Chapter 2

Linear Programming

1. INTRODUCTION

The main feature of a linear programming problem (LPP) is that all functions involved, the objective function and those expressing the constraints, must be linear. The appearance of a single nonlinear function, either on the objective or in the constraints, suffices to reject the problem as an LPP.

Definition 2.1 *(General form of an LPP) An LPP is an optimization problem of the general form*

$$\text{Minimize} \quad cx = \sum_i c_i x_i$$

subject to

$$\sum_i a_{ji} x_i \le b_j, \quad j = 1, \dots, p,$$

$$\sum_i a_{ji} x_i \ge b_j, \quad j = p+1, \dots, q,$$

$$\sum_i a_{ji} x_i = b_j, \quad j = q+1, \dots, m,$$

where c_i, b_j, a_{ji} are data of the problem. Depending on the particular values of p and q we may have inequality constraints of one type and/or the other, and equality restrictions as well.

We can gain some insight into the structure and features of an LPP by looking at one simple example.

Example 2.2   *Consider the LPP*

$$\text{Maximize} \quad x_1 - x_2$$

subject to

$$x_1 + x_2 \le 1, \quad -x_1 + 2x_2 \le 2,$$
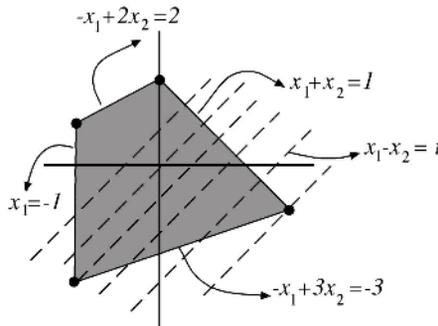$$x_1 \ge -1, \quad -x_1 + 3x_2 \ge -3.$$

Figure 2.1. The feasible region and level curves in an LPP.

It is interesting to realize the shape of the set of vectors in the plane satisfying all the requirements that the constraints express: Each inequality represents a "half-space" at one side of the line corresponding to changing the inequality to an equality. Thus the intersection of all four half-paces will be the "feasible region" for our problem. Notice that this set has the form of a polygon or polyhedron. See Figure 2.1.

On the other hand, the cost, being linear, has level curves that are again straight lines of equation $x_1 - x_2 = t$, a constant. When t moves, we obtain parallel lines. The question is then how big t can become so that the line of equation $x_1 - x_2 = t$ meets the above polygon somewhere. Graphically, it is not hard to realize that the optimal vector corresponds to the vertex $(-1/2, 3/2)$, and the value of the maximum is 2.

Note that regardless of what the cost is, as long as it is linear, the optimal value will always correspond to one of the four vertices of the feasible set. These vertices play a crucial role in the understanding of LPP, as we will see.

An LPP can adopt several equivalent forms. The initial form usually depends on the particular formulation of the problem, or the most convenient way in which the constraints can be represented. The fact that all possible formulations correspond to the same underlying optimization problem enables us to fix one reference format, and refer to this form of any particular problem for its analysis.

Definition 2.3 *(Standard form of an LPP) An LPP in standard form is*

$$\text{Minimize} \quad cx \quad \text{under} \quad Ax = b, \quad x \geq 0. \tag{P}$$

Thus, the ingredients of every LPP are:
1. an $m \times n$ matrix A, with $n > m$ and typically n much greater than m;
2. a vector $b \in \mathbf{R}^m$;
3. a vector $c \in \mathbf{R}^n$.

Notice that cx is the inner product of the two vectors c and x, while Ax is the product of the matrix A and the vector x. We will not make the distinction between these possibilities, since it will be clear from the context. It is therefore a matter of finding the minimum value the inner product cx can take on as x runs through all feasible vectors $x \in \mathbf{R}^n$ with nonnegative components ($x \geq 0$) satisfying the additional, and important restriction $Ax = b$. We are

also interested in one vector x (or all vectors x) where this minimum value is achieved.

We have argued that any LPP can in principle be transformed into the standard form. It is therefore desirable that readers understand how this transformation can be accomplished. We will proceed in three steps.

