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László Székelyhidi Jr. (Leipzig, Germany)

Global solutions of the 2D Euler equations, starting with the work of Witold Wolibner

1. Introduction

The motion of incompressible ideal fluids can be described in two ways. The Lagrangian description gives the flow map $X(x, t)$, denoting the position of the fluid particle at time $t > 0$ whose position was at x initially, such that the map $x \mapsto X(x, t)$ is a volume-preserving homeomorphism. The Eulerian description gives a vector field $v(x, t)$, denoting the velocity of the flow at time $t > 0$ at the point x, such that $\operatorname{div} v = 0$.

It is possible to formulate an existence theory for the Cauchy problem based solely on either the Lagrangian or the Eulerian formulation. If the functions involved are sufficiently regular, the different theories coincide, but this need not be the case if one lowers the regularity assumptions. In the 1920s the local-in-time existence of classical solutions was proved in the pioneering work of L. Lichtenstein [18] and N. M. Günther [15]. The first global existence result in two space dimensions was proved by W. Wolibner in [27], a few months later a somewhat different proof by E. Hölder appeared in [16] in the same journal. The paper [27] not only contains the existence of global classical solutions, but also a considerable part of what is now known as the Yudovich theory for bounded vorticity. Indeed, a careful reading shows that Wolibner in fact proves the global existence of a weak solution for bounded initial vorticity.

In this note our purpose is to give an account of Wolibner's work, and survey some recent developments concerning global existence results for weak solutions in two dimensions.

2. Notions of solution

Let us fix some notation. From now on we will consider – for simplicity – a simply connected bounded domain $\Omega \subset \mathbb{R}^2$ with C^1 boundary $\partial\Omega$. For a given time $T > 0$ let $Q_T = \Omega \times (0, T)$.

2.1. *Classical solution.* By a classical solution one usually means a vectorfield $v \in C^1(\overline{Q_T})$ and a scalar function $p \in C^1(Q_T)$ such that

$$\begin{aligned} \partial_t v + v \cdot \nabla v + \nabla p &= 0, \\ \operatorname{div} v &= 0 \end{aligned} \tag{E}$$

in Q_T, together with the boundary condition $v \cdot \nu = 0$ on $\partial\Omega$ and the initial condition $v(x, 0) = v_0(x)$, where $v_0 \in C^1(\overline{\Omega})$ is a given initial velocity such that $\operatorname{div} v_0 = 0$.

It is a classical fact that, as a consequence of the divergence-free condition, the flow-map $X = X(x, t)$, defined by

$$\frac{\mathrm{d}}{\mathrm{d}t} X(x, t) = v(X(x, t), t), \tag{1}$$

$$X(x, 0) = x, \tag{2}$$

is an area-preserving diffeomorphism $\overline{\Omega} \to \overline{\Omega}$ for each t.

2.2. *Weak solution.* Although distributions were not yet developed in the 1930s, it was certainly recognized at the time, that one needs a notion of solution that allows discontinuities in the vorticity (vortex patches) and in the velocity (vortex sheets), see [23] and [19]. Wolibner defines weak solutions in [27] as a pair $(v, p) \in C(\overline{Q_T})$ such that for any simply connected region $U \subset \Omega$ with C^1 boundary and any $t \in (0, T)$

$$\int_U v(x, t)\, \mathrm{d}x - \int_U v(x, 0)\, \mathrm{d}x + \int_0^t \int_{\partial U} v(v \cdot \nu) + p\, \mathrm{d}l\, \mathrm{d}s = 0, \tag{W}$$

where ν is the unit outward normal to U, and moreover the flow-map X is an area-preserving homeomorphism $\overline{\Omega} \to \overline{\Omega}$ for each t. It is easy to see that if $(v, p) \in C^1$ is a solution of (W) then it is a solution of (E).

