SOME INFINITE GROUPS GENERATED BY INVOLUTIONS HAVE KAZHDAN'S PROPERTY (T)

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Abstract. We construct a large family of infinite groups acting on contractible simplicial 2-complexes, which are generated by involutions, but in general are not Coxeter groups. Our construction develops an idea of M. Bourdon [Bou], section 1.E.3. We derive presentations of the constructed groups, and show that many of these groups satisfy Kazhdan's Property (T), in contrast with the fact that infinite Coxeter groups never satisfy this property [BJS].

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Introduction. A well known class of groups generated by involutions are Coxeter groups, extensively studied and exhibiting connections with various areas of mathematics. In this paper we are interested in groups generated by involutions which are not Coxeter groups. The paper arouse from contemplating the examples of such groups constructed by M. Bourdon in [Bou], section 1.E.3, and from an attempt to generalize them.

To each Coxeter group there is associated a polyhedral complex (the so called Davis complex [Da]), which is contractible and on which the group acts properly discontinuously and cocompactly. In fact, a Coxeter group can be recovered from the quotient space of its action on this complex, provided that some additional local information about the action is given. This local information is called an orbihedron structure on the quotient space (see [Hae] or section 4 below for more details concerning this notion). In this paper we use the technique of orbihedra for constructing groups generated by involutions that are geometrically very similar to Coxeter groups. In particular, it follows from the construction that these groups act properly discontinuously and cocompactly on some infinite contractible simplicial 2-complexes.

Coxeter groups have canonical presentations, with each relation involving one or two generators only. The groups constructed in this paper also admit natural presentations, with generators being involutions. Relations involving two generators have the same form as in Coxeter groups, but in general there are also relations involving three or more generators. Such additional relations make these groups isomorphic to quotients of Coxeter groups.

An important motivation for the author to study groups acting on infinite contractible simplicial complexes is that one meets among them groups satisfying Kazhdan's Property (T) (see [BS], [Bou], [Swi] for the examples, and [HV] for more informations about Kazhdan's Property (T)). This is also the case for groups constructed in this paper. The new

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examples of Kazhdan groups described in section 6 seem rather surprising, since they are so similar to infinite Coxeter groups which never satisfy Property (T) [BJS].

Organization of the paper. The main construction of groups generated by involutions is performed in section 5. It is preceded by the discussion of finite groups generated by involutions, since such groups are the important building blocks in the construction. Thus, in section 1 we recall some properties of the (properly defined) Cayley graphs of such groups. Sections 2 and 3 provide few classes of interesting examples. In section 4 we recall basic facts concerning orbihedra, which are necessary to understand the construction. In section 6 we show that some families of the constructed groups satisfy Kazhdan's Property (T), making use of the examples of section 3. Section 8 is devoted to deriving presentations of the constructed groups, and it is preceded by a useful technique for getting presentations, described in section 7.

1. Cayley graphs of groups generated by involutions.

Given a group H with a generating set S, define its *Cayley graph* $C(H,S)$ as follows. Take the set H as the vertex set, and for $h_1, h_2 \in H$ let $\{h_1, h_2\}$ be an (unoriented) edge of $C(H,S)$ iff $h_2 = h_1 s$ for some $s \in S$.

Beware that the above definition of Cayley graph differs slightly from the standard one (compare e.g. [BH], Chapter I, section 1.9, p. 8). According to it Cayley graphs are the ordinary (unoriented) graphs without loops and multiple edges. For the purpose of this paper such a variant of the notion of Cayley graph is more useful than the standard one.

Note that, like in the standard case, a group H acts on its Cayley graph $C(H,S)$ from the left by combinatorial automorphisms.

In this section we consider groups H generated by sets S of *involutions*, i.e. elements of order 2. Below we describe some properties of the actions of such groups H on their Cayley graphs $C(H,S)$. These properties will be exploited in the construction of section 5.

1.1. Properties. Consider the barycentric subdivision $C'(H,S)$ of the Cayley graph $C(H,S)$. The quotient $H\backslash C(H,S)$ can be then identified with the star $st(e, C'(H,S))$ of the vertex e in $C'(H,S)$ corresponding to the neutral element in H. Moreover, there is a 1-1 correspondence between the generators $s \in S$ and the vertices in $st(e, C'(H,S))$ distinct from e. For a vertex x corresponding to s the stabilizing subgroup of H at x is $\langle s \rangle$, and s acts locally around x as a reflection (s reverses the edge of $C(H,S)$ having x as the barycenter).

1.2. Example. A pair (H,S) is a *Coxeter system* if S is a generating set for the group H and if H has a presentation

$$H = \langle S | (s_i s_j)^{m_{ij}} : i,j \in I \rangle,$$

where $S = \{s_i : i \in I\}$ and (m_{ij}) is a symmetric matrix with values in $N \cup \{\infty\}$ such that $m_{ii} = 1$ for each $i \in I$. An infinite exponent m_{ij} in the expression $(s_i s_j)^{m_{ij}}$ means that for those i,j there is no relation of the above form in the presentation.