1. Variables not restricted in sign. For the variables not restricted in sign, we use the decomposition into positive and negative parts according to the identities

$$x = x^+ - x^-, \quad |x| = x^+ + x^-,$$

where

$$x^+ = \max\{0, x\} \geq 0, \quad x^- = \max\{0, -x\} \geq 0.$$

What we mean with this decomposition is that a variable x_i not restricted in sign can be written as the difference of two new variables that are nonnegative:

$$x_i = x_i^{(1)} - x_i^{(2)}, \quad x_i^{(1)}, x_i^{(2)} \geq 0.$$

2. Transforming inequalities into equalities. Quite often, restrictions are formulated in terms of inequalities. In fact, an LPP will come many times in the form

$$\text{Minimize} \quad cx \quad \text{under} \quad Ax \leq b, \quad A'x = b', \quad x \geq 0.$$

Notice that by using multiplication by minus signs we can change the direction of an inequality. In this situation, the use of "slack variables" permits the passage from inequalities to equalities in the following way. Introduce new variables by putting

$$y = b - Ax \geq 0.$$

If we now set

$$X = (x \quad y), \quad \tilde{A} = (A \quad \mathbf{1}),$$

where $\mathbf{1}$ is the identity matrix of the appropriate size, the inequality restrictions are written now as

$$\tilde{A}X = b,$$

so all constraints are now in the form of equalities, but we have a greater number of variables (one more for each inequality).

3. Transforming a max into a min. If the LPP asks for a maximum instead of for a minimum, we can keep in mind that

$$\max(\text{expression}) = -\min(-\text{expression});$$

or more explicitly,

$$\max\{cx : Ax = b, x \geq 0\} = -\min\{(-c)x : Ax = b, x \geq 0\}.$$

An example will clarify any doubt about these transformations.

Example 2.4 *Consider the LPP*

$$\text{Maximize} \quad 3x_1 - x_3$$

subject to

$$x_1 + x_2 + x_3 = 1,$$
$$x_1 - x_2 - x_3 \leq 1,$$
$$x_1 + x_3 \geq -1,$$
$$x_1 \geq 0, \quad x_2 \geq 0.$$

1. Since there are variables not restricted in sign, we must set

$$x_3 = y_1 - y_2, \quad y_1 \geq 0, y_2 \geq 0,$$

so that the problem will change to

$$\text{Maximize} \quad 3x_1 - y_1 + y_2$$

subject to

$$x_1 + x_2 + y_1 - y_2 = 1,$$
$$x_1 - x_2 - y_1 + y_2 \leq 1,$$
$$x_1 + y_1 - y_2 \geq -1,$$
$$x_1 \geq 0, \quad x_2 \geq 0,$$
$$y_1 \geq 0, \quad y_2 \geq 0.$$

2. We use slack variables so that inequality restrictions may be transformed into equalities: $z_1 \geq 0$ and $z_2 \geq 0$ are used to transform

$$x_1 - x_2 - y_1 + y_2 \leq 1, \quad x_1 + y_1 - y_2 \geq -1,$$

respectively, into

$$x_1 - x_2 - y_1 + y_2 + z_1 = 1, \quad z_1 \geq 0,$$

and

$$x_1 + y_1 - y_2 - z_2 = -1, \quad z_2 \geq 0.$$

The problem will now have the form

$$\text{Maximize} \quad 3x_1 - y_1 + y_2$$

subject to

$$x_1 + x_2 + y_1 - y_2 = 1,$$
$$x_1 - x_2 - y_1 + y_2 + z_1 = 1,$$
$$x_1 + y_1 - y_2 - z_2 = -1,$$
$$x_1 \geq 0, \quad x_2 \geq 0,$$
$$y_1 \geq 0, \quad y_2 \geq 0,$$
$$z_1 \geq 0, \quad z_2 \geq 0.$$

3. Finally, we easily change the maximum to a minimum:

$$\text{Minimize} \quad -3x_1 + y_1 - y_2$$

subject to

$$x_1 + x_2 + y_1 - y_2 = 1,$$
$$x_1 - x_2 - y_1 + y_2 + z_1 = 1,$$
$$x_1 + y_1 - y_2 - z_2 = -1,$$
$$x_1 \geq 0, \quad x_2 \geq 0,$$
$$y_1 \geq 0, \quad y_2 \geq 0,$$
$$z_1 \geq 0, \quad z_2 \geq 0,$$

bearing in mind that once the value of this minimum is found, the corresponding maximum will have its sign changed.