This definition still includes the pressure. On the other hand it is well known that the pressure can be recovered (uniquely, up to an additive constant) from (E) via the equation

$$-\Delta p = \operatorname{div} \operatorname{div}(v \otimes v) \tag{3}$$

together with appropriate Neumann boundary conditions for p obtained from (E). Therefore it makes sense to eliminate the pressure from the equation. This can be done in several ways.

One way is to (formally) project the first equation of (E) onto divergence-free fields. For instance, one obtains

$$\iint_{Q_T} \partial_t\varphi \cdot v + \nabla\varphi : v \otimes v \,\mathrm{d}x\mathrm{d}t + \int_\Omega \varphi(x,0) \cdot v_0(x)\,\mathrm{d}x = 0 \tag{D}$$

for all $\varphi \in C_c^\infty(\Omega \times [0,T); \mathbb{R}^2)$ with $\operatorname{div}\varphi = 0$. Accordingly, the weakest possible notion of solution in the Eulerian description is given by a vectorfield $v \in L^2_{\mathrm{loc}}(Q_T)$ with $\operatorname{div} v = 0$ in the sense of distributions such that (D) holds.

2.3. *The vorticity equation.* The vorticity of the fluid flow is defined as[1]

$$\omega = \operatorname{curl} v. \tag{4}$$

From a given vorticity it is possible to recover a unique velocity field using the Biot–Savart law. Indeed, given ω, the velocity v is a solution to the elliptic system consisting of (4), $\operatorname{div} v = 0$ and the boundary condition $v \cdot \nu = 0$ on $\partial\Omega$. The solution can be written using the logarithmic potential as $v = K\omega$, where

$$K\omega(x) := \nabla^\perp \frac{1}{2\pi} \int_\Omega \omega(y) \log|x-y|\,\mathrm{d}y + \nabla\Phi(x) \tag{5}$$

where $\nabla^\perp = (-\partial_2, \partial_1)$ and Φ is a harmonic function on Ω with a Neumann boundary condition so that $K\omega \cdot \nu = 0$ on $\partial\Omega$ (see e.g. [27, § 3] or [19, p. 483]). It is not difficult to see that, if Ω is bounded and simply-connected, then v is uniquely determined. On the other hand, if either the domain is unbounded or not simply-connected, further conditions are necessary. Usually one asks for decay conditions at infinity and that the circulation along interior boundary components is zero.

By taking the curl of (E) one obtains the formulation of the Euler equations for the vorticity:

$$\begin{aligned} \partial_t\omega + v\cdot\nabla\omega &= 0, \\ v &= K\omega. \end{aligned} \tag{V}$$

[1] In the 1930s the vorticity was defined to be $\frac{1}{2}\operatorname{curl} v$, since this is the correct magnitude of the rotation represented by the vorticity, cf. [19, p. 190] or [22, p. 7].

Notice that, analogously to (D), the system (V) can be formulated in the sense of distributions, since $v \cdot \nabla\omega = \operatorname{div}(v\omega)$. Here the weakest possible requirement is that $v\omega \in L^1_{\mathrm{loc}}(Q_T)$. Using L^p estimates for K (as in (10) below) and Sobolev embedding, this condition will be satisfied if $\omega \in L^2_{\mathrm{loc}}\big(0,T;L^{4/3}_{\mathrm{loc}}(\Omega)\big)$. Moreover, in this case a solution ω of (V) yields a solution $v := K\omega$ of (D).

2.4. *Lagrangian solution.* Equation (V) means that the vorticity is simply transported by the flow:

$$\omega(X(x,t),t) = \omega(x,0)$$

for all $t \geq 0$. Then, using the operator K from (5) we can obtain a closed system of equations for X. Regarding the notation, it helps here to denote the dependence on time t as a subscript, i.e. $X_t = X(\cdot,t)$, $v_t = v(\cdot,t)$, etc. The equations for X_t then take the form

$$\begin{aligned} X_t(x) &= x + \int_0^t v_s(X_s)\,\mathrm{d}s, \\ v_t &= K\omega_t, \\ \omega_t &= \omega_0(X_t^{-1}). \end{aligned} \tag{L}$$