Coxeter systems form an important class of groups generated by involutions, but there are many other examples. In the next two sections we describe such examples of finite groups with particularly interesting Cayley graphs. We will use them later in sections 5 and 6 in the construction of infinite groups generated by involutions and satisfying Kazhdan's Property (T).

2. Dihedral graphs.

2.1. Definition. Denote by P_n the boundary of a regular n-gon in the plane, viewed as a graph consisting of n vertices and n edges. We say that a graph L is *dihedral*, if for some natural number $n \geq 3$ the following conditions are satisfied:

(1) L is a bipartite graph with the set of vertices split into two subsets V_1 and V_2 with equal cardinalities n (so that each edge of L connects a vertex from V_1 with a vertex from V_2);

(2) vertices of L can be identified (bijectively) with the centers of the edges of the barycentric subdivision P_n' of P_n so that
   (a) the centers of adjacent edges of P_n' always belong to distinct sets V_i;
   (b) the dihedral group D_n of all symmetries of P_n lifts to a group of automorphisms of the graph L.

2.2. Lemma. Each connected dihedral graph is a Cayley graph of a dihedral group D_n with respect to some generating set S consisting of involutions.

2.3. Remark. Dihedral group D_n admits a set of generators consisting of precisely two involutions, with which it forms a Coxeter system. Generating sets occuring in the lemma contain as many elements as the valency of the corresponding graph, which is in general more than 2. Therefore a pair (D_n, S) resulting from the lemma in general is not a Coxeter system.

Proof of Lemma 2.2: Let L be a connected dihedral graph and P_n the corresponding regular polygon. The dihedral group D_n of all symmetries of P_n is then a group of automorphisms of L acting *simply transitively* (i.e. transitively and with trivial stabilizers) on the set of vertices of L. Fix a vertex p of L and note, that for any edge e of L adjacent to p there is an axial symmetry s_e of P_n which, after lifting to an automorphism of L, reverses the edge e. Put then

$$S = \{s_e : e \text{ is an edge of } L \text{ adjacent to } p\},$$

and note that S is a set of involutions in D_n.

To prove that S generates D_n, it is sufficient to prove that the group $\langle S \rangle$ generated by S acts transitively on the vertices of L. This follows however easily from the connectivity and D_n-invariance of L.

An isomorphism ϕ of the Cayley graph $C(D_n, S)$ to the graph L is given by the explicit formula $\phi(h) = h(p)$. Thus the lemma follows.

3. Examples of dihedral graphs.

3.1. Complete bipartite graphs. A *complete bipartite graph* $K_{n,m}$ is the graph with $n + m$ vertices split into two subsets V_1 and V_2 of cardinalities n and m respectively, with the edge set $\{\{u, v\} : u \in V_1, v \in V_2\}$. In the case when $n = m$ the corresponding complete bipartite graph $K_{n,n}$ is a dihedral graph. This can be seen by identifying its vertices with the centers of edges in P'_n in any way satisfying condition (2)(a) in Definition 2.1.

3.2. Bipartite doubles of complete graphs. Given a graph L, a *bipartite double* $\tilde{L}$ of L is the graph with two vertices v_+ and v_- for each vertex v of L, such that (v_δ, w_ϵ) is an edge of $\tilde{L}$ iff $\delta \neq \epsilon$ and (v, w) is an edge of L.

Let K_n be the *complete graph* with n vertices, i.e. the graph with an edge between any two of its vertices. Then the bipartite double $\tilde{K}_n$ is a dihedral graph, as it can be seen by identifying its vertices with the centers of edges in P'_n so that:

(1) if v_δ and w_ϵ correspond to the centers of adjacent edges of P'_n then $\delta \neq \epsilon$;
(2) the centers of any two edges of P'_n obtained by subdividing an edge of P_n correspond to the vertices v_+ and v_-, for some vertex v of K_n.

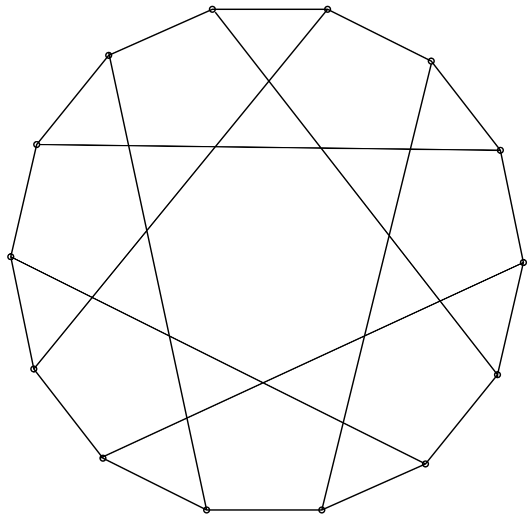

Figure 1. The Heawood graph.