If we uniformize the notation by writing

$$(X_1, X_2, X_3, X_4, X_5, X_6) = (x_1, x_2, y_1, y_2, z_1, z_2),$$

the problem will obtain its standard form

$$\text{Minimize} \quad X_3 - X_4 - 3X_1$$

subject to

$$X_1 + X_2 + X_3 - X_4 = 1,$$
$$X_1 - X_2 - X_3 + X_4 + X_5 = 1,$$
$$X_1 + X_3 - X_4 - X_6 = -1,$$
$$X \geq 0.$$

Once this problem has been solved and we have an optimal solution X and the value of the minimum m, the answer to the original LPP would be as follows: The maximum is $-m$, and it is achieved at the point $(X_1, X_2, X_3 - X_4)$. Or if you like, the value of the maximum will be the value of the original linear cost function at the optimal solution $(X_1, X_2, X_3 - X_4)$. Notice how the slack variables do not enter into the final answer, since they are auxiliary variables.

Concerning the optimal solution of an LPP, all situations can actually happen:
1. the set of admissible vectors is empty;
2. it can have no solution at all, because the cost cx can decrease indefinitely toward $-\infty$ for feasible vectors x;
3. it can admit a single optimal solution, and this is the most desirable situation;
4. it can also have several, in fact infinitely many, optimal solutions; indeed, it is very easy to check that if x_1 and x_2 are optimal, then any convex combination

$$tx_1 + (1-t)x_2, \quad t \in [0,1],$$

is again an optimal solution.

In the next section, we will see how to solve an LPP in its standard form by the simplex method. Though interior-point methods are becoming more and more important in mathematical programming, in both versions, linear and nonlinear, we tend to believe that they are the subject of a second course on mathematical programming. The fact is that the simplex method helps greatly in understanding the special structure of linear programming as well as duality.

2. THE SIMPLEX METHOD

We look more closely at an LPP in its standard form, and describe the simplex method, which is one of the most successful approaches for finding the optimal solution for such problems. Let us concentrate, then, on the problem of finding a vector x solving

$$\text{Minimize} \quad cx$$

subject to

$$Ax = b, \quad x \geq 0.$$

There is no restriction in assuming that the linear system $Ax = b$ is solvable, for otherwise, there would be no feasible vectors. Moreover, if A is not a full-rank matrix, we can select a submatrix A' by eliminating several rows of A, and the corresponding components of b, so that the new matrix A' has full rank. In this case we obtain the new, equivalent, LPP

$$\text{Minimize} \quad cx$$

subject to

$$A'x = b', \quad x \geq 0,$$

where b' is the subvector of b obtained by eliminating the components corresponding to the rows of A we have previously discarded. This new LPP is equivalent to the initial one in the sense that they both have the same optimal solutions, but the matrix A' for the reduced problem is a full-rank matrix. We shall therefore assume, without loss of generality, that the rank of the $m \times n$ matrix A is m (remember $m \leq n$) and that the linear system $Ax = b$ is solvable.

There are special feasible vectors that play a central role in the simplex method. These are the solutions of the linear system $Ax = b$ with nonnegative and at least $n-m$ null components. In fact, all of these extremal points or basic solutions, as they are typically called, can, in principle, be obtained by solving all square $m \times m$ linear systems $Ax = b$ where $n - m$ components of x are set to zero, and discarding those solutions with at least one negative component. The very special linear structure of an LPP enables us to concentrate on these basic solutions when looking for optimal solutions.

Lemma 2.5   *If the LPP*

$$\text{Minimize} \quad cx$$

subject to

$$Ax = b, \quad x \geq 0,$$

admits an optimal solution, then there is also an optimal solution that is a basic solution.