3. Wolibner's global existence result

In [27] W. Wolibner constructs a global classical solution (v,p) with given initial data given by $v_0 = K\omega_0$, where $\omega_0 \in C^\alpha(\overline{\Omega})$. Moreover he proves

Theorem 3.1 [27, § 2][2]. *Let* $\omega_0 \in L^\infty(\Omega)$ *and* $T > 0$. *The Lagrangian system* (L) *admits a unique solution* X_t *on* $[0,T]$, *which is an area-preserving homeomorphism* $\overline{\Omega} \to \overline{\Omega}$.

This is achieved with the following estimates, proved in [27, § 1].

3.1. *The estimates of Wolibner.* Let $\omega \in L^\infty(\Omega)$ and $v = K\omega$. Then

$$|v(x) - v(y)| \leq H\,|x-y| \log \frac{1}{|x-y|} \tag{6}$$

[2] Wolibner does not at this stage specify the regularity assumptions on ω_0. However, all his estimates depend solely on the supremum-norm of the vorticity. Therefore one should attribute this existence theorem to him – at variance with the prevailing opinion in the literature (cf. [22, Chapter 8], [5, Chapter 5]), attributing it to V. Yudovich in [28].

for any $x, y \in \Omega$ with $|x - y| \le \frac{1}{2}$, where the constant H only depends on the supremum-norm of ω. Moreover, given $v \in C(\overline{\Omega})$ with (6), the ODE

$$\frac{dX_t}{dt} = v(X_t)$$
$$X_0(x) = x$$

admits a unique solution $X_t(x)$ satisfying

$$|x - y|^{e^{Ht}} \le |X_t(x) - X_t(y)| \le |x - y|^{e^{-Ht}}. \tag{7}$$

The deformation estimate (7) is the key point. It follows from (6) via the differential inequality

$$f'(t) \le c f(t) \log \tfrac{1}{f(t)} \tag{8}$$

for $0 < f \le 1/2$. Wolibner concludes (without proof) that (8) implies $f(t) \le f(0)^{e^{-ct}}$ ([27, pp. 713 and 719]). E. Hölder writes in [16] that this estimate is the key point in his paper as well, and he attributes the technique of obtaining the estimate from integrating (8) to E. Kamke [17].

It should be noted that the deformation estimate (7) is optimal: an example was constructed by H. Bahouri and J.Y. Chemin, where $|X_t(x) - X_t(0)| \ge |x|^{e^{-t}}$, see [5, § 5.4].

3.2. *Sketch Proof of Theorem* (3.1). The construction involves discretizing the time-dependence of v_t in (L). More precisely, given a partition $0 = t_0 < t_1 < \cdots < t_n = T$, a map X_t can be inductively defined with $X_0(x) = x$ and such that

$$X_t(x) = X_{t_k}(x) + \int_{t_k}^{t} v_k(X_s)\, ds,$$

for $t_k < t < t_{k+1}$, where $v_k = v_k(x)$ is defined by

$$v_k = K\omega_k,$$
$$\omega_k = \omega_0(X_{t_k}^{-1}).$$

Because the estimate (7) only depends on the supremum of ω_k, it holds uniformly for $t \in [0, T]$, and by standard compactness arguments (Arzèla–Ascoli) one can pass to the limit in a sequence of increasingly refined partitions.

The global existence and uniqueness statement of Wolibner for classical solutions follows from the following assertions:

Proposition 3.2 [27, § 3]. *Let $\omega_0 \in L^\infty(\Omega)$ and X_t the solution of* (L). *Then $(v,p) \in C(\overline{Q_T})$ satisfies* (W).

Proposition 3.3 ([16; 27, § 4–5]). *Let $\omega_0 \in C^\alpha(\Omega)$ and X_t the solution of* (L). *Then ω satisfies*

$$|\omega(x,t) - \omega(y,t)| \leq C|x-y|^{\alpha e^{-Ht}}$$

provided $|x-y| \leq 1/2$.

