A class of automorphism groups of polygonal complexes

Jacek Świątkowski

0. Introduction.

In this paper we construct a class of groups together with their actions on certain spaces called polygonal complexes. In fact, we give a complete description of the class of all groups which act by automorphisms simply transitively on the oriented edges of nonpositively curved polygonal complexes, see Theorem 1 below. This family of group actions is very natural from a geometrical viewpoint, being a kind of higher dimensional analogue of the actions of groups on their Cayley graphs (which are simply transitive on vertices). The construction is based on the notion of a nonpositively curved orbihedron and its fundamental group and described in section 2. It considerably generalizes the earlier ideas by W. Ballmann and the author from [2].

The obtained class of groups seems to be very rich and possibly worth further study. As it is shown in section 3, it contains rotation groups of regular tesselations, some special 2-dimensional Coxeter groups and some natural finite extensions of Coxeter groups. A subclass consisting of groups satisfying Kazhdan's property (T) is indicated at the end of section 3 (see also Theorem 2 below). The way of proving property (T) for those groups is that of [2].

To give precise statements of the main results of the paper it is necessary to introduce some terminology and notation. A *polygonal complex* is a 2-dimensional polyhedral cell complex. If X is a polygonal complex and $v \in X$ is a vertex, the link of X at v is the graph $L(X,v)$ whose vertices represent the edges of X adjacent to v, where two vertices are connected by an edge if the corresponding two edges in X are adjacent to a common face. This generalizes the corresponding notion for simplicial complexes. Throughout this paper it will be assumed that links at vertices in polygonal complexes are all connected.

Equip X with the length metric for which each 2-cell of X turns into a regular euclidean polygon with side length 1. This metric induces a length metric on the link $L(X,v)$, by declaring the length of each edge in $L(X,v)$ to be equal to the angle measure in the 2-cell in X adjacent to v and corresponding to this edge. We say that a polygonal complex X is *nonpositively curved* (shortly NPC) if the sum of the lengths of edges in any nontrivial circuit in any link $L(X,v)$ is not smaller than 2π. The reader is referred to [5] for more details concerning nonpositively curved complexes.

A group Γ of automorphisms of a polygonal complex X is said to act *simply transitively* on the set of oriented edges in X, if it acts on this set transitively and if its stabilizer at some (and hence any) oriented edge is trivial. Groups of rotations of regular tesselations of the euclidean or hyperbolic plane are the examples of such groups.

The next few notions having rather technical flavor, are in fact very well motivated geometrically, as it will be shown in section 1. Given a finite group H, let S be a generating set for H which is *symmetric*, i.e. $s^{-1} \in S$ whenever $s \in S$. Consider a permutation μ of the set S which satisfies the property that $\mu^{-1}(s^{-1}) = (\mu(s))^{-1}$. Each such permutation will be called a *monodromy function* for H and S. Given a monodromy function μ, put $l_\mu(s)$

to be the length of a cycle in μ containing s, i.e. the smallest positive n with $\mu^n(s) = s$. Then l_μ is easily seen to be invariant under the action of μ on S and under the operation of taking the inverse.

Let $c : S \to N \setminus \{0\}$ be another function satisfying properties $c(\mu(s)) = c(s)$ and $c(s^{-1}) = c(s)$. Suppose also that for each $s \in S$ we have $l_\mu(s) \cdot c(s) \geq 3$. A triple (S, μ, c) as above will be called a *system* for the group H. Given a system (S, μ, c) for H and a generator $s \in S$, put

$$\alpha(s) = \frac{l_\mu(s) \cdot c(s) - 2}{l_\mu(s) \cdot c(s)} \cdot \pi$$

and call it an *angle measure* at a generator s. By the properties of l_μ and c we have then $\alpha(s^{-1}) = \alpha(s)$, $\alpha(\mu(s)) = \alpha(s)$ and $\alpha(s) > 0$ for each $s \in S$.

Denote by $C(H, S)$ the Cayley graph of H with respect to S, i.e. the graph with the vertex set H and with vertices h_1, h_2 connected by an (unoriented) edge iff $h_1 = h_2 s$ for some $s \in S$. Put a length metric ρ on the Cayley graph $C(H, S)$ by declaring the length of an edge (h, hs) to be equal to the angle measure $\alpha(s)$ (which is well defined since $\alpha(s) = \alpha(s^{-1})$). A system (S, μ, c) for a group H is said to be NPC (nonpositively curved) if the length of any closed embedded polygonal path in the Cayley graph $C(H, S)$, with respect to the length metric ρ, is not smaller than 2π.

Theorem 1. (a) Given a finite group H with a presentation $\langle S|R\rangle$ and a system (S, μ, c) for H which is NPC, there exists a polygonal complex X such that the group Γ given by the presentation

$$(*) \qquad \Gamma = \langle S \cup \{\tau\} | R \cup \{\tau^2\} \cup \{[\prod_{i=0}^{l_\mu(s)-1} (\tau\mu^i(s))]^{c(s)} | s \in S\}\rangle$$

acts simply transitively on the set of oriented edges of X, the group H is isomorphic to the stabilizer of Γ at any vertex of X, and the Cayley graph $C(H, S)$ is isomorphic to the link of X at any of its vertices. This polygonal complex X is nonpositively curved and contractible.
(b) Any group acting simply transitively by automorphisms on the set of oriented edges of a locally finite nonpositively curved contractible polygonal complex has a form as in (a).

The author's initial motivation for studying groups as in Theorem 1 was to find new examples of finitely presented groups satisfying Kazhdan's property (T). A theorem of C. Champetier [7] says that property (T) is generic (in certain precise sense) among finitely presented groups. This genericity is not at all reflected in the amount of so far known examples. Classical examples are the lattices in the isometry groups of symmetric spaces of noncompact type and rank ≥ 2, of the quaternionic hyperbolic space and of the Cayley hyperbolic plane [12]. Except for them and some exotic automorphism groups of Euclidean buildings of type $\widetilde{A}_2$ (see [6]), the other known examples are some automorphism groups of hyperbolic buildings in dimension 2 (see [4], [10]) and some Kac-Moody groups acting on hyperbolic buildings in dimensions up to 4, see [8]. A family of finitely presented groups with property (T) acting on complexes that are not buildings was constructed by

W. Ballmann and the author in [2] (see also Example 3.5 in this paper). The examples provided by Theorem 2 below generalize the family from [2].

In order to state Theorem 2, some terminology related to the notion of the combinatorial Laplacian on finite graphs is necessary. For a finite graph L with the set of vertices $\mathcal{V}_L$, the *Laplacian* Δ on the space of real valued functions on $\mathcal{V}_L$ is defined by

$$\Delta f(v) = f(v) - Af(v),$$

where $Af(v)$ is the mean value of f on the vertices adjacent to v. Clearly, Δ is a self adjoint operator. We denote by $\kappa = \kappa(L)$ its smallest positive eigenvalue.

Theorem 2. If a system (S, μ, c) for a finite group H satisfies the following assumptions:
(0) S consists of at least two elements;
(1) $l_\mu(s) \cdot c(s) = 3$ for each $s \in S$;
(2) each closed embedded polygonal path in the Cayley graph $C(H, S)$ consists of at least six edges;
(3) $\kappa(C(H, S)) > \frac{1}{2}$;
then the group Γ given by the presentation (*) as in Theorem 1 is infinite and satisfies Kazhdan's property (T).

