a symplectic fibration  $F \rightarrow E \rightarrow P$  and compose it with  $P \rightarrow S^{2k}$ . Suppose that this composition is a symplectic fibration with fiber  $F \rightarrow Q \rightarrow M$ . Then the classes  $\mu_k(E) \in H^{2k}(\text{BSymp}_0(Q))$  are nonzero.

**6.9. Chern-Weil approach.** Recall from Section 4.4 that the universal configuration fibration admits a connection with curvature form  $\Theta$ . It is an  $ad$ -invariant two-form on the total space  $\text{Symp}(M, \mathbb{C}P^\infty)$  of the universal principal fibration with values in the Lie algebra of  $\text{Symp}(M, \omega)$  which is identified with closed one-forms on  $M$ . Since the reduced holonomy group is Hamiltonian (Theorem 2.14 (1)), the curvature form has values in the Lie algebra of  $\text{Ham}(M, \omega)$  which is identified with  $C_0^\infty(M)$ , the space of functions on  $M$  with zero mean value.

Let  $\mathcal{P}: C_0^\infty(M) \rightarrow \mathbb{R}$  be an invariant polynomial. It means that  $\mathcal{P}(h_1 \circ \psi, \dots, h_k \circ \psi) = \mathcal{P}(h_1, \dots, h_k)$ , where  $\psi \in \text{Symp}(M, \omega)$  and  $h_i \in C_0^\infty(M)$ . Then

$$\chi_{[f]}^{\mathcal{P}}(X_1, Y_1, \dots, X_k, Y_k) := \mathcal{P}(\Theta_f(X_1^h, Y_1^h), \dots, \Theta_f(X_k^h, Y_k^h))$$

defines a  $2k$ -form on the base  $B$ . The definition does not depend on the choice of  $f$  in the fiber since the polynomial  $\mathcal{P}$  is invariant. It is a standard fact that these forms are closed [KN96b]. We define universal characteristic classes to be cohomology classes of the form  $\chi^{\mathcal{P}}$ .

The following formula gives a sequence of invariant polynomials.

$$\mathcal{P}_k(h_1, \dots, h_k) := \int_M h_1 \dots h_k \omega^n$$

The proof of the next lemma is analogous to the proof of Lemma 3.9 in Kędra-McDuff [KM05].

**Lemma 6.10.**  $[\chi^{\mathcal{P}_k}] = \text{const } \chi_k$  for  $k > 1$ . □

#### REFERENCES

- [ABKP00] J. Amorós, F. Bogomolov, L. Katzarkov, and T. Pantev. Symplectic Lefschetz fibrations with arbitrary fundamental groups. *J. Differential Geom.*, 54(3):489–545, 2000. With an appendix by Ivan Smith.
- [EM02] Y. Eliashberg and N. Mishachev. *Introduction to the h-principle*, volume 48 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2002.
- [GLS96] Victor Guillemin, Eugene Lerman, and Shlomo Sternberg. *Symplectic fibrations and multiplicity diagrams*. Cambridge University Press, Cambridge, 1996.
- [GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.
- [Gom95] Robert E. Gompf. A new construction of symplectic manifolds. *Ann. of Math. (2)*, 142(3):527–595, 1995.
- [Gro86] Mikhael Gromov. *Partial differential relations*, volume 9 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1986.
- [HV04] S. Haller and C. Vizman. Non-linear grassmannians as coadjoint orbits. *Math. Ann.*, 329(4):771–785, 2004.
- [JK] Tadeusz Januszkiewicz and Jarek Kędra. Characteristic classes of smooth fibrations. *math.SG/0209288*.
- [KM05] Jarek Kędra and Dusa McDuff. Homotopy properties of Hamiltonian group actions. *Geom. Topol.*, 9:121–162 (electronic), 2005.
- [KN96a] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996.
- [KN96b] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. II*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996.
- [McD] Dusa McDuff. Enlarging the hamiltonian group. *math.SG/0503268*.
- [MM65] John W. Milnor and John C. Moore. On the structure of Hopf algebras. *Ann. of Math. (2)*, 81:211–264, 1965.
- [MS98] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

- [NR61] M. S. Narasimhan and S. Ramanan. Existence of universal connections. *Amer. J. Math.*, 83:563–572, 1961.
- [Seg70] Graeme Segal. Cohomology of topological groups. In *Symposia Mathematica, Vol. IV (INDAM, Rome, 1968/69)*, pages 377–387. Academic Press, London, 1970.
- [Spa66] Edwin H. Spanier. *Algebraic topology*. McGraw-Hill Book Co., New York, 1966.
- [tD87] Tammo tom Dieck. *Transformation groups*, volume 8 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.

GAL:

MATHEMATICAL INSTITUTE, UNIVERSITY OF WROCLAW, PL. GRUNWALDZKI 2/4, 50-384 WROCLAW, POLAND  
INSTITUTE OF MATHEMATICS, UNIVERSITY OF WARSAW, UL. BANACHA 2, 02-097 WARSZAWA, POLAND  
*E-mail address:* [sgal@math.uni.wroc.pl](mailto:sgal@math.uni.wroc.pl)  
*URL:* <http://www.math.uni.wroc.pl/~sgal>

KĘDRA:

INSTITUTE OF MATHEMATICS, UNIVERSITY OF SZCZECIN, WIELKOPOLSKA 15, 70-451 SZCZECIN, POLAND  
DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ABERDEEN, MESTON BUILDING, KING'S COLLEGE,  
ABERDEEN AB24 3UE, SCOTLAND, UK  
*E-mail address:* [kedra@maths.abdn.ac.uk](mailto:kedra@maths.abdn.ac.uk)  
*URL:* <http://www.maths.abdn.ac.uk/~kedra>