3.3. The Heawood graph. Let F_2P^2 denote the projective plane over the field F_2 of order 2. The *Heawood graph* is the incidence graph of the plane F_2P^2, i.e. the graph with vertices representing all points and lines in F_2P^2 and with edges corresponding to incident pairs of points and lines.

The Heawood graph is a dihedral graph, as it follows from Figure 1 presenting a useful view of this graph. The role of polygon P_n is played in this case by the 7-gon.

4. Developable rigid 2-orbihedra.

In this section we recall the notion of rigid orbihedron in the special case of dimension 2. The related notions of developability and nonpositive curvature of an orbihedron are also recalled. We refer the reader to [Hae] for the detailed exposition of these subjects.

Given a simplicial complex K and its simplex σ, the *link* of K at σ, denoted $L(K,\sigma)$, is the subcomplex of K consisting of all simplices of K not containing σ but contained in a simplex of K containing σ.

An action of a group G by automorphisms on a simplicial complex K is *rigid*, if there is no nontrivial element $g \in G$ $(g \neq e)$ fixing pointwise some simplex σ in K of dimension $\dim\sigma = \dim K$.

4.1. Rigid 2-orbihedron. *Rigid* 2-*orbihedron* is a simplicial 2-complex K together with a family of triples (G_x, L_x, p_x), one for each vertex x of K, satisfying the following conditions.

(1) L_x is a graph and G_x is a group acting simplicially, rigidly and without inversions on L_x.

(2) $p_x : L_x \to L(K,x)$, called a *projection*, is a G_x-invariant simplicial map which induces an isomorphism of the quotient $G_x\backslash L_x$ on the link $L(K,x)$.

(3) For an edge e in K containing a vertex x, let x_e be the vertex in the link $L(K,x)$ contained in e, and let $\tilde{x}_e$ be a vertex of L_x with $p_x(\tilde{x}_e) = x_e$. Then the pair of a set $L_{e,x} := L(L_x, \tilde{x}_e)$ and a group $G_{e,x} := \mathrm{Stab}(G_x, \tilde{x}_e)$ with the induced action on $L_{e,x}$ (which is free due to rigidity of the action of G_x on L_x) does not depend on the choice of $\tilde{x}_e$, up to an equivariant isomorphism. Projection p_x induces then (in a canonical way, which turns out to be independent on the choice of $\tilde{x}_e$) a $G_{e,x}$-invariant map $p_{e,x} : L_{e,x} \to L(K,e)$. It is then required that for each edge $e = (x,y)$ in K the pairs $(L_{e,x}, G_{e,x})$ and $(L_{e,y}, G_{e,y})$ are equivariantly isomorphic by an isomorphism which projects by $p_{e,x}$ and $p_{e,y}$ to the identity of the link $L(K,e)$.

Remark. The above definition of rigid 2-orbihedron is a simplified version of the general definition of orbihedron in [Hae]. Simplifications are due to the assumptions of rigidity and dimension 2.

4.2. Developability. Let $\widetilde{K}$ be a simplicial 2-complex and G a group acting on $\widetilde{K}$ simplicially, without inversions and rigidly. Then the quotient $K = G\backslash\widetilde{K}$ has the natural structure of a simplicial complex, and one defines the so called *quotient orbihedron* $G(\widetilde{K})$ over K as follows. For each vertex $x \in K$ let $\tilde{x} \in \widetilde{K}$ be a vertex in the preimage of x by the quotient map $q : \widetilde{K} \to K$. Put $L_x := L(\widetilde{K}, \tilde{x})$ and $G_x := \mathrm{Stab}(G, \tilde{x})$, where G_x acts on L_x in a canonical way. Let $p_x : L(\widetilde{K}, \tilde{x}) \to L(K,x)$ be the map induced on links by the map q. The quotient orbihedron $G(\widetilde{K})$ as above is then an example of a rigid 2-orbihedron.

A rigid 2-orbihedron Q is *developable*, if it is isomorphic to a quotient orbihedron of the form $G(\widetilde{K})$, where $\widetilde{K}$ is simply connected. It is known [Hae] that if Q is developable, then the corresponding pair $(\widetilde{K}, G)$ is unique up to an equivariant isomorphism. The complex $\widetilde{K}$ is called the *universal covering* of the orbihedron Q, and the group G is called the *fundamental group* of Q.

4.3. Nonpositive curvature. Let K be a rigid 2-orbihedron, and assume that its underlying simplicial complex is equipped with a piecewise euclidean metric which turns each 2-simplex of K into a euclidean triangle. This metric induces angular length metrics on the links $L(K, x)$ in an obvious way, and those metrics can be further lifted by the projections p_x to the graphs L_x. Denote such lifted length metrics on the links L_x by ρ_x.