Examples of Cayley graphs $C(H, S)$ satisfying the assumptions of Theorem 2 can be found in the literature, as it is exhibited at the end of section 3. Various ways of equipping such Cayley graphs with appropriate functions μ and c follow easily from Lemma 3.1 of section 3. It is worth noting that the earlier examples of Kazhdan's groups constructed by W. Ballmann and the author in [2] correspond to those in Theorem 2 which have trivial monodromy.

The appendix to this paper describes efficient presentations of groups acting transitively on the vertices of simply connected polygonal complexes, which is a slight addition to the description of W. Ballmann & M. Brin [1]. Results of the appendix are used in the paper to derive presentations of the constructed groups

1. Structure of groups acting simply transitively on oriented edges of polygonal complexes.

In this section a structure of groups that act simply transitively on oriented edges of simply connected polygonal complexes is investigated. It is shown that such groups have the form (*) of Theorem 1, and the geometric meaning of the objects S, μ and c appearing in this theorem is explained. It is also explained how the condition NPC for the resulting system (S, μ, c) follows naturally from the assumption that a polygonal complex acted upon by a group as above is nonpositively curved. This gives a proof of part (b) in Theorem 1 of the introduction.

Let X be a simply connected polygonal complex, and let Γ be a group of its automorphisms acting simply transitively on the set of oriented edges in X. Fix a vertex $v \in X$

and consider the link $L(X, v)$ together with the induced action of the stabilizing subgroup $H = \mathrm{Stab}(\Gamma, v)$ on $L(X, v)$. By the assumption about the transitivity of Γ, the induced action of H on $L(X, v)$ is simply transitive on the vertices of $L(X, v)$. Since Γ is transitive on the vertices of X, the pair $(L(X, v), H)$ does not depend on the choice of v up to an equivariant isomorphism.

Fix a vertex $p \in L(X, v)$. For each vertex $q \in L(X, v)$ adjacent to p denote by s_q the unique element of H which maps q to p. Denote by S the set of all such elements s_q. This set is independent on the choice of p up to conjugacy in H and moreover satisfies the following.

Proposition 1.1. (1) The set S is a symmetric generating set for the group H.
(2) The link $L(X, v)$ is equivariantly isomorphic to the Cayley graph $C(H, S)$ with the canonical action of the group H on it from the left.

Proof: S is a generating set by the transitivity of H on the vertices of $L(X, v)$ and by the connectivity of $L(X, v)$ (which is a general assumption for polygonal complexes in this paper). S is symmetric since $(s_q)^{-1} = s_{s_q(p)}$. This proves assertion (1) of the proposition. To prove assertion (2), observe that the map $\varphi : C(H, S) \to L(X, v)$ defined by $\varphi(h) := h(p)$ is a required equivariant isomorphism. This finishes the proof.

Denote by e the edge in X adjacent to v, which is represented in the link $L(X, v)$ by the vertex p. Denote by w the second vertex of e and by τ the unique automorphism from Γ which reverses orientation on e, i.e. for which $\tau(v) = w$ and $\tau(w) = v$. (The uniqueness of τ follows from simple transitivity of Γ on the oriented edges of X.) Let $\mathcal{F}$ be the set of all faces of X adjacent to the edge e. Since the faces of $\mathcal{F}$ are represented in the link $L(X, v)$ by the edges adjacent to p, it follows that $\mathcal{F}$ is in one-to-one correspondence with the generating set S. Denote by $f_s \in \mathcal{F}$ the face corresponding as above to a generator $s \in S$.

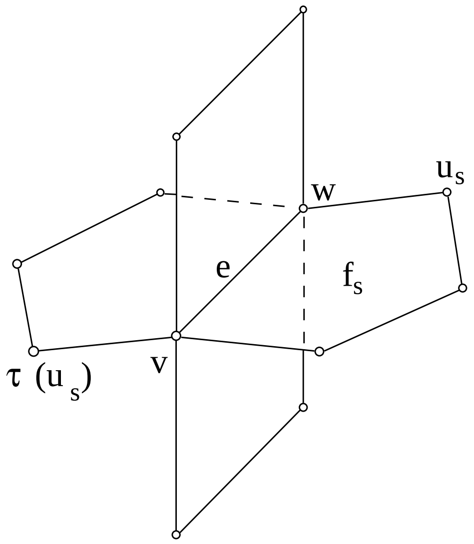

Figure 1.

Define a monodromy function $\mu : S \to S$ for Γ as follows. For each $s \in S$ denote by u_s the vertex in the face f_s which is adjacent to w and distinct from v (see Figure 1). Put $\mu(s)$ to be the unique element of S for which $\mu(s)(w) = \tau(u_s)$ (viewing elements of S as automorphisms of the whole complex X). The next proposition establishes crucial properties of the function μ.

Proposition 1.2. The function μ is a permutation of the generating set S and $\mu^{-1}(s^{-1}) = (\mu(s))^{-1}$.

It is necessary to precede the proof of Proposition 1.2 by several comments and a lemma.

Labellings in the barycentrically subdivided complexes $C'(H,S)$ and X'. Consider first the barycentric subdivision $C'(H,S)$ of the Cayley graph $C(H,S)$. If b is the barycenter of an edge (h,hs) of $C(H,S)$ then label the (nonoriented) edge (b,hs) in $C'(H,S)$ with the label $s \in S$. This determines uniquely the labelling of all edges in $C'(H,S)$ by the elements of the generating set S.

Recall that the link $L(X,v)$ is naturally identified with the Cayley graph $C(H,S)$. This determines the identification of the link $L(X',v)$ of the barycentrically subdivided complex X' with the subdivided graph $C'(H,S)$. The induced action of Γ on the subdivided complex X' determines then the identifications of the links $L(X',u)$ at the vertices of X with the same subdivided graph $C'(H,S)$. These identifications are not unique, but they differ from each other by automorphisms of $C'(H,S)$ which preserve labels. In particular, they allow to induce the labelling of edges in $C'(H,S)$ to edges in the links $L(X',u)$, for any vertex u of X. Now, since the edges in $L(X',u)$ correspond to the 2-simplices of X' adjacent to u, the labelling of edges in $C'(H,S)$ determines uniquely a labelling of all 2-simplices of X'. Denote this labelling by Λ and note that by definition it is Γ-invariant.

Types of vertices and edges in X'. Each vertex in X' is a barycenter of a unique cell of X. Define a *type* of a vertex of X' to be the dimension of the corresponding cell of X. If this dimension is equal to i, the vertex will be called an *i-vertex*. Note that the vertices contained in any edge of X' have always distinct types. This allows to define types of edges in X' by means of types of their vertices. Accordingly, the set of edges in X' splits into disjoint subsets of $(0,1)$-, $(0,2)$- and $(1,2)$-edges. It is also clear that the three vertices of each 2-simplex of X' have distinct types and consequently the same is true for the three edges of each 2-simplex.

The next lemma describes some properties of the labelling Λ, and in particular the relation of labels of adjacent 2-simplices of X' in terms of types of their common edges.

Lemma 1.3.
(a) The labelling Λ establishes a one-to-one correspondence between the set of 2-simplices of X' adjacent to any fixed $(0,1)$-edge and the set S.
(b) For each $(0,2)$-edge of X' there are exactly two 2-simplices in X' adjacent to this edge, and their labels are inverse to each other.
(c) For any $s \in S$ the group Γ acts simply transitively on the set of all 2-simplices in X' labelled with s.
(d) For each $(1,2)$-edge of X' there are exactly two 2-simplices in X' adjacent to this edge, and if one of them is labelled with s then the other one is labelled with $\mu(s)^{-1}$.

Proof of Lemma 1.3: Parts (a) and (b) follow directly from the definition of the labelling Λ and the properties of the Cayley graph $C(H, S)$. Part (c) follows from the Γ-invariance of Λ together with part (a) and simple transitivity of Γ on $(0,1)$-edges of X'.