This is quite evident if we realize that the feasible set of an LPP is some kind of "polyhedron," and therefore minimum (or maximum) values of linear functions must be taken on at a vertex. See Figure 2.2, and remember the comments on Example 2.2.
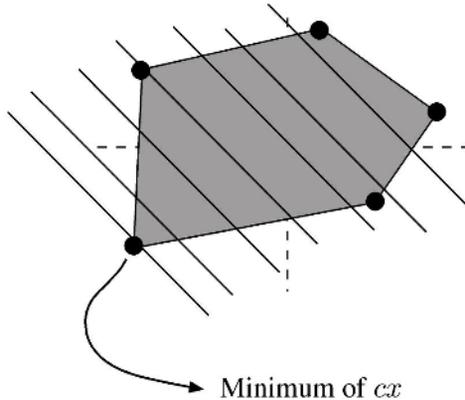

Figure 2.2. Optimal basic solution.

For the proof of the lemma, assume that x is an optimal solution with at least $m + 1$ strictly positive components, and let d be a nonvanishing vector in the kernel of A with the property that $x_i = 0$ implies $d_i = 0$. If x has at least $m + 1$ strictly positive components, such a vector d can always be found (why?).

We claim that necessarily $cd = 0$. For otherwise, if t is small enough so that $x + td$ is feasible (i.e., $x + td \geq 0$) and $tcd < 0$, then the cost of the vector $x + td$ is stricly smaller than that of x itself, which is impossible if x is optimal. Therefore, $cd = 0$, and the vectors $x + td$ are also optimal as long as they are feasible. All that remains to be done is to move t away from zero (either positive or negative) until some of the components of $x + td$ hit zero for the

first time. For such value of t we would have an optimal solution with at least one more vanishing component than x. This process can be repeated as long as the vector d is not the zero vector, i.e., until the optimal solution has at least $n - m$ vanishing components.

As an immediate consequence of Lemma 2.5, we can find optimal solutions for an LPP by looking at all solutions of the system $Ax = b$ with at least $n - m$ zeros, discarding those with some negative component, and, by computing the cost of the remaining ones, decide on the optimal vector. This process would indeed lead us to one optimal solution, but the simplex method aims to organize these computations in a judicious way so that we can reach the optimal solution as soon as possible without having to go through an exhaustive analysis of all extremal points. In some cases, though, the simplex method actually goes through all basic solutions before finding an extremal solution. This situation is, however, rare.

The SM starts at one particular extremal feasible vector x, which, after an appropriate permutation of indices, can be written as

$$x = (\, x_B \quad 0 \,), \quad x_B \in \mathbf{R}^m, \quad 0 \in \mathbf{R}^{n-m}, \quad x_B \geq 0.$$

The basic iterative step consists in setting one of the components of x_B to zero (the so-called "leaving variable"), and letting a vanishing component of $0 \in \mathbf{R}^{n-m}$ (the "entering variable") become positive. In this way, we have moved from an extremal point to an adjacent one. The key issue is to understand how to make these choices (leaving and entering variables) in such a way that we lower the cost as much as possible. Furthermore, we need a criterion to decide when the cost cannot be decreased any more, so that we have actually found an optimal solution and no more iterative steps are needed. We discuss this procedure more precisely in what follows.

Let

$$x = (\, x_B \quad 0 \,), \quad x_B \in \mathbf{R}^m, \quad 0 \in \mathbf{R}^{n-m}, \quad x_B \geq 0,$$

be a feasible extremal vector. In the same way, the matrix A, after the same permutation of columns, can be decomposed as

$$A = (\, B \quad N \,).$$

The equation $Ax = b$ is equivalent to

$$(\, B \quad N \,) \begin{pmatrix} x_B \\ 0 \end{pmatrix} = b, \quad x_B = B^{-1}b.$$

The cost of such a vector x is

$$cx = \begin{pmatrix} c_B & c_N \end{pmatrix} \begin{pmatrix} x_B \\ 0 \end{pmatrix} = c_B x_B = c_B B^{-1} b.$$

The basic step of the simplex method consists in moving to another feasible (adjacent) extremal point so that the cost has been lowered in such a movement. The change from $x = \begin{pmatrix} x_B & 0 \end{pmatrix}$ to $\overline{x} = \begin{pmatrix} \overline{x}_B & x_N \end{pmatrix}$, where x_N is at our disposal, will take place if we can ensure three requirements:

1. $A\overline{x} = b$;
2. $c\overline{x} < cx$;
3. $\overline{x} \geq 0$.