A rigid 2-orbihedron K with a metric as above is *nonpositively curved*, if for each vertex x of K the length (with respect to the induced length metric ρ_x as above) of any closed embedded polygonal path in L_x is not smaller than 2π.

The following theorem will play crucial role in the construction of the next section.

4.4. Theorem [Hae]. Let K be a rigid 2-orbihedron, and assume that for some piecewise euclidean metric on the underlying complex it is nopositively curved. Then K is developable.

5. Construction.

This section contains the announced construction of a class of groups acting by automorphisms on infinite contractible simplicial 2-complexes. The construction is based on the technique of developable orbihedra described in the previous section.

5.1. An outline of the construction. The construction consists of assembling (out of some proper initial data) a compact nonpositively curved rigid 2-orbihedron Q. By Theorem 4.4, such orbihedron is developable, and its fundamental group will be the required group.

The underlying complex of an orbihedron Q is the simplicial cone cL over a bipartite graph L (called the *base graph*), and we identify L with a subcomplex of cL in the obvious way. The vertices of L of the two types play different roles in an orbihedron structure. Vertices of the first type, called *face vertices*, determine "faces of reflection" in cL. Namely, for each such vertex w, the star $st(w, L)$ (viewed as a subcomplex of cL) is a mirror for a reflection which determines the local orbihedron structure at w. Vertices of the second type, at which several "faces of reflection" meet, are called *corner vertices*. The local orbihedron structure at each corner vertex v is modelled on the action of a finite group H_v generated by involutions on its Cayley graph, as described in section 1 (Properties 1.1). We will call the corresponding groups H_v *corner groups*.

5.2. Example. Consider a hyperbolic reflection group G acting in H^2, and let P be the fundamental polygon for G. Denote by P' the barycentric subdivision of P, and by $\widetilde{Q}$ the G-equivariant extension of this subdivision to the whole plane H^2. Consider the quotient orbihedron $Q = G(\widetilde{Q})$. Q is then an example of an orbihedron as outlined above and described in detail below in subsection 5.3.

To be more precise, note that the underlying simplicial complex of the orbihedron Q is isomorphic to P'. Let L be the graph isomorphic to the boundary of P'. P' is then clearly isomorphic to the simplicial cone cL. Corner vertices in L are then the vertices of P and face vertices are the barycenters of edges. The group G_o stabilizing the barycenter o of P is clearly trivial, while the groups stabilizing face vertices are all cyclic of order 2.

Corner groups (corresponding to stabilizers of corner vertices) are the dihedral groups, and the links of $\widetilde{Q}$ at the corresponding corner vertices are isomorphic to the Cayley graphs of these groups (with respect to standard generating pairs of reflections).

5.3. Orbihedra Q. We now pass to a more detailed description of a class of orbihedra Q. To get an orbihedron from this class we need the following data:

(1) a finite bipartite graph L with the set of vertices partitioned according to its bipartite structure into two subsets V_1 and V_2 (these are the sets of face and corner vertices respectively);

(2) for each vertex $v \in V_2$, a finite group H_v and a generating set S_v of involutions in this group, such that the cardinality of S_v equals the *valence* of the vertex v, i.e. the number of edges of the graph L that contain v;

(3) for each vertex $v \in V_2$, a combinatorial isomorphism $\pi_v : H_v \backslash C'(H_v, S_v) \to st(v, L)$ (compare Properties 1.1).

As the underlying complex for an orbihedron Q take the *simplicial cone* cL, i.e. the 2-complex with a distinguished vertex o, with the link $L(cL, o)$ isomorphic to the graph L, and such that $st(o, cL) = cL$. The graph L can be then identified with the subcomplex in cL consisting of all simplices not containing the vertex o.

We define an orbihedron structure on cL as follows.

(a) The triple (G_o, L_o, p_o) is trivial, i.e. G_o is the trivial group, $L_o = L(cL, o) = L$, and $p_o : L_o \to L(cL, o)$ is the identity.

(b) For each vertex $w \in V_1$ let L_w be the graph with the vertex set $\{w_+, w_-\} \cup \{v \in V_2 : v \text{ is adjacent to } w \text{ in } L\}$, and with the edge set consisting of all possible edges of the form $(w_\pm, v)$. Let G_w be the cyclic group of order 2 with the generator acting on L_w by transposing the vertices $w_\pm$ and fixing the vertices v. Finally, let $p_w : L_w \to st(w, L) \cong L(cL, w)$ be the simplicial map determined by $p_w(w_\pm) = w$ and $p_w(v) = v$ for each $v \in V_2$ adjacent to w in L. Here and below we identify $st(w, L)$ and $L(cL, w)$ by sending w to o and mapping all other vertices to themselves.