To prove part (d), recall the notation of Figure 1 and denote by b the barycenter of the edge e of X. Note that the 2-simplex of X' contained in the face f_s and adjacent to the edge (v, b) is labelled with s. By the Γ-invariance of Λ, it is sufficient to find the label of the 2-simplex of X' adjacent to above 2-simplex along a $(1,2)$-edge, i.e. the 2-simplex contained in f_s and adjacent to the edge (b, w) (this 2-simplex is labelled with "?" at Figure 2).

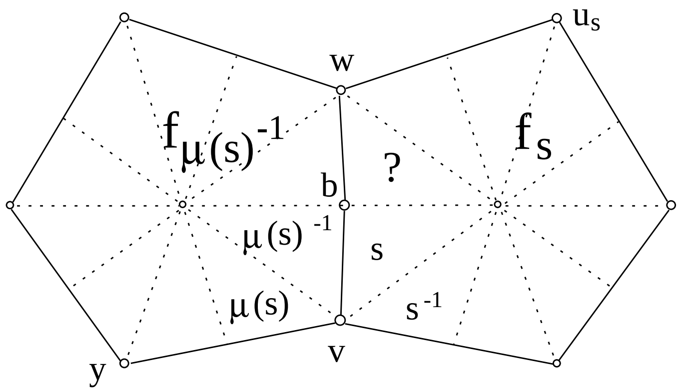

Figure 2.

Let $y = \mu(s)(w)$, where $\mu(s)$ is viewed as an automorphism of X. Then $\mu(s)^{-1}(y) = w$ and hence the vertices y, v and w are all contained in the face $f_{\mu(s)^{-1}}$ (see Figure 2). Since the automorphism τ transposes the vertices v and w, and moreover $\tau(y) = \tau \circ \mu(s)(w) = \tau \circ \tau(u_s) = u_s$, the face $f_{\mu(s)^{-1}}$ is mapped by τ on the face f_s. The 2-simplex of X' contained in $f_{\mu(s)^{-1}}$ and adjacent to the edge (v, b) is clearly labelled with $\mu(s)^{-1}$. Since it is mapped by τ on the above mentioned 2-simplex adjacent along a $(1,2)$-edge to a simplex labelled with s, part (d) of the lemma follows from the Γ-invariance of the labelling Λ.

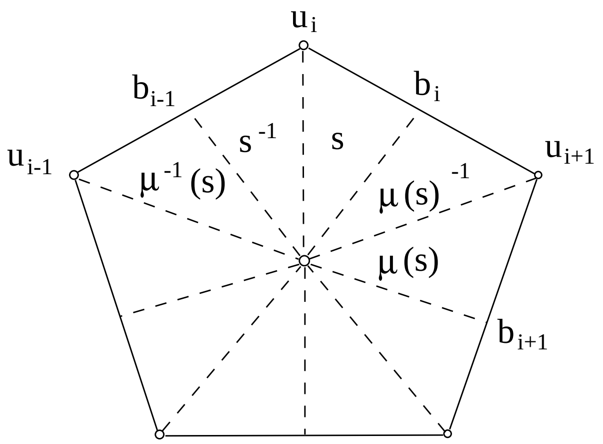

Figure 3. The labelling Λ restricted to a cell.

Proof of Proposition 1.2: Let f be a face of X and $u_1, u_2, \ldots, u_k$ be all the vertices of f in a natural order of occurence around f (with any orientation). For each $i = 1, 2, \ldots, k$ denote by b_i the barycenter of the edge (u_i, u_{i+1}), where $u_{k+1} = u_1$. Denote also by b the barycenter of f. It follows from Lemma 1.3 (b) and (d) that if a 2-simplex (u_i, b_i, b) is labelled with s, then the 2-simplex (u_{i+1}, b_{i+1}, b) is labelled with $\mu(s)$ (see Figure 3). The inverse function μ^{-1} is then well defined by putting $\mu^{-1}(s)$ to be the label of the 2-simplex (u_{i-1}, b_{i-1}, b). Hence μ is a permutation.

The equality $\mu^{-1}(s^{-1}) = \mu(s)^{-1}$ is a direct consequence of the above interpretation of μ and μ^{-1} and part (b) of Lemma 1.3. This finishes the proof of the proposition.

For $s \in S$ denote by k_s the number of edges (or equivalently vertices) in the polygonal face f_s. It follows from Lemma 1.3 that the faces f_s, $f_{s^{-1}}$ and $f_{\mu(s)}$ are in the same orbit of the action of Γ on the faces of X. In particular, those faces have the same number of edges and thus $k_{s^{-1}} = k_{\mu(s)} = k_s$ for any $s \in S$. Recall from the introduction that $l_\mu(s)$ is the length of the cycle in the permutation μ which contains s. The interpretation of μ as in the proof of Proposition 1.2 implies that $\mu^{k_s}(s) = s$, and hence k_s is a multiple of $l_\mu(s)$. Put $c(s) = k_s/l_\mu(s)$ and observe the following.

Fact 1.4. c is a function on the set S with values in positive integers and $c(s^{-1}) = c(\mu(s)) = c(s)$.

The essential role of the functions μ and c in getting a presentation of the group Γ is established in the following.

Proposition 1.5. The group Γ has a presentation

$$\Big\langle S \cup \{\tau\} | R \cup \{\tau^2\} \cup \{[\prod_{i=0}^{l_\mu(s)-1} (\tau\mu^i(s))]^{c(s)} | s \in S\}\Big\rangle,$$

where the meaning of H, S, τ, μ and c is as described above in this section, and R is the set of relations in a presentation $\langle S|R\rangle$ of the group H.

Proof: The proof is based on Proposition A.2 of the appendix, which is a refined version of Proposition 2.1 in [1].

Since the group $H = \mathrm{Stab}(\Gamma, v)$ is transitive on the vertices of X adjacent to v, it follows that the group Γ is generated by the set $S \cup \{\tau\}$. Since moreover H is simply transitive on this set, there is no relation of type 1, as described in the appendix. It is easily seen that the automorphism τ is an involution, and therefore the only relation of type 2 is $\tau^2 = e$. To get sufficiently many relations of type 3, note that the set $\mathcal{F}$ contains representatives of all orbits of the action of H on the faces of X adjacent to v. In view of the properties of μ described in Lemma 1.3 and in the proof of Proposition 1.2, a word of form

$$[\tau\mu(s)\tau\mu^2(s)\tau \ldots \mu^{l_\mu(s)-1}(s)\tau s]^{c(s)}$$

is a relation of type 3 corresponding to the face $f_s \in \mathcal{F}$. Since a cyclic rearrangement of this word gives the form as required, the proposition follows from Proposition A.2 of the appendix.

Remark 1.6. It is possible to reduce the above presentation of the group Γ as follows. Let S_0 be a subset of S consisting of all elements of S with $s^{-1} = s$, and containing exactly one of the two elements s, s^{-1} for each $s \in S$ with $s^{-1} \neq s$. Clearly, S_0 is a generating set for the group H, and consequently the set $S_0 \cup \{\tau\}$ generates Γ. Consider the equivalence relation in the set S determined by the equivalences $s \sim s^{-1}$ and $s \sim \mu(s)$ for $s \in S$. Let Σ be a subset of S_0 containing one representative for each equivalence class of this relation. It follows from part (c) of Lemma 1.3 that the set $\{f_s | s \in \Sigma\}$ contains representatives of all orbits of the action of H on the faces of X adjacent to v. Therefore, according to Proposition A.2 of the appendix, a more efficient presentation for the group Γ has a form

$$\langle S_0 \cup \{\tau\} | R \cup \{\tau^2\} \cup \{[\prod_{i=0}^{l_\mu(s)-1} (\tau\mu^i(s))]^{c(s)} | s \in \Sigma\}\rangle.$$

Assume now that the polygonal complex X on which the group Γ acts is nonpositively curved. Since Γ is transitive on the vertices in X, this is equivalent to the condition that the sum of angles at the vertex v for faces corresponding to any closed embedded polygonal path in the link $L(X, v)$ is not smaller than 2π.