The first one forces us to take

$$\overline{x}_B = x_B - B^{-1} N x_N.$$

Indeed, notice that

$$\begin{pmatrix} B & N \end{pmatrix} \begin{pmatrix} \overline{x}_B \\ x_N \end{pmatrix} = b$$

implies

$$\overline{x}_B = B^{-1}(b - N x_N) = x_B - B^{-1} N x_N.$$

Consequently, the new cost will be

$$\begin{pmatrix} c_B & c_N \end{pmatrix} \begin{pmatrix} x_B - B^{-1} N x_N \\ x_N \end{pmatrix} = c_B(x_B - B^{-1} N x_N) + c_N x_N$$
$$= (c_N - c_B B^{-1} N) x_N + c_B x_B.$$

We see that the sign of

$$(c_N - c_B B^{-1} N) x_N$$

will dictate whether we have been able to decrease the cost by moving to the new vector

$$\begin{pmatrix} x_B - B^{-1} N x_N & x_N \end{pmatrix}.$$

The so-called vector of reduced costs

$$r = c_N - c_B B^{-1} N$$

will play an important role in deciding whether we can move to a new basic solution and lower the cost. Since $x_N \geq 0$ (by requirement 3 above), two situations may occur:

1.Stopping criterion. If all components of r turn out to be nonnegative, there is no way to lower the cost, and the present extremal point is indeed optimal. We have found a solution for our problem.

2. Iterative step. If r has some negative components, we can, in principle, lower the cost by letting those components of x_N become positive. However, we must exercise caution in this change in order to ensure that the vector

$$x_B - B^{-1}Nx_N \tag{2--1}$$

is feasible, i.e., will always have nonnegative coordinates. If this is not the case, even though the cost will have a smaller value in the vector

$$(\, x_B - B^{-1}Nx_N \quad x_N \,),$$

it will not be feasible and therefore wil not be admissible as an optimal solution of the LPP. We must ensure the nonnegativity of the extremal vectors.

Instead of looking for more general choices of x_N, the simplex method focuses on taking $x_N = tv$, where $t \geq 0$ and v is a basis vector having vanishing coordinates in all but one place, where it has 1. This means that we will change one component at a time. The chosen component is precisely the "entering variable." How is this variable to be selected? According to our previous discussion, we are trying to ensure that the product

$$rx_N = trv$$

will be as negative as it possibly can. Since v is a basis vector, rv is a component of r, and therefore v must be chosen as the basis vector corresponding to the most negative variable of r. Once v has been selected, we have to examine

$$x_B - B^{-1}Nx_N = B^{-1}b - tB^{-1}Nv \tag{2--2}$$

in order to determine the leaving variable. The idea is the following. When $t = 0$, we are sitting on our basic solution x_B. What can happen if t starts to move away from zero to the positive part? At this point three situations may occur. We discuss them succesively.

1. Infeasible solution. As soon as t becomes positive, the vector in (2–2) is not feasible any longer because one of its components is less than zero. In this case, we cannot use the chosen variable to reduce the cost, and we must turn to the next negative variable in r; or alternatively, we can simply take this variable as the leaving variable in spite of the fact that the cost will not decrease. This second choice is usually preferred due to coherence of the whole process.

2. Leaving variable. There is a positive threshold value of t at which one of the coordinates of (2–2) vanishes for the first time. We choose precisely this one as the leaving variable, and compute a new extremal point with smaller cost than the previous one.

3. No solution. No matter how big t becomes, we can lower and lower the cost, and none of the components in (2–2) will ever reach zero. The problem does not admit an optimal solution because we can reduce the cost indefinitely.
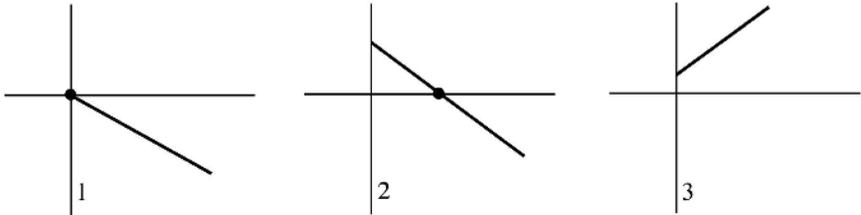

Figure 2.3. Three possibilities in choosing the leaving variable.