(c) For each vertex $v \in V_2$ put $L_v = C'(H_v, S_v)$ and $G_v = H_v$. Define the mapping $p_v : L_v \to L(cL, v)$ by $p_v = \tilde{\pi}_v$, where $\tilde{\pi}_v : C'(H_v, S_v) \to st(v, L)$ is the lift of π_v.

5.4. Nonpositive curvature. We describe now a convenient metric on an orbihedron Q, and express the nonpositive curvature condition for this metric in combinatorial terms.

A *girth* g_L of a graph L is the number of edges in the shortest closed circuit of edges in L. Note that the 2-simplices of Q all have the form (o, v, w), where $w \in V_1$ and $v \in V_2$. Put on each of them a euclidean metric with the right angle at the vertex w and with the angle of $2\pi/g_L$ at the vertex o. Note that this can be done only if $g_L > 4$, which for bipartite graphs is equivalent to $g_L \geq 6$.

With this metric Q satisfies the nonpositive curvature condition at the vertex o and at all the vertices $w \in V_1$. To have this condition fulfilled at the vertices $v \in V_2$ we need additional assumptions. Note that now the angle at v in any 2-simplex of Q equals $\alpha := (g_L - 4)\pi/2g_L$. The nonpositive curvature condition at v reads then as $2g_{C_v}\alpha \geq 2\pi$, where C_v is a short notation for the Cayley graph $C(H_v, S_v)$. After appropriate substitutions and simplifications we get the purely combinatorial condition $g_{C_v} \geq 2g_L/(g_L - 4)$.

5.5. Proposition. Let Q be an orbihedron as described in subsection 5.3, and suppose that Q is nonpositively curved. Suppose also that the graph L occuring in the description of Q contains a nontrivial circuit of edges. Then the universal covering $\widetilde{Q}$ of Q is an infinite contractible simplicial complex, and the fundamental group $G = \pi_1(Q)$ is an infinite group generated by involutions acting properly discontinuously and cocompactly on $\widetilde{Q}$.

Proof: Since Q is a nonpositively curved orbihedron, it follows from Theorem 4.4 that it is developable, and hence its universal covering $\widetilde{Q}$ and the fundamental group G are well defined. By definition, G acts properly discontinuously on $\widetilde{Q}$, and the action is cocompact by the compactness of Q. Moreover, by Theorem 6.4 of [Hae], $\widetilde{Q}$ equipped with the lifted piecewise euclidean metric is a CAT(0) space (see [Hae], section 6 or [BH], Chapter II.1 for the definition of CAT(0) spaces), and hence it is contractible ([BH], Chapter II.1, Corollary 1.5, p. 161). The fact that G is generated by involutions follows from the presentation of G derived in section 8. It follows also by observing that all the vertex groups in the orbihedron structure of Q are generated by involutions.

It remains to prove that the complex $\widetilde{Q}$ is infinite. Together with the cocompactness of the action of G on $\widetilde{Q}$, this will imply that the group G is infinite.

Let J be a subgraph consisting of all edges of a nontrivial circuit* in the graph L. Then the simplicial cone cJ is a subcomplex in the simplicial cone cL, and hence in (the underlying complex of) the orbihedron Q. Let $\pi : \widetilde{Q} \to Q$ be the covering map of the orbihedron Q. Consider the subcomplex $\Omega = \pi^{-1}(cJ)$ and note that each 1-simplex of Ω is contained in precisely two 2-simplices of Ω. Therefore, if the subcomplex Ω is finite, we may view it as a simplicial 2-cycle ω in $\widetilde{Q}$, where coefficients are in the group Z_2. Moreover, ω determines a nonzero homology element $[\omega] \in H_2(\widetilde{Q}, Z_2)$, becouse the complex $\widetilde{Q}$ is 2-dimensional. But since $\widetilde{Q}$ is contractible, we have $H_2(\widetilde{Q}, Z_2) = 0$, and hence the subcomplex Ω is infinite. Thus $\widetilde{Q}$ is infinite either.

6. Groups with Kazhdan's Property (T).

To prove Property (T) for some of the groups constructed in the previous section we will refer to a theorem from [BS] (see also [Żuk]). We recall this theorem below, after introducing necessary terminology.

For a finite graph L with the set of vertices $\mathcal{V}_L$, the *Laplacian* Δ on the space of real valued functions on $\mathcal{V}_L$ is defined by

$$\Delta f(v) = f(v) - Af(v),$$

where $Af(v)$ is the mean value of f on the vertices adjacent to v. Clearly, Δ is a self adjoint operator. We denote by κ_L its smallest positive eigenvalue.

6.1. Theorem [BS]. Let X be a contractible 2-dimensional simplicial complex and Γ be a properly discontinuous and cocompact group of automorphisms of X. Assume that for any vertex v the link $L(X, v)$ is connected and $\kappa_L > 1/2$. Then Γ satisfies Property (T).