The Hölder-continuity of ω_t in turn implies that $(v,p) \in C^1(Q_T)$, i.e. it is a classical solution.

Independently of Wolibner and at the same time, E. Hölder in [16] also showed that there exists a global classical solution. The observation of Hölder was precisely the statement of Proposition 3.3, following from the estimates (6)-(7). This can then be combined with the local existence result of L. Lichtenstein (e.g. [18, p. 481]) to show that the local solution can be continued indefinitely.

3.3. *Vortex Patches.* The existence result in Wolibner allows one to define a further notion of solution. Assuming that ω_0 is the characteristic function of a smooth subdomain $U \subset \Omega$, the vorticity ω_t of the solution to (L) obtained in Theorem 3.1 will be the characteristic function of the transported set $U(t) = X_t(U)$. Accordingly, one can formulate the problem solely for the evolution of the boundary curve $\partial U(t)$. For simplicity let us assume that $\Omega = \mathbb{R}^2$ and let $\gamma_t \colon S^1 \to \mathbb{R}^2$ be the parametrization of the boundary curve at time t. From (5) and integrating by parts we obtain

$$\partial_t \gamma_t(s) = \frac{1}{2\pi} \int_0^{2\pi} \log|\gamma_t(s) - \gamma_t(r)| \partial_r \gamma_t(r) \, dr. \tag{P}$$

The problem of global regularity of γ_t was proposed by A. Majda [21] and solved by J. Y. Chemin in 1991. The result can be stated as

Theorem 3.4 ([4; 5, § 5.4]). *Assume that $\gamma_0 \in C^{1,\alpha}(S^1; \mathbb{R}^2)$ is an embedding. Then* (P) *admits a unique global solution $\gamma_t \in C^{1,\alpha}(S^1; \mathbb{R}^2)$ which remains an embedding.*

Furthermore, in this case (i.e. when $\gamma_t \in C^{1,\alpha}$) the log-Lipschitz estimate (6) can be strengthened to a true Lipschitz estimate for the velocity v_t, see [2].

4. Uniqueness

The uniqueness part of Theorem 3.1 follows essentially from the estimate (7). To pass from the uniqueness for (L) to the uniqueness for the Eulerian formulation (E) or (W), one needs to be able to show that the vorticity is transported by the flow. More precisely, one needs to show that the velocity v is given by

$$v_t = K\omega_t,$$

where $\omega_t := \omega_0 \circ X_t^{-1}$!. This is far from obvious and is indeed false for (D) and even for (W) without additional assumptions, as we will see below in Section 5.2.

4.1. *The Kelvin circulation theorem.* Let us assume that $(v, p) \in C^1(Q_T)$ is a solution of (E). The classical circulation theorem of Kelvin states that the circulation

$$\oint_{\partial U(t)} v_t \cdot \mathrm{d}l$$

is independent of time, where $U \subset \Omega$ is a simply connected domain with C^1 boundary and $U(t) := X_t(U)$ is the image of U under the flow-map associated to v. Next, define $\omega_t := \omega_0 \circ X_t^{-1}$ and $w_t := K\omega_t$, where $\omega_0 = \operatorname{curl} v_0$. Then

$$\oint_{\partial U(t)} w_t \cdot \mathrm{d}l = \int_{U(t)} \omega_t \, \mathrm{d}x = \int_U \omega_0 \, \mathrm{d}x$$

essentially by definition. Therefore, using the circulation theorem we find that v_t and w_t have the same circulation along any $\partial U(t)$. It is then a classical fact (e.g. Morera's theorem in complex analysis) that $v_t - w_t$ is the gradient of a harmonic function. By using the boundary conditions we can deduce that $v_t = w_t$. This is the essence of Wolibner's argument in [27, § 5]. Combining this with Theorem 3.1 and Proposition 3.3 leads to the existence and uniqueness of classical solutions for initial datum $v_0 \in C^{1,\alpha}(\Omega)$.