The issue is how we can decide in each particular situation whether we are in case 1, 2, or 3, above, and how to proceed accordingly. Notice that each expression in (2–2) represents a straight line as a function of t. The three possibilities are drawn in Figure 2.3.

Assume that we have chosen an entering variable identified with a basis vector v. We proceed as follows:

1. If there is one vanishing component of $x_B = B^{-1}b$ corresponding to a positive component of $B^{-1}Nv$ (diagram 1 in Figure 2.3), then as soon as t becomes positive this coordinate will be smaller than zero in (2–2), and the vector will not be feasible. We might resort to a different entering variable (a different basis vector v), which corresponds to another negative component of r, if available. If r does not have more negative components, we already

have the optimal solution, and the simplex method stops. Alternatively, and this choice is typically preferred for coherence, we can consider the vanishing ratio as one candidate for the process in 2 below.

2. Examine the ratios of the vectors $B^{-1}b$ over $B^{-1}Nv$ componentwise, and choose as leaving variable the one corresponding to the least of those ratios among the strictly positive ones, including, as remarked earlier, the vanishing ratios with positive denominators. These would certainly be chosen, if present, since they are smaller than the strictly positive ones. Disregard the quotients over zero including $0/0$. Start the whole process with the new extremal vector. Notice that these ratios correspond to the values of t when t intersects the horizontal axis in diagram 2 of Figure 2.3.

3. If there are no positive ratios, the LPP does not admit an optimal solution, since the cost can be indefinitely lowered by increasing the entering variable. This situation occurs when all diagrams are of the type 3 in Figure 2.3.

Since the set of feasible extremal points is finite, after a finite number of steps, the simplex method leads us to an optimal solution or to the conclusion that there is no optimal vector. In some very peculiar instances, the simplex method can enter a cyclic, infinite process. Such cases are so rare that we will pay no attention to them. One easy example is proposed as an exercise at the end of the chapter.

In practice, the computations can be organized in the following algorithmic fashion.

1. Initialization. *Find a square $m \times m$ submatrix B such that the solution of the linear system $Bx_B = b$ is such that $x_B \geq 0$.*

2. Stopping criterion. *Write*

$$c = (\, c_B \quad c_N \,), \quad A = (\, B \quad N \,).$$

Solve

$$z^T B = c_B$$

and look at the vector

$$r = c_N - z^T N.$$

If $r \geq 0$, stop: We already have an optimal solution. If not, choose the entering variable corresponding to the most negative component of r.

3. Main iterative step.   *Solve*

$$Bw = y,$$

where y is the entering column of N corresponding to the entering variable, and look at the ratios x_B/w componentwise. Among these ratios select those with positive denominators. Choose as leaving variable the one corresponding to the smallest ratio among the selected ones. Go to step 2. If there is no variable to select from, the problem does not admit an optimal solution.

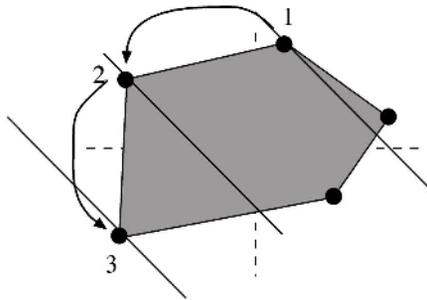We have tried to reflect the main iterative step of the simplex method in Figure 2.4.

Figure 2.4. Several iterative steps in the simplex method.

In order to ensure that our readers understand the strategy in the simplex method, how the entering and leaving variables are chosen and the stopping criterion, we are going to look briefly at several simple examples.