The next proposition establishes some sufficient conditions for the groups constructed in the previous section to satisfy Kazhdan's Property (T).

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* I.e. a subgraph homeomorphic to a circle.

6.2. Proposition. Suppose that L, H_v and S_v are some data as in subsection 5.3, necessary to construct an orbihedron Q. Suppose that the graph L is connected, nonsimply connected, and that $g_L \geq 6$ and $\kappa_L > 1/2$. Furthermore, suppose that each Cayley graph $C_v := C(H_v, S_v)$ satisfies conditions $g_{C_v} \geq 2g_L/(g_L - 4)$ and $\kappa_{C_v} > 1/2$. Then the fundamental group Γ of the corresponding orbihedron Q is infinite and satisfies Property (T).

Proof: The assumption on girths g_L and g_{C_v} allows to make Q a nonpositively curved orbihedron, as in subsection 5.4. By Proposition 5.5, Γ is then infinite and acts by automorphisms, properly discontinuously and cocompactly, on the universal covering $\widetilde{Q}$.

To apply Theorem 6.1, modify slightly the simplicial structure on the universal cover $\widetilde{Q}$. Namely, arrange all 2-simplices of $\widetilde{Q}$ in pairs in such a way that the simplices in each pair are adjacent along an edge which projects by the covering projection $\widetilde{Q} \to Q$ to an edge in the subcomplex $L \subset Q$. This can be done in a unique way. Since the edge connecting the 2-simplices in each pair is contained only in these two 2-simplices, we can join them into a single 2-cell. But the original 2-simplices both carry geometric structures of right angled triangles, and therefore their union is geometrically a triangle. We then view each such union as a 2-simplex, in a way consistent with its geometry. In this way we obtain a new simplicial complex X, which has the same underlying space as $\widetilde{Q}$. The induced action of Γ on X is clearly simplicial, proper discontinuous and cocompact.

Now, it is easy to realize that each link of the complex X is isomorphic either to the base graph L or to the Cayley graph $C(H_v, S_v)$ of one of the corner groups H_v, depending on the image of the corresponding vertex by the covering projection $X \to Q$. Since these graphs are assumed to satisfy the spectral estimate from Theorem 6.1, the proposition follows.

6.3. Sources of data for groups with Property (T). Kazhdan groups constructed by M. Bourdon in [Bou], section 1.E.3, have the form as in Proposition 6.2, with corner graphs C_v isomorphic to a complete bipartite graph $K_{n,n}$ (Example 3.1 above shows that $K_{n,n}$ is the Cayley graph as required). For the base graph L Bourdon uses the so called *generalized m-gons*, with m equal to 6 or 8. Both complete bipartite graphs and generalized m-gons are also known as spherical buildings of dimension 1 (see [Ron], Chapter 3.2, p. 28–30). The corresponding complexes X on which Bourdon's groups act are 2-dimensional *hyperbolic* buildings, i.e. the buildings with appartments isomorphic to tesselations of the hyperbolic plane. Bourdon notes, by referring to Theorem 6.1 as we also do, that for L with sufficiently large valence the corresponding groups have Property (T).

We mention here more examples of graphs L and Cayley graphs $C(H_v, S_v)$ appropriate to get groups with Property (T) by means of Proposition 6.2. In many cases the corresponding simplicial 2-complexes X acted upon by these groups are not buildings. In fact, they are not buildings whenever at least one of the graphs L and $C(H_v, S_v)$ is not a spherical building (compare [Ron], Theorem 3.5, p. 30). These non-building examples seem interesting as most of the so far known examples of Kazhdan automorphism groups are directly related to buildings.

Observe that the dihedral graphs mentioned in section 3 satisfy the required spectral condition, since we have $\kappa_{K_{n,n}} = 1$ ([CDS], section 2.6, 5°), $\kappa_{\tilde{K}_n} = 1 - 1/(n-1)$ ([BCN], theorem 1.11.1(v) and [CDS], section 2.6, 2°), and finally, denoting by $\mathcal{H}$ the Heawood graph, we have $\kappa_{\mathcal{H}} = 1 - \sqrt{2}/3$ ([CDS], Table 4 in Appendix, graph 4.19). Since, according to Lemma 2.2, these graphs are the Cayley graphs of groups generated by involutions, they can be used as corner graphs C_v.

Note that, except the Heawood graph, the graphs mentioned above have girth 4. The usage of these graphs in the construction requires the choice of graphs L with girth $g_L \geq 8$. Except for the graphs used in [Bou], generalized 4-gons with sufficiently large girth are also appropriate, see e.g. [BS], section 3.1. Bourdon doesn't mention these graphs since he is interested in groups acting on hyperbolic builings only, and the usage of generalized 4-gons, together with complete bipartite graphs as corner graphs, leads to euclidean buildings. There are also various families of appropriate bipartite Ramanujan graphs (see [Val] for an overview), many of which have sufficiently large girth.