Observe that a face in X containing a 2-simplex of X' labelled with s is a regular k_s-gon, with $k_s = l_\mu(s) \cdot c(s)$. It follows that the measure of each angle in this face is equal to

$$\alpha(s) := \frac{k_s - 2}{k_s}\pi = \frac{l_\mu(s) \cdot c(s) - 2}{l_\mu(s) \cdot c(s)}\pi.$$

Since $\alpha(s) = \alpha(s^{-1})$, the function α induces a length metric ρ on the Cayley graph $C(H, S)$, by putting the length of an edge of form (h, hs) to be equal to $\alpha(s)$. This length metric, defined solely in terms of the data H, S, μ and c, allows to express the nonpositive curvature condition for the complex X in terms of these data. After identifying the Cayley graph $C(H, S)$ with the link $L(X, v)$, the length of any edge of $L(X, v)$ (with respect to ρ) equals the angle measure at v in the face of X corresponding to this edge. This leads to the following.

Lemma 1.7. Let ρ be the length metric in the Cayley graph $C(H, S)$, defined as above in terms of the data H, S, μ and c. Then the complex X is nonpositively curved iff the length of each closed embedded polygonal path in $C(H, S)$, with respect to ρ, is not smaller than 2π.

2. Construction.

This section contains the proof of part (a) of Theorem 1 of the introduction. The proof consists of a construction of an appropriate group of automorphisms of a polygonal complex, out of the initial data as in the theorem. The construction is based on the notion of nonpositively curved orbihedron, an example of which is constructed using these initial data. The group as asserted in the theorem arises then as the fundamental group of this orbihedron.

The section starts with recalling the notion of rigid orbihedron in the special case of dimension 2. The related notions of developability and nonpositive curvature of an orbihedron are also recalled.

Rigid 2-orbihedron. *Rigid 2-orbihedron* is a simplicial 2-complex K together with a family of triples (G_x, L_x, p_x), one for each vertex x of K, satisfying the following conditions.

(1) L_x is a graph and G_x is a group acting simplicially without inversions on L_x; the action of G_x on L_x is *rigid*, i.e. if $g \in G_x$ fixes an edge of L_x then $g = id$.

(2) $p_x : L_x \to L(K,x)$, called a *projection*, is a G_x-invariant simplicial map which induces an isomorphism of the quotient $G_x \backslash L_x$ on the link $L(K,x)$.

(3) For an edge e in K containing a vertex x, let x_e be the vertex representing e in the link $L(K,x)$, and let $\tilde{x}_e$ be a vertex of L_x with $p_x(\tilde{x}_e) = x_e$. Then the pair of a set $L_{e,x} := L(L_x, \tilde{x}_e)$ and a group $G_{e,x} := \mathrm{Stab}(G_x, \tilde{x}_e)$ with the induced action on $L_{e,x}$ (which is free due to rigidity of the action of G_x on L_x) does not depend on the choice of $\tilde{x}_e$, up to an equivariant isomorphism. Projection p_x induces then (in a canonical way, which turns out to be independent on the choice of $\tilde{x}_e$) a $G_{e,x}$-invariant map $p_{e,x} : L_{e,x} \to L(K,e)$, where $L(K,e)$ is the link of the edge e in K, i.e. a 0-complex consisting of vertices representing 2-simplices of K adjacent to e. It is then required that for each edge $e = (x,y)$ in K the pairs $(L_{e,x}, G_{e,x})$ and $(L_{e,y}, G_{e,y})$ are equivariantly isomorphic by an isomorphism which projects by $p_{e,x}$ and $p_{e,y}$ to the identity of the link $L(K,e)$.

Remarks. (1) Observe that, in a rigid 2-orbihedron, G_x acts freely on the edges of L_x but not necessarily freely on the vertices.

(2) The above definition of rigid 2-orbihedron is a simplified version of the general definition of orbihedron in [11]. Simplifications are due to the assumptions of rigidity and dimension 2.

Developability. Let $\widetilde{K}$ be a simplicial 2-complex and G a group acting on $\widetilde{K}$ simplicially, without inversions and rigidly. Then the quotient complex $K = G \backslash \widetilde{K}$ is well defined, and a rigid 2-orbihedron $G(\widetilde{K})$ over K is well defined as follows. For each vertex $x \in K$ let $\tilde{x} \in \widetilde{K}$ be a vertex in the preimage of x by the quotient map $q : \widetilde{K} \to K$. Put $L_x := L(\widetilde{K}, \tilde{x})$ and $G_x := \mathrm{Stab}(G, \tilde{x})$, where G_x acts on L_x in a canonical way. Put also $p_x : L(\widetilde{K}, \tilde{x}) \to L(K,x)$ to be the map induced on links from the quotient map q. An orbihedron of the form as above will be called a *quotient orbihedron*.

A rigid 2-orbihedron Q is *developable*, if it is isomorphic to a quotient orbihedron of the form $G(\widetilde{K})$, where $\widetilde{K}$ is simply connected. It is known [11] that if Q is developable, then the corresponding pair $(\widetilde{K}, G)$ is unique up to an equivariant isomorphism. The complex $\widetilde{K}$ is called the *universal covering* of the orbihedron Q, and the group G is called the *fundamental group* of Q.

Nonpositive curvature. Let K be a rigid 2-orbihedron, and assume that its underlying simplicial complex is equipped with a piecewise euclidean geodesic metric which turns each 2-simplex of K into a euclidean triangle. This metric induces length metrics on the links $L(K,x)$, and those metrics can be further lifted by the projections p_x to the graphs L_x. Denote such lifted length metrics on the links L_x by ρ_x.

A rigid 2-orbihedron K with a geodesic metric as above is *nonpositively curved*, if for each vertex x of K the length (with respect to the induced length metric ρ_x as above) of any closed embedded polygonal path in L_x is not smaller than 2π.

The following fundamental theorem for nonpositively curved orbihedra plays crucial role in the construction of this section.

Theorem 2.1 [11]. Let K be a rigid 2-orbihedron, and assume that for some geodesic piecewise euclidean metric on the underlying complex it is nonpositively curved. Then K is developable.

Construction of an orbihedron $Q_{H,S,\mu,c}$. Let H be a finite group and (S,μ,c) be a system for H, as defined in the introduction (before Theorem 1). Consider the simplicial complex $\Delta := (0,1) * S$, where $(0,1)$ is a 1-simplex with vertices 0 and 1, the set S is viewed as a 0-complex, and $*$ denotes the operation of simplicial join. Let $Q := \Delta/\sim$ be the quotient of Δ by the equivalence relation $\sim$ determined by the following identifications of edges in Δ:
(a) $(0,s) \sim (0,s^{-1})$ in such a way that $s \sim s^{-1}$;
(b) $(1,s) \sim (1,\mu(s)^{-1})$ in such a way that $s \sim \mu(s)^{-1}$.
The image of any simplex σ of Δ in the qoutient complex Q will be denoted by $[\sigma]$.