4.2. *The Energy Method.* If the initial vorticity ω_0 is merely bounded, the solution v constructed in Theorem 3.1 is not necessarily differentiable (only satisfies the log-Lipschitz estimate (6)), therefore the proof of the circulation theorem is not applicable. The uniqueness for the class of weak solutions (v, p) with bounded vorticity was obtained several decades later by V. Yudovich in [28]. The theorem of Yudovich can then be reformulated as:

Theorem 4.1 [28, § 3]. *Let $\omega \in L^\infty(Q_T)$ be a weak solution of* (V). *Then*

$$\omega_t = \omega_0(X_t^{-1}),$$

where X_t is solution obtained in Theorem 3.1.

The essential element of Yudovich's proof which is not contained in Wolibner's paper is the energy of the flow, defined as $E(t) = \frac{1}{2}\int_\Omega |v(x,t)|^2 \mathrm{d}x$. If the vorticity ω is bounded, the first equation of (E) can be written as

$$\partial_t v + \omega v^\perp + \nabla\Big(p + \tfrac{|v|^2}{2}\Big) = 0$$

in the sense of distributions, where $v^\perp = (-v_2, v_1)$. Taking the scalar product with v and integrating in x leads to $\frac{\mathrm{d}E}{\mathrm{d}t} = 0$, i.e. the energy is conserved in time. In a similar vain one can consider the difference between two solutions v, $\tilde{v}$. Setting $f(t) = \frac{1}{2}\|v_t - \tilde{v}_t\|^2_{L^2(\Omega)}$, one is lead to

$$\begin{aligned} f'(t) &\leq \int_\Omega |\nabla v_t||v_t - \tilde{v}_t|^2 \,\mathrm{d}x \\ &\leq \|\nabla v_t\|_{L^p(\Omega)}\|v_t - \tilde{v}_t\|^{2/p}_{L^\infty(\Omega)} f(t)^{1-1/p} \end{aligned} \tag{9}$$

for any $1 \leq p \leq \infty$. In order to use that $v_t = K\omega_t$ with $\omega_t \in L^\infty$, Yudovich derives the following potential-theoretic estimates on K for any $1 < p < \infty$ and $0 < \alpha < 1$:

$$\begin{aligned} \|\nabla K\omega\|_{L^p(\Omega)} &\leq Cp\|\omega\|_{L^p(\Omega)}, \\ \|K\omega\|_{C^\alpha(\Omega)} &\leq \tfrac{C}{1-\alpha}\|\omega\|_{L^\infty(\Omega)}, \end{aligned} \tag{10}$$

where the constant C is independent of p and α. In particular we have

$$f'(t) \leq Cpf(t)^{1-1/p} \tag{11}$$

for any $p < \infty$. Yudovich concludes the uniqueness by integrating this inequality and then letting $p \to \infty$. Alternatively, one can optimize the exponent p. Indeed, it is easy to check that for any fixed t the left-hand side of (11) is minimized when $p = -\log f(t)$. With this choice of p we are lead to the differential inequality (8) of Wolibner. Consequently, we deduce

$$\|v_t - \tilde{v}_t\|_{L^2(\Omega)} \leq \|v_0 - \tilde{v}_0\|^{e^{-Ct}}_{L^2(\Omega)},$$

provided the right-hand side is less than 1/2. This version of Yudovich's proof can be found for instance in [5, Chapter 5].

By the same reasoning the second estimate in (10) is equivalent to (6).

4.3. *Renormalized solutions.* There is another route to uniqueness, via the theory of renormalized solutions developed by R. DiPerna and P. L. Lions in [7] and L. Ambrosio in [1]. Let us briefly recall, that if $v \in W^{1,2}(\Omega)$, a *regular Lagrangian flow* for the ODE (1) is defined to be a map X_t for almost every t, which preserves Lebesgue-null sets and satisfies (1) in the sense of distributions.