It's likely that there are many other graphs L as required. For example, the graphs constructed in [Fri] satisfy the required spectral estimate, but we don't know which of them have sufficiently large girth.

6.4. $\widetilde{A}_2$-groups generated by involutions. We exhibit here two kinds of groups generated by reflections acting on euclidean buildings of type $\tilde{A}_2$ and thickness 3. Recall that such buildings can be characterized as simply connected simplicial 2-complexes links of which at all vertices are isomorphic to the Heawood graph (see [Ron], Example 4 on page 4 and the last paragraph of page 115).

Examples of the first kind are obtained according to the scheme of section 5, where the base graph L and all of corner graphs C_v are isomorphic to the Heawood graph. There is some possible choice in performing the construction using these data, so that a number of nonisomorphic groups (and perhaps buildings) may be obtained, but we do not touch these phenomena here.

A unique example of the second type has different nature and requires a separate description. Take as the underlying space of an orbihedron R the sum of three 2-simplices (v, w, b_i), $i = 1, 2, 3$, all adjacent along the edge (v, w). For an orbihedron structure at the vertex w take $G_w = Z_2$ acting locally by reflection, precisely like the groups at vertices $w \in V_1$ in the orbihedron structure of orbihedra Q of section 5. For a structure at v take the dihedral group D_7 in its action on (the barycentric subdivision of) the Heawood graph, as described in 3.4. Finally, for structures at vertices b_i take the symmetry group of the equilateral triangle in its canonical action.

Put a metric on R in which the 2-simplices (v, w, b_i) have the right angle at w, the angle of $\pi/6$ at v, and the angle of $\pi/3$ at b_i. Then the orbihedron R is nonpositively curved and it makes sense to consider its universal cover $\widetilde{R}$ and its fundamental group Γ. Note that $\widetilde{R}$ is isomorphic to the barycentric subdivision of an $\widetilde{A}_2$-building X of thickness 3. The barycenters of the 2-simplices of X correspond to those vertices of $\widetilde{R}$ which project by the covering projection $\widetilde{R} \to R$ on the vertices b_i.

Note that both kinds of reflection $\widetilde{A}_2$-groups described above satisfy all the assumptions of Theorem 6.1, and hence satisfy Property (T). The last described group is moreover a representative of the class of groups described in [Swi] (see also section 4 of [BS]). It follows from the results of [Swi] that this group has the presentation

$$\langle s_1, s_2, s_3, t | \{s_i^2\}, t^2, \{(s_i t)^3\}, \{(s_i s_j)^7 : i \neq j\}, s_3 s_2 s_1 s_2 s_1 s_2 \rangle.$$

7. Presentations of vertex-transitive groups.

This section is a preparation for the task (that we undertake in section 8) of getting presentations for groups obtained by the construction of section 5. It describes presentations of groups acting transitively on vertices of simply connected polygonal complexes. Proofs of the statements from this section can be found in the appendix to the paper [Swi], compare also section 2 of [BB].

A *polygonal complex* is a 2-dimensional polyhedral cell complex. We will refer to polygonal 2-cells of a polygonal complex as *faces*. Let Γ be a group of automorphisms of a simply connected polygonal complex X, and assume that Γ acts transitively on the set of vertices of X. Fix a vertex $v \in X$ and denote by $H = \mathrm{Stab}(\Gamma, v)$ the stabilizing subgroup of Γ at v. Let $\langle S|R\rangle$ be a presentation for the group H. We find a finite presentation for Γ which can be obtained by extending S to a generating set $S \cup T$, and by extending R to a sufficient set $R \cup P$ of relations for Γ with respect to $S \cup T$.

To describe T consider first the set $N := \{g \in \Gamma : g(v) \text{ is adjacent to } v \text{ in } X\}$. Consider also the equivalence relation in N defined by $g \sim g'$ iff $g' = h_1 g h_2$ or $g' = h_1 g^{-1} h_2$ for some $h_1, h_2 \in H$. Let T be a set of representatives of equivalence classes of this relation.

Lemma 7.1 [Swi]. The set $S \cup T$ generates the group Γ.

From now on fix a set T as above. Consider three types of relations for Γ with respect to $S \cup T$, as described below.

Type 1: $h_2 t h_1 t$ for any $t \in T$ and any $h_1 \in H$ with $h_1 t(v) = t^{-1}(v)$, where $h_2 = t^{-1} h_1^{-1} t^{-1} \in H$.

Type 2: $h_2 t^{-1} h_1 t$ for any $t \in T$ and any $h_1 \in H$ with $h_1 t(v) = t(v)$, where $h_2 = t^{-1} h_1^{-1} t \in H$.