Define an atlas $\{(G_x, L_x, p_x)\}$ of a rigid 2-orbihedron $Q_{H,S,\mu,c}$ having Q as the underlying complex, as follows.

(1) For the vertex $[0] \in Q$ put $L_{[0]} := C'(H,S)$ and $G_{[0]} := H$, where the action of $G_{[0]}$ on $L_{[0]}$ coincides with the natural action of H induced on the barycentric subdivision of the Cayley graph $C(H,S)$. Define also projection $p_{[0]} : L_{[0]} \to L(Q,[0])$ to be the unique simplicial map determined by the following requirements:
   (a) each element $h \in H$ (viewed as a vertex in C(H,S), and hence also in $C'(H,S)$) is mapped on the vertex representing the edge $[(0,1)]$;
   (b) if b is the barycenter of an edge (h,hs) in $C(H,S)$ then the edge (b,hs) of $C'(H,S)$ is mapped on the edge representing the 2-simplex $[(0,1,s)]$.
(2) For the vertex $[1] \in Q$ put $L_{[1]} := \{u_+, u_-\} * S$ and $G_{[1]} := \langle \tau | \tau^2 \rangle$ (i.e. a cyclic group of order 2), where $u_\pm$ are two distinct vertices and $\tau : L_{[1]} \to L_{[1]}$ is the automorphism determined by the transposition in the set $\{u_+, u_-\}$ and the involution $s \to \mu(s)^{-1}$ in the set S. Define projection $p_{[1]} : L_{[1]} \to L(Q,[1])$ to be the simplicial map uniquely determined by the following requirements:
   (a) vertices u_+ and u_- are mapped on the vertex representing the edge $[(0,1)]$;
   (b) for any $s \in S$ the edge (u_+, s) is mapped on the edge representing the 2-simplex $[(0,1,s)]$;
   (c) for any $s \in S$ the edge (u_-, s) is mapped on the edge representing the 2-simplex $[(0,1,\mu(s)^{-1})]$.
(3) For the vertex $[s] \in Q$ put $L_{[s]}$ to be the graph isomorphic to the boundary of a $2k_s$-gonal cell, where $k_s = l_\mu(s) \cdot c(s)$ (in graph theory such a graph is called a $2k_s$-*gon*). By the properties of μ and c assumed in Theorem 1, the number k_s, and hence also the graph $L_{[s]}$, depend only on the equivalence class $[s] \in Q$ of the vertex $s \in \Delta$. View $L_{[s]}$ as a bipartite graph and denote by V_1 and V_2 the vertex subsets of this bipartition. Label the edges of the graph $L_{[s]}$, uniquely up to an automorphism of

$L_{[s]}$, according to the following two rules:

(i) if an element $t \in S$ occurs as a label, then $[t] = [s]$;

(ii) if an edge is labelled with t then the edge adjacent to it through the vertex from V_1 is labelled with t^{-1}, while the edge adjacent through the vertex from V_2 - with $\mu(t)^{-1}$.

Put $G_{[s]}$ to be the group of all label preserving automorphisms of $L_{[s]}$ which furthermore preserve both vertex subsets V_1 and V_2. $G_{[s]}$ is then either a dihedral or a cyclic group, depending on the pattern of labels of edges in $L_{[s]}$. Define also a projection $p_{[s]} : L_{[s]} \to L(Q, [s])$ to be the unique simplicial map determined by the following requirements:

(a) any vertex from the subset V_1 adjacent to an edge labelled with an element $t \in S$ is mapped on the vertex representing the edge $[(0, t)]$;

(b) any vertex from V_2 adjacent to an edge labelled with t is mapped on the vertex representing the edge $[(1, t)]$.

A straightforward verification that the above described atlas satisfies all the conditions from the definition of rigid 2-orbihedron is omitted.

Lemma 2.2. If a system (S, μ, c) for H is NPC then the orbihedron $Q_{H,S,\mu,c}$ admits a metric for which it is nonpositively curved.

Proof: Let α be the angle measure related to the system (S, μ, c), as described in the introduction. Consider a piecewise euclidean length metric ρ_Q on the underlying complex Q uniquely determined by the requirements that for each $s \in S$ the restriction of ρ_Q to the 2-simplex $[(0, 1, s)]$ turns this 2-simplex into a euclidean triangle with:

(1) the edge $[(0, 1)]$ of length $1/2$;

(2) the right angle at the vertex $[1]$;

(3) the angle at the vertex $[0]$ equal to $\alpha(s)/2$.

A straightforward observation shows that a system (S, μ, c) is NPC iff the orbihedron $Q_{H,S,\mu,c}$ equipped with the metric ρ_Q is nonpositively curved.

Proof of Theorem 1.

Part (b) of the theorem is proved in section 1. Here we prove part (a).

Given data H, S, μ, c as assumed, consider the orbihedron $Q = Q_{H,S,\mu,c}$ as described above and the metric ρ_Q as in the proof of Lemma 2.2. Since Q is nonpositively curved with respect to ρ_Q, it is developable (see Theorem 2.1). Denote by $\widetilde{Q}$ the universal cover of Q, and by Γ the corresponding group of covering transformations (i.e. the fundamental group of Q together with its natural action on $\widetilde{Q}$). The simplicial complex $\widetilde{Q}$ can be identified with the barycentric subdivision of a polygonal complex X uniquely determined by the condition that, if $\pi : \widetilde{Q} \to Q$ is the covering map, then any vertex of $\widetilde{Q}$ in the preimage $\pi^{-1}(\{[s] : s \in S\})$ is the barycenter of a (2-dimensional) face of X. It is also easy to observe that the action of the group Γ on $\widetilde{Q}$ induces an action of this group on X by combinatorial automorphisms.

We will show now that Γ acts simply transitively on the oriented edges in X. Note first that any half-edge (x, b) in X, where x is a vertex and b is the barycenter of an adjacent edge in X, is mapped by π to the edge $[(0,1)]$ in Q, in such a way that x is mapped to $[0]$. It follows that Γ is transitive on oriented edges in X. The stabilising subgroup in Γ of an oriented edge in X is then naturally isomorphic to the subgroup $G_{[(0,1)],[0]}$ in the group $G_{[0]}$ belonging to the structure of the orbihedron Q. Simple transitivity follows then from the fact that this subgroup is trivial.

By what was said above, X and Γ is a pair as studied in section 1. A direct checking shows that the set H, S, μ, c of data derived for X and Γ as in section 1 coincides with the corresponding initial data used in the construction of X and Γ. A helpful observation for this checking is the coincidence of the following two labellings:

(1) the labelling Λ as in Lemma 1.3;

(2) the labelling of 2-simplices of $\widetilde{Q}$ lifted from the one which assigns label s to a 2-simplex $[(0,1,s)]$ in Q, for each $s \in S$.

As a consequence of the above coincidence of data, we get that the group Γ has a presentation of the form (*). We take this group, together with its action on the above constructed complex X, as a group Γ asserted in the theorem. To prove the theorem we need to show that X and Γ satisfy the other assertions.

A straightforward calculation of angles shows that the metric ρ_Q lifted to $\widetilde{Q}$ and then induced on X turns X into a piecewice euclidean nonpositively curved polygonal complex whose faces are regular and have side lengths 1. The remaining assertions of the theorem follow immediately from the definition of the universal covering of an orbihedron and the action of the fundamental group on it, and from the construction of Q.

3. Examples.

The aim of this section is to show that the construction of the previous section provides a wealth of examples of groups acting on polyhedral 2-complexes. Some of those examples are (more or less) classical, whereas others seem to be new. The most prominent new examples are the ones which satisfy Kazhdan's property (T), described in the last subsection of this section.