Example 2.6   *(Unique solution)*

$$\text{Minimize} \quad 3x_1 + x_2 + 9x_3 + x_4$$

subject to

$$x_1 + 2x_3 + x_4 = 4$$
$$x_2 + x_3 - x_4 = 2,$$
$$x_i \geq 0.$$

In this particular instance,

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad c = (\,3 \quad 1 \quad 9 \quad 1\,).$$

1. Initialization. Choose

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad c_B = (\,3 \quad 1\,), \quad c_N = (\,9 \quad 1\,).$$

2. Checking the stopping criterion. It is trivial to find

$$x_B = b = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

such that the initial vertex is $(4, 2, 0, 0)$ with cost 14. On the other hand,

$$z = c_B = (\,3 \quad 1\,), \quad r = (\,9 \quad 1\,) - (\,3 \quad 1\,)\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = (\,2 \quad -1\,).$$

Since not all components of r are nonnegative, we must go through the iterative process in the simplex method.

3. Iterative step. Choose x_4 as the entering variable, since this is the one associated with the negative component of r. Moreover,

$$w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \frac{x_B}{w} = \{4, -2\},$$

so that x_1 is the leaving variable, being the one corresponding to the least ratio among the ones we would select (ratios with positive denominators).

4. Checking the stopping criterion. These computations lead us to the new choice

$$B = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad c_B = (\,1 \quad 1\,), \quad c_N = (\,3 \quad 9\,).$$

It is easy to find

$$x_B = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

and the new extremal vector $(0, 6, 0, 4)$ with associated cost 10. The new vectors z and r are

$$z = (\, 2 \quad 1 \,), \quad r = (\, 3 \quad 9 \,) - (\, 2 \quad 1 \,) \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = (\, 1 \quad 4 \,).$$

Since all components of r are nonnegative, we have ended our search: The minimum cost is 10, and it is taken on at the vector $(0, 6, 0, 4)$.

Example 2.7   (Degenerate example)

$$\text{Minimize} \quad 3x_1 + x_2 + 9x_3 + x_4$$

subject to

$$x_1 + 2x_3 + x_4 = 0,$$
$$x_2 + x_3 - x_4 = 2,$$
$$x_i \geq 0.$$

In this particular case,

$$A = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad c = (\, 3 \quad 1 \quad 9 \quad 1 \,).$$

1. Initialization. Choose

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad c_B = (\, 3 \quad 1 \,), \quad c_N = (\, 9 \quad 1 \,).$$

2. Checking the stopping criterion. It is trivial to find

$$x_B = b = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

such that the initial vertex is $(0, 2, 0, 0)$ with cost 2. On the other hand,

$$z = c_B = (\, 3 \quad 1 \,), \quad r = (\, 9 \quad 1 \,) - (\, 3 \quad 1 \,) \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = (\, 2 \quad -1 \,).$$

Since not all components of r are nonnegative, we must go through the iterative process in the simplex method.

3. Iterative step. Choose x_4 as the entering variable, since it is the one associated with the negative component of r. Moreover,

$$w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \frac{x_B}{w} = \{0, -2\},$$

so that x_1 is the leaving variable, being the one corresponding to the least ratio among the ones we would select (ratios with positive denominators). We can predict, however, that because our only choice is a vanishing ratio, we will not be able to lower the cost in spite of going through an iterative step of the simplex method. In other words, the vertex $(0, 2, 0, 0)$ is already an optimal solution. Since for this optimal solution the stopping criterion does not hold for our original choice of B (r has negative coordinates), we must, for the sake of coherence of the scheme, go through an iterative step of the simplex method.

4. Checking the stopping criterion. The new choice

$$B = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad c_B = (1 \quad 1), \quad c_N = (3 \quad 9),$$

leads us to find

$$x_B = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

and the new extremal vector is again $(0, 2, 0, 0)$ with associated cost 2. The vectors z and r are

$$z = (2 \quad 1), \quad r = (3 \quad 9) - (2 \quad 1) \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = (1 \quad 4).$$

Since all components of r are nonnegative, we have ended our search as we had anticipated: The minimum cost is 2 and it is taken on at the vector $(0, 2, 0, 0)$.