Theorem 4.2 ([1, 7], see also [8]). *Let $\omega \in L^2(Q_T)$ be a weak solution of* (V). *Then there exists a unique regular Lagrangian flow X_t for* (1) *such that*

$$\omega_t = \omega_0(X_t^{-1}).$$

In other words ω induces a weak solution X_t of (L). If $\omega \in L^\infty$, X_t agrees with the solution of Wolibner, implying uniqueness. In contrast, for $\omega \in L^p$ with $p < \infty$ there is currently no proof of uniqueness for (L). Moreover, observe that we require $\omega \in L^2$ (and consequently $v \in W^{1,2}$) for the theory of renormalized solutions, and thus a gap remains between this result and the weakest possible requirement for defining solutions of (V) (cf. Section 2.3).

4.4. *Admissibility and weak-strong uniqueness.* The inequality (9) still holds if $\tilde{v}$ is only a solution of (D) (since only the derivative of v is involved in the estimate) with non-increasing energy. Using this observation one is lead to the concept of admissibility, borrowed from the theory of hyperbolic conservation laws. A solution $v \in L^2_{\text{loc}}(Q_T)$ of (D), is said to be *admissible*, if in addition

$$\begin{aligned} t \mapsto v_t \text{ is a continuous map } [0, T] \to L^2_w(\Omega), \\ \|v_t\|_{L^2(\Omega)} \leq \|v_0\|_{L^2(\Omega)}, \end{aligned} \tag{A}$$

where L^2_w denotes the space L^2 endowed with the weak topology. The energy method then yields

Theorem 4.3. *Let ω be a weak solution of* (V) *with $\omega \in L^\infty(Q_T)$. Then $v := K\omega$ is* unique *in the class of admissible solutions of* (D).

This weak-strong uniqueness property is also the basis for the definition of *dissipative solutions*, introduced by P. L. Lions in [20]. Moreover, using the same idea one can even prove that v is unique in the class of admissible *measure-valued solutions*, see [3].

5. Weaker notions of solution

5.1. *Compactness.* Since Theorem 3.1 provides a global classical solution for any initial velocity $v_0 \in C^{1,\alpha}$, a way to obtain solutions to (D) with rougher initial data is by regularizing the data and using compactness arguments. This route to existence (which obviously does not tell anything about uniqueness) has been first considered by R. DiPerna and A. Majda in [13].

The argument start with the observation, that if X_t is a solution of (L) with corresponding vorticity ω_t, then

$$\int_\Omega |\omega(x,t)|^p \, \mathrm{d}x = \int_\Omega |\omega_0(x)|^p \, \mathrm{d}x.$$

It follows using (10), that if $\omega_0 \in L^p(\Omega)$ with $p > 1$ and $\omega_0^\varepsilon \to \omega_0$ in L^p for smooth initial data v_0^ε, then the corresponding solutions v^ε satisfy the uniform bound

$$\|\nabla v_t^\varepsilon\|_{L^p(\Omega)} \le C\|\omega_0\|_{L^p(\Omega)}.$$

By using the Rellich compact embedding together with the Aubin–Lions lemma, this estimate can be used to show the existence of a subsequence $v^{\varepsilon'}$ converging strongly to a limit v in $L^2_{\mathrm{loc}}(Q_T)$, which is therefore a solution of (D). See for instance [22, Chapter 10] for details.

For the limit case $p = 1$ both the *a priori* estimate on ∇v and the compact embedding fail. On the other hand this case holds special interest because of connections to classical situations such as the Kelvin–Helmholtz instability, e.g. when the initial velocity v_0 is given by[3]

$$v_0(x) = \begin{cases} e_1 & \text{if } x_1 > 0, \\ -e_1 & \text{if } x_1 < 0, \end{cases} \tag{12}$$

where e_1 is the vector $(1,0)$. Thus v_0 is discontinuous and ω_0 is a positive measure of finite mass, $\omega_0 \in \mathcal{M}_+(\Omega)$. The existence proof for this case is due to J. M. Delort:

Theorem 5.1 [6, § 2]. *Let* $v_0 \in L^2(\Omega)$ *such that* $\omega_0 \in \mathcal{M}_+(\Omega)$. *Then there exists a weak solution* $v \in L^\infty(0,T;L^2(\Omega))$ *of* (D) *such that* $\omega_t \in \mathcal{M}_+(\Omega)$ *with mass* $\|\omega_t\|_{\mathcal{M}} \le \|\omega_0\|_{\mathcal{M}}$.