Description of monodromy functions.

Recall that given a symmetric generating set S in a finite group H, a monodromy function is a permutation μ of the set S such that $\mu^{-1}(s^{-1}) = (\mu(s))^{-1}$ for each $s \in S$. The next lemma gives a characterization of monodromy functions in terms of cycles into which they are decomposed as permutations. It is clear from this characterization that one can easily construct many such functions. A straightforward proof of the lemma is omitted.

Lemma 3.1. A permutation μ of the set S is a monodromy function iff each cycle in it has one of the following forms:

(1) $(t_1, \dots, t_k, t_k^{-1}, \dots, t_1^{-1})$, where $k \geq 1$ and $t_i \neq t_i^{-1}$ for $i = 1, \dots, k$;

(2) $(s, t_1, \dots, t_k, t_k^{-1}, \dots, t_1^{-1})$, where $k \geq 0$, $s = s^{-1}$ and $t_i \neq t_i^{-1}$ for $i = 1, \dots, k$;

(3) $(s, t_1, \dots, t_k, r, t_k^{-1}, \dots, t_1^{-1})$, where $k \geq 0$, $s = s^{-1}$, $r = r^{-1}$ and $t_i \neq t_i^{-1}$ for $i = 1, \dots, k$;

(4) $(t_1, \dots, t_k)$, where $k \geq 1$, $t_i \neq t_i^{-1}$ for $i = 1, \dots, k$ and $(t_k^{-1}, \dots, t_1^{-1})$ is also a cycle in μ.

A sufficient condition for nonpositive curvature.

Denote by $g_{H,S}$ the *girth* of the Cayley graph $C(H,S)$, i.e. the minimal number of edges in a closed embedded polygonal path in $C(H,S)$. This number corresponds to the minimal length of a nontrivial cyclically reduced relation in the group H with respect to the generating set S, except for relations of form s^2 with $s \in S$. The next lemma describes a practical sufficient condition for a system (S,μ,c) to be NPC, expressed in terms of girth of the graph $C(H,S)$.

Lemma 3.2. Let (S,μ,c) be a system for a finite group H, and assume that

$$l_\mu(s)\cdot c(s) \geq 2g_{H,S}/(g_{H,S}-2) \text{ for each } s\in S. \tag{3.2.1}$$

Then this system is NPC.

Proof: Recall that an edge in $C(H,S)$ corresponding to $s \in S$ is given the length $\alpha(s) = \pi\cdot[l_\mu(s)\cdot c(s)-2]/[l_\mu(s)\cdot c(s)]$. Note also that the condition (3.2.1) is equivalent to inequalities $[l_\mu(s)\cdot c(s)-2]/[l_\mu(s)\cdot c(s)] \geq 2/g_{H,S}$ for all $s\in S$. It follows that the length L of a closed embedded polygonal path in $C(H,S)$ can be estimated as follows:

$$L \geq g_{H,S}\cdot \min\{\alpha(s) : s \in S\} \geq g_{H,S}\cdot\frac{2\pi}{g_{H,S}} = 2\pi.$$

This finishes the proof.

Theorem 1 together with Lemmas 3.1 and 3.2 allows to produce many explicit examples of groups acting simply transitively on the oriented edges of simply connected nonpositively curved polygonal complexes. Below we discuss several of such examples obtained by some particular choices of the data H,S,μ,c.

Examples with trivial monodromy.

Given an arbitrary finite group H with a symmetric generating set S, and a monodromy function $\mu = id_S$, choose a function c so that

$$c(s) \geq \frac{2g_{H,S}}{g_{H,S}-2} \text{ for each } s\in S$$

(for example a constant function). In view of Lemma 3.2 the system H,S,μ,c is then NPC, and hence satisfies assumptions of Theorem 1. The next examples are special cases of this general class.

Example 3.3 - groups of rotations of regular tesselations. Let $H = \langle s | s^k\rangle$ be a cyclic group of order $k \geq 3$, with a symmetric generating set $S = \{s, s^{-1}\}$. Note that $g_{H,S} = k$ and for each $m \geq 2k/(k-2)$ the system H,S,μ,c with $\mu = id$ and $c(s) = m$ is NPC. The corresponding group $\Gamma = \langle s,\tau | s^k, \tau^2, (\tau s)^m\rangle$ as in Theorem 1 is then a well known group of rotations for a regular tesselation of the euclidean or hyperbolic plane, and the corresponding polygonal complex X can be canonically identified with this tesselation.

Example 3.4 - a class of 2-dimensional Coxeter groups. Let (H,S) be a Coxeter system for a finite Coxeter group H, i.e. $H = \langle S | \{s_i^2 : i\in I\}\cup\{(s_is_j)^{m_{ij}} : i,j\in I\}\rangle$, where $S = \{s_i : i \in I\}$. Consider the trivial monodromy function $\mu = id_S$ and a function c on S defined by $c(s_i) = k_i$, where $k_i \geq 2g_{H,S}/(g_{H,S} - 2)$ for each $i \in I$, and note that the system H, S, μ, c is NPC. The corresponding group Γ as in Theorem 1, together with the generating set $S \cup \{\tau\}$ of reflections, gives then a new Coxeter system, as it can be easily read from the presentation (*). The corresponding polygonal complex X of Theorem 1 is in this case an example of the so called Bestvina complex for the Coxeter system $(\Gamma, S \cup \{\tau\})$, see [3]. As it was shown in [3], the dimension of this complex is equal to the virtual cohomological dimension of the group Γ. Therefore we have $vcd(\Gamma) = 2$ for Coxeter groups Γ constructed in this example.

Example 3.5 - a class for which X is simplicial. Let H be a finite group with $g_{H,S} \geq 6$, and let $\mu = id_S$ and $c(s) = 3$ for each $s \in S$, and note that then the system H, S, μ, c is NPC. The group Γ obtained from this system as in Theorem 1 acts then on the polygonal complex X which is simplicial, i.e. its all 2-cells are triangles. This class of groups was constructed and studied in [2], where it was shown that it contains many examples satisfying Kazhdan's property (T). Below in the last subsection we refer to the arguments of [2] to prove property (T) for a much bigger family of examples.

Examples with nontrivial monodromy.

In the rest of this section we indicate examples of groups Γ arising from Theorem 1 by choosing a nontrivial monodromy μ. The two classes of such examples below turn out to be semidirect extensions of Coxeter groups by some finite groups, and before getting to them we need a definition.

Definition 3.6. Let (W, Σ) be a Coxeter system with canonical presentation $W = \langle \Sigma | \{(\sigma_i \sigma_j)^{m_{ij}}\} \rangle$, where $\Sigma = \{\sigma_i : i \in I\}$. Consider a group K of permutations of the set I, which preserves the Coxeter matrix (m_{ij}), i.e. $m_{k(i),k(j)} = m_{ij}$ for each $k \in K$. Then for each $k \in K$ the map $s_i \to s_{k(i)}$ extends to an automorphism $\phi(k)$ of the group W, so that $\phi : K \to Aut(W)$ is a homomorphism. Let $H = W \times_\phi K$ be a semidirect product of the groups W and K with respect to the homomorphism ϕ. Any group H obtained in this way will be called a *natural semidirect extension* of the Coxeter system (W, Σ).