Example 2.8   (No solution)

$$\text{Minimize} \quad -3x_1 + x_2 + 9x_3 + x_4$$

subject to

$$x_1 - 2x_3 - x_4 = -2,$$
$$x_2 + x_3 - x_4 = 2,$$
$$x_i \geq 0.$$

In this case,

$$A = \begin{pmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ 2 \end{pmatrix}, \quad c = (-3 \quad 1 \quad 9 \quad 1).$$

1. Initialization. If we were to choose

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} -2 & -1 \\ 1 & -1 \end{pmatrix}, \quad c_B = (-3 \quad 1), \quad c_N = (9 \quad 1),$$

then we would obtain

$$x_B = b = \begin{pmatrix} -2 \\ 2 \end{pmatrix}$$

which is not a feasible vector, since there is one negative coordinate. Let us take instead (second and fourth columns of A)

$$B = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad c_B = (1 \quad 1), \quad c_N = (-3 \quad 9).$$

2. Checking the stopping criterion. It is easy to find

$$x_B = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

such that the initial vertex is $(0, 4, 0, 2)$ with cost 6. On the other hand,

$$z = (-2 \quad 1), \quad r = (-3 \quad 9) - (-2 \quad 1) \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = (-1 \quad 4).$$

Since not all components of r are nonnegative, we must go through the iterative process in the simplex method.

3. Iterative step. Choose x_1 as the entering variable, since it is the one associated with the negative component of r. Moreover,

$$w = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \frac{x_B}{w} = \{-4, -2\}.$$

In this situation, we have no choice for the leaving variable: no positive denominator. This means that the proposed LPP does not admit an optimal

solution, i.e., the cost can be lowered indefinitely. This can be easily checked by considering the feasible vectors

$$\begin{pmatrix} t - 2 \\ t + 2 \\ 0 \\ t \end{pmatrix}, \quad t > 0.$$

The cost associated with such points is $8 - t$, which can clearly be sent to $-\infty$ by taking t sufficiently large.

Example 2.9   *(Multiple solution)*

$$\text{Minimize} \quad 3x_1 + 2x_2 + 8x_3 + x_4$$

subject to

$$x_1 - 2x_3 - x_4 = -2,$$
$$x_2 + x_3 - x_4 = 2,$$
$$x_i \geq 0.$$

In order to argue that there are infinitely many optimal solutions for this LPP, we will use the equality constraints to "solve" for x_1 and x_2 and take these expressions back into the objective function. Namely,

$$x_1 = 2x_3 + x_4 - 2 \geq 0, \quad x_2 - x_3 + x_4 + 2 \geq 0,$$

and the cost function becomes

$$6(2x_3 + x_4) - 2.$$

Since the first constraint reads

$$2x_3 + x_4 \geq 2,$$

it is clear that the minimum value of the cost will be achieved when

$$2x_3 + x_4 = 2.$$

We have the two basic solutions $(0, 1, 1, 0)$ and $(0, 4, 0, 2)$. Any convex combination of these two will also be an optimal solution

$$t(0, 1, 1, 0) + (1 - t)(0, 4, 0, 2), \quad t \in [0, 1].$$

We believe that it is elementary to understand the way in which the simplex method works after several examples. Computations can, however, be organized in tables (tableaux) to facilitate the whole process without having to explicitly write down the different steps as we have done in the previous examples. We will treat some of these practical issues in a subsequent section.

3.   DUALITY

Duality is a concept that intimately links the two following LPP:

$$\text{Minimize} \quad cx \quad \text{subject to} \quad Ax \geq b, \quad x \geq 0;$$
$$\text{Maximize} \quad yb \quad \text{subject to} \quad yA \leq c, \quad y \geq 0.$$

We will identify the first problem as the primal, and the second one as its associated dual. Notice how the same elements, the matrix A and the vectors b and c, determine both problems.

Definition 2.10 *(Dual problem) The dual problem of the LPP*

$$\text{Minimize} \quad cx \quad \text{subject to} \quad Ax \geq b, \quad x \geq 0$$

is the LPP

$$\text{Maximize} \quad yb \quad \text{subject to} \quad yA \leq c, \quad y \geq 0.$$

Although this format is not the standard one we have utilized in our discussion of the simplex method, it allows us to see in a more transparent fashion that the dual of the dual is the primal. This is, in fact, very easy to check by transforming minima to maxima and reversing inequalities by appropriately using minus signs (this is left to the reader).

If the primal problem is formulated in the standard form

$$\text{Minimize} \quad cx \quad \text{under} \quad Ax = b, \quad x \geq 0,$$