[3] v_0 does not satisfy the boundary condition $v \cdot \nu = 0$ on $\partial\Omega$; this could be remedied for instance by considering periodic boundary conditions instead.

A crucial observation of Delort was that in formulating (D) it suffices to consider the traceless part of the matrix $v \otimes v$. In order to pass to the limit, Delort showed that after extracting a subsequence,

$$\left(v^\varepsilon \otimes v^\varepsilon - \tfrac{1}{2}|v^\varepsilon|^2 I\right) \to \left(v \otimes v - \tfrac{1}{2}|v|^2 I\right)$$

in the sense of distributions. In particular v^ε does not necessarily converge to v strongly in L^2_{loc}. It is only the special nonlinear structure of the equation that allows one to pass to the limit. This is an instance of compensated compactess, see also [14].

5.2. *Non-uniqueness.* The first "clearly unphysical" weak solution to the Euler equations was constructed by V. Scheffer in [24]. A different proof was subsequently given by A. Shnirelman in [25].

Theorem 5.2 [9, 24, 25]. *There exist infinitely many compactly supported weak solutions $v \in L^\infty(Q_T)$ of* (D) *in any space dimension $n \geq 2$.*

In [9] and [10] a framework using the technique of convex integration was developed, which places Theorem 5.2 in the context of Gromov's *h-principle.* In a sense convex integration is closely related to *lack of compactness.* Indeed, as is well known in the turbulence literature, without an *a priori* bound on the vorticity, weakly convergent sequences of solutions or approximate solutions of (E) do not converge to a solution of (D). In general there will be a defect usually called the Reynolds stress. By using localized oscillatory perturbations, this defect can be iteratively removed (in a highly non-unique way) to produce solutions of the type in Theorem 5.2. For details see [11]. Recently, in [12] the convex integration technique was extended to produce continuous solutions (v, p) to (W).

The solutions constructed in Theorem 5.2 violate the second law of thermodynamics, and a natural condition (other than regularity) to impose on the solution is admissibility. Although this condition does not recover uniqueness in general (see [10] and compare with Theorem 4.3), certain constraints do appear; e.g. we have

Theorem 5.3 [26]. *Let v_0 as in* (12)*. There exist infinitely many admissible solutions to* (D)*. These solutions do not necessarily conserve the energy, but do satisfy* $0 \geq \frac{dE}{dt} \geq -1/6$.

Without admissibility, the total energy could be dissipated in an arbitrarily short time interval (cf. [10]).

6. Conclusion

The incompressible Euler equations have a very rich mathematical structure, especially so in dimension two (indeed, in this brief survey we have not even touched upon the variational structure). Our aim was to show that there are many ways of defining solutions, although the different definitions do not necessarily coincide. The paper [27] had solved an outstanding open problem of the time, and even though many of the techniques had to be "rediscovered", the idea of using the Lagrangian formulation (L) and thereby solving an ODE with a log-Lipschitz vectorfield remains up to today one of the central points of the theory. Although the Eulerian formulation (E) is more easily handled with functional-analytical techniques (such as the formulation (D), Sobolev spaces, compactness or relaxation), which were unavailable in the 1930s, recent non-uniqueness results indicate that a link to the Lagrangian formulation should still be desirable.

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László Székelyhidi Jr.
Institut für Mathematik
Universität Leipzig
laszlo.szekelyhidi@math.uni-leipzig.de