Let $\langle K_0, P \rangle$ be a presentation for a group K. Then a natural semidirect extension $H = W \times_\phi K$ described above has a presentation

$$H = \langle \Sigma \cup K_0 | \{(\sigma_i \sigma_j)^{m_{ij}}\} \cup P \cup \{k \sigma k^{-1} = \phi(k) : k \in K_0, \sigma \in \Sigma\} \rangle. \tag{3.6.1}$$

Example 3.7 - semidirect extension of a Coxeter group by the group H. Let H, S, μ, c be a system such that $s \neq s^{-1}$, $\mu(s) = s^{-1}$ and $c(s) \geq g_{H,S}/(g_{H,S} - 2)$ for each $s \in S$. Then, since clearly $l_\mu(s) = 2$ for any $s \in S$, in view of Lemma 3.2 the system is NPC. A straightforward observation shows that the group Γ obtained from this system by Theorem 1 is a natural semidirect extension of a Coxeter system (W, Σ), with $\Sigma = \{s_h : h \in H\}$ and $s_h = h \tau h^{-1}$, by the group H which acts on Σ by $\phi(h)(s_{h'}) = s_{hh'}$. This can be realized by comparing presentations (*) and (3.6.1).

Example 3.8 - Coxeter group extensions of index 2. Another natural Coxeter group extension is obtained by choosing a system H, S, μ, c in such a way that (H, S) is a Coxeter system, $\mu^2 = id_S$, $c(s)$ is even for all s with $\mu(s) = s$, and the condition (3.2.1) holds. The group Γ obtained by Theorem 1 from this system is a natural semidirect extension of a Coxeter system (W, Σ) with $\Sigma = S \cup \{\bar{s} : s \in S\}$ and $\bar{s} = \tau s \tau$, and the group $\langle \tau | \tau^2 \rangle$ acting on Σ by $\tau(s) = \bar{s}$ and $\tau(\bar{s}) = s$. This follows immediately from comparing presentations (*) and (3.6.1).

New groups with Kazhdan's property (T).

This subsection contains the proof of Theorem 2 of the introduction, together with few sources of the initial data satisfying the assumptions of this theorem. A crucial technical tool for proving Theorem 2 is the following result from [2] (compare also [14]). Recall from the introduction that $\kappa(L)$ denotes the smallest positive eigenvalue for the combinatorial Laplacian on a finite graph L.

Theorem 3.9 [2]. Let X be a contractible 2-dimensional simplicial complex and Γ be a properly discontinuous and cocompact group of automorphisms of X. Assume that for any vertex v the link L of the complex X at v is connected and $\kappa(L) > 1/2$. Then Γ satisfies Kazhdan's property (T).

Proof of Theorem 2: Note that in view of Lemma 3.2, assumptions (1) and (2) of the theorem imply that the system H, S, μ, c is NPC. Therefore, we can use this system to construct a pair (Γ, X) as in Theorem 1. Assumption (1) implies then that the polygonal complex X is simplicial, and by Theorem 1 its links at vertices are isomorphic to the Cayley graph $C(H, S)$. Since the group Γ clearly acts properly discontinuously and cocompactly on X, the pair (Γ, X) satisfies all the assumptions of Theorem 3.9, and hence Γ satisfies Kazhdan's property (T). It remains to prove that Γ is infinite. By the assumption (0) the complex X has no free edges. Since moreover it is simply connected and nonpositively curved, it has to be unbounded. The group Γ is then infinite by transitivity on the vertices of X. This finishes the proof.

Cayley graphs satisfying the assumptions of Theorem 2.

A class of examples of Cayley graphs $C(H, S)$ satisfying the assumptions of Theorem 2 are the graphs constructed by Lubotzky-Philips-Sarnak and Margulis. They are the examples of the so called Ramanujan graphs, i.e. graphs with very sharp estimate for the second eigenvalue. They are described in detail in Chapter 3 of [13], from where it is also clear that most of them satisfy all the other assumptions of Theorem 2.

Another family of examples, with weaker but still sufficient eigenvalue estimates, can be found in [9]. It is however not clear to the author which of them satisfy the other assumptions of Theorem 2.

Appendix

This appendix is devoted to describing presentations of groups of automorphisms of simply connected polygonal complexes, which act transitively on vertices in these complexes. Presentations of such groups have been already described in section 2 of [1]. The exposition here is more precise in the sense that it provides reasonably small sets of generators and relations. Also, the proof given here is less sketchy than that in [1].

Let Γ be a group of automorphisms of a simply connected polygonal complex X, and assume that Γ acts transitively on the set of vertices of X. Fix a vertex $v \in X$ and denote by $H = \mathrm{Stab}(\Gamma, v)$ the stabilizing subgroup of Γ at v. Let $\langle S|R \rangle$ be a presentation for the group H. A presentation for Γ will be constructed by extending S to a generating set $S \cup T$, and by extending R to a sufficient set $R \cup P$ of relations for Γ with respect to $S \cup T$.

To describe T consider first the set $N := \{g \in \Gamma : g(v) \text{ is adjacent to } v \text{ in } X\}$. Consider also the equivalence relation in N defined by $g \sim g'$ iff $g' = h_1 g h_2$ or $g' = h_1 g^{-1} h_2$ for some $h_1, h_2 \in H$. Let T be a set of representatives of equivalence classes of this relation.

Lemma A.1. The set $S \cup T$ generates the group Γ.

Proof: By definition of N and by connectedness of X, it is clear that Γ is generated by $N \cup H$. Since each element of N can be expressed by the elements of $T \cup H$, and H is generated by S, it follows that $S \cup T$ is a generating set for Γ.

From now on fix a set T as above. Consider three types of relations for Γ with respect to $S \cup T$, as described below.

Type 1: $h_2 t h_1 t$ for any $t \in T$ and any $h_1 \in H$ with $h_1 t(v) = t^{-1}(v)$, where $h_2 = t^{-1} h_1^{-1} t^{-1} \in H$.

Type 2: $h_2 t^{-1} h_1 t$ for any $t \in T$ and any $h_1 \in H$ with $h_1 t(v) = t(v)$, where $h_2 = t^{-1} h_1^{-1} t \in H$.

Type 3: $h_1 t_1 h_2 t_2 \dots h_k t_k = h$ for $h_i \in H$ and $t_i \in T \cup T^{-1}$ such that the vertices

$$v_0 = v, v_1 = h_1 t_1(v), v_2 = h_1 t_1 h_2 t_2(v), \dots, v_k = h_1 t_1 \dots h_k t_k(v) = v = v_0$$

are the consecutive vertices of a face f of X adjacent to v. For such h_i and t_i the element h as above is also an element of H.

In all three types of relations above it is understood that elements of the group H which appear in relations represent some reduced words on the set S.

To get a small set of relations for Γ consider the action of the group H on the set of faces of X adjacent to the vertex v. Let $\mathcal{F}_0$ be a set of representatives of orbits of this action. For each face $f \in \mathcal{F}_0$ choose a relation of type 3 corresponding to f, and denote it by p_f. Define a set P of relations for Γ to consist of all relations of types 1 and 2, and the chosen relations p_f for all $f \in \mathcal{F}_0$.

Proposition A.2. Let P be a set of relations for Γ with respect to a generating set $S \cup T$, as described above. Then the pair $\langle S \cup T | R \cup P \rangle$ is a presentation for the group Γ.

Proof: To prove that $\langle S \cup T | R \cup P \rangle$ is a presentation for Γ, it is sufficient to prove that any word on the set $S \cup T$ which gives the trivial element in Γ can be reduced to the trivial word using the relations from $R \cup P$. To do this, fix a word as above and note that after cyclic reduction it has a form $w = h_1 t_1 h_2 t_2 \dots h_n t_n$, where $t_i \in T \cup T^{-1}$ and h_i are reduced words on the set S (some of them could be empty).

Consider a sequence $u_1, u_2, \dots, u_n$ of vertices in the complex X, defined by $u_0 = v$ and $u_k = h_1 t_1 \dots h_k t_k(v)$ for $k = 1, \dots, n$. Note that this sequence defines a closed polygonal path in (the 1-skeleton of) X, i.e. vertices u_i and u_{i+1} are adjacent for $i = 0, 1, \dots, n-1$, and $u_0 = v = u_n$. We will call this path the *induced path* of a word w, and denote it by $(u_0, u_1, \ldots, u_n)$.

Claim A.3. By inserting subwords of form $h_1 t_1 \ldots h_k t_k t_k^{-1} h_k^{-1} \ldots t_1^{-1} h_1^{-1}$ one can convert a word w into a combination $w_1 w_2 \ldots w_l$ of words of form $h_1 t_1 \ldots h_m t_m$ with the induced polygonal paths $(u_0, u_1, \ldots, u_m)$ satisfying the following properties:
(1) $u_i = u_{m-i}$ for $0 \le i \le d$ and for some $d < m/2$;
(2) $(u_d, u_{d+1}, \ldots, u_{m-d})$ is a closed polygonal path around a single cell of X.

Proof: The claim follows from the fact that the complex X is simply connected by the standard argument in algebraic topology. Namely, there is a way of describing the fundamental group of a simplicial complex in combinatorial terms, where loops consisting of edges are (combinatorially) homotopic iff they differ by a sequence of "lassoes", as in the assertion of the claim.

Note that each word w_i as in Claim A.3 represents in the group Γ an element belonging to the subgroup H. It follows that, expanding h_1 in each of the words w_i into a product of the appropriate elements of H if necessary, we can express the full word $w_1 w_2 \ldots w_l$ as a new combination $w_1' w_2' \ldots w_l'$, where each w_i' has a form $h' t_1 h_2 t_2 \ldots h_m t_m h''$ and represents a trivial element in Γ. Now, our next goal is to modify separately each word w_i', using relations of types 1 and 2, into a word of form $w_i'' = h_1 t_1 \ldots h_d t_d p_i t_d^{-1} h_d^{-1} \ldots t_1^{-1} h_1^{-1}$, where p_i is a relation of type 3 from P or a cyclic modification of such a relation, and the induced polygonal path for w_i'' is the same as the one for w_i. To do this, note first that a word w_i'' as above indeed exists. Choose as p_i the appropriate cyclic modification of that relation $p_f \in P$ for which the face $f \in \mathcal{F}_0$ lies in the same Γ-orbit as the face f_i surrounded by the subpath $(u_d, u_{d+1}, \ldots, u_{m-d})$ of the path induced by w_i. Choose also a word $h_1 t_1 \ldots h_d t_d$ in such a way that its induced path coincides with the path $(u_0, u_1, \ldots, u_d)$, and so that the corresponding element of Γ maps f onto f_i. Then w_i'' is as required. A modification of w_i' into w_i'' by means of relations is equivalent to reducing the combined word $(w_i')^{-1} w_i''$ to a trivial word. To see that this can be done using relations of types 1 and 2, apply as many times as necessary the following.

Lemma A.4. Let $(u_0, u_1, \ldots, u_p)$ be the induced polygonal path in X for a word $z = h_1 t_1 \ldots h_p t_p$, and assume that $u_q = u_{q+2}$ for some $0 \le q \le p-2$. Then z can be reduced to a word $z' = h_1 t_1 \ldots h_q t_q h_{q+3}' t_{q+3} \ldots h_p t_p$, for some $h_{q+3}' \in H$, using a relation of type 1 or type 2.

Proof: Since $h_1 t_1 \ldots h_q t_q(v) = u_q = u_{q+2} = h_1 t_1 \ldots h_q t_q(h_{q+1} t_{q+1} h_{q+2} t_{q+2}(v))$, we have $h_{q+1} t_{q+1} h_{q+2} t_{q+2}(v) = v$, i.e. $h_{q+1} t_{q+1} h_{q+2} t_{q+2} \in H$. It follows that $h h_{q+1} t_{q+1} h_{q+2} t_{q+2} = e$ for some $h \in H$, and hence $t_{q+1} = t_{q+2}$ or $t_{q+1} = t_{q+2}^{-1}$ (because $t_{q+1} \sim t_{q+2}$). In any case, putting $h h_{q+1} = \bar{h}$, we get that $\bar{h} t_{q+1} h_{q+2} t_{q+2}$ is a relation of type 1 or type 2 in P. The lemma then follows by expanding h_{q+3} as a product $h_{q+3} \bar{h}^{-1} \bar{h}$, putting $h_{q+3}' = h_{q+3} \bar{h}^{-1}$ and cancelling $\bar{h} t_{q+1} h_{q+2} t_{q+2}$.

To finish the proof of Proposition A.2, note that a new combination $w_1'' \ldots w_m''$, obtained from $w_1' \ldots w_m'$ in a way described above, can be further reduced by first cancelling relations p_i, and then subsequently cancelling the rest of the word. Thus the proposition follows.

Acknowledgements

The author was supported by the Polish Scientific Research Commitee (KBN) grant 2 P02A 023 14.

References

[1] W. Ballmann, M. Brin, *Polygonal complexes and combinatorial group theory,* Geom. Dedicata 50 (1994), 165–191.

[2] W. Ballmann & J. Świątkowski, *On L^2-cohomology and property (T) for automorphism groups of polyhedral cell complexes,* GAFA 7 (1997), 615–645.

[3] M. Bestvina, *The virtual cohomological dimension of Coxeter groups,* Geometric group theory (G. Niblo & M. Roller, eds.), Vol. 1, London Math. Soc. Lect. Notes Series 181, Cambridge Univ. Press, 1993.

[4] M. Bourdon, *Sur les immeubles fuchsiens et leur type de quasiisometrie,* Ergodic Theory and Dynamical Systems 20 (2000), 1–23.

[5] M. Bridson *Geodesics and curvature in metric simplicial complexes,* Group theory from a geometrical viewpoint (E. Ghys, A. Haefliger & A. Verjovsky, eds.), Proceedings ICTP Trieste, World Scientific, 1991.

[6] D. Cartwright, W. Młotkowski, T. Stger, *Property (T) and $\tilde{A}_2$-groups,* Ann. Inst. Fourier 44 (1993), 213–248.

[7] C. Champetier, *L'escpace des groupes de type fini,* Prepublication de l'Institut Fourier Grenoble no. 231, 1993.

[8] J. Dymara, T. Januszkiewicz, *New Kazhdan groups,* to appear in Geometriae Dedicata.

[9] J. Friedman, *Some graphs with small second eigenvalue,* Combinatorica 15 (1) (1995), 31–42.

[10] D. Gaboriau, F. Paulin, *Sur les immeubles hyperboliques,* to appear in Geometriae Dedicata.

[11] A. Haefliger, *Complexes of groups and orbihedra,* Group theory from a geometrical viewpoint (E. Ghys, A. Heafliger & A. Verjovsky, eds.), Proceedings ICTP Trieste, World Scientific, 1991.

[12] P. de la Harpe & A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts,* Astérisque 175, Soc. Math. France, 1989.

[13] P. Sarnak, *Some applications of modular forms,* Cambridge Tracts in Math. 99, Cambridge Univ. Press, 1990.

[14] A. Zuk, *La propriete (T) de Kazhdan pour les groupes agissant sur les polyedres,* CRAS 323, Serie I (1996), 453–458.

Instytut Matematyczny
Uniwersytet Wrocławski
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: swiatkow@math.uni.wroc.pl

Instytut Matematyczny PAN
ul. Sniadeckich 8
00-950 Warszawa, Poland