Type 3: $h_1 t_1 h_2 t_2 \dots h_k t_k = h$ for $h_i \in H$ and $t_i \in T \cup T^{-1}$ such that the vertices $v_0 = v, v_1 = h_1 t_1(v), v_2 = h_1 t_1 h_2 t_2(v), \dots, v_k = h_1 t_1 \dots h_k t_k(v) = v = v_0$ are the consecutive vertices of a face f of X adjacent to v. For such h_i and t_i the element h as above is also an element of H.

In all three types of relations above it is understood that elements of the group H which appear in relations represent some reduced words on the set S.

To get a small set of relations for Γ consider the action of the group H on the set of faces of X adjacent to the vertex v. Let F be a set of representatives of orbits of this action. For each face $f \in F$ choose a relation of type 3 corresponding to f, and denote it by p_f. Take the set P of relations for Γ consisting of all relations of types 1 and 2, and the chosen relations p_f for all $f \in F$.

Proposition 7.2 [Swi]. Let P be a set of relations for Γ with respect to a generating set $S \cup T$, as described above. Then the pair $\langle S \cup T | R \cup P\rangle$ is a presentation for the group Γ.

8. Presentations for groups constructed in section 5.

In this section we derive presentations for the groups constructed in section 5. To do this, we first modify the simplicial 2-complex X corresponding to a group Γ from this class, to get a simply connected polygonal complex X' on which Γ acts vertex-transitively. We then apply the method of getting presentations described in the previous section.

Modification of the complex X. Define the *o-skeleton* X_o of the complex X as a 1-dimensional subcomplex consisting of all edges of X with both vertices in the preimage $\pi^{-1}(o)$ of the vertex o by the covering projection $\pi : X \to Q$. Equivalently, X_o is the preimage under π of the subcomplex in Q consisting of all edges $(o, w) : w \in V_1$. Note that this subcomplex disconnects X into components coinciding with the open stars $st(\tilde{v}, X)$, where each $\tilde{v} \in \pi^{-1}(V_2)$. Observe also that if $\tilde{v} \in \pi^{-1}(V_2)$ then the link $L(X, \tilde{v})$ is contained in X_o and is canonically isomorphic to the Cayley graph $C_v = C(H_v, S_v)$, where $v = \pi(\tilde{v})$. Choose a set P_v of cyclically reduced relations for the group H_v with respect to S_v, which together with the relations $\{s^2 : s \in S_v\}$ give a presentation for H_v. Let K_v be the *Cayley complex* of the presentation $\langle S_v | \{s^2 : s \in S_v\} \cup P_v \rangle$, i.e. a polygonal complex obtained by glueing 2-cells corresponding to the relations of P_v to the Cayley graph C_v. More precisely, for each vertex $h \in H_v$ and each cyclically reduced word $\omega = s_1 s_2 \ldots s_k \in P_v$ consider a closed unoriented polygonal path $p_{\omega,h}$ in C_v determined by the vertices $h, hs_1, hs_1s_2, \ldots, hs_1s_2\ldots s_k$. Each such path may be viewed as a k-gonal 2-cell glued to C_v according to this path. Consider the set $\mathcal{P}$ of all closed unoriented paths in C_v of the form $p_{\omega,h}$. (Note that in general there are different pairs ω, h for which the corresponding paths $p_{\omega,h}$ are the same.) The 2-complex K_v is then obtained by attaching to the graph C_v all the 2-cells represented by the paths from $\mathcal{P}$. Note that according to this definition, the complex K_v contains no 2-cells corresponding to relations s^2, since these relations do not belong to P_v.

The above version of the Cayley complex is slightly different from the standard one used in combinatorial group theory. It is especially useful for groups containing involutions, and in this context it is used e.g. in [ZVC], 4.13.8. It is important to note that the complex K_v as defined above is simply connected and that the action of H_v on the Cayley graph C_v extends to K_v (compare [ZVC], 4.13.9).

To get a polygonal complex X', replace each open star $st(\tilde{v}, X)$ by the Cayley complex K_v glued to X_o according to the above mentioned identification of $C_v \subset K_v$ with $L(X, \tilde{v}) \subset X_o$. Since X is contractible and the contractible stars in it are replaced by the simply connected complexes, it follows from the Seifert-van Kampen Theorem that the modified complex X' is simply connected. The only vertices of X' are those of the o-skeleton $X_o \subset X'$. Hence the induced action of Γ on X' is transitive on the vertices. These facts allow to apply the method described in section 7 to the action of Γ on X'.

Presentation. Fix a vertex $\tilde{o} \in \pi^{-1}(o)$ in X and note that it is also a vertex of X'. Since the group G_o in the orbihedron structure of Q is trivial, the stabilizing subgroup $H = Stab(\tilde{o}, \Gamma)$ in both X and X' is trivial. It follows that the action of Γ on X' is simply transitive on vertices.

Consider the canonical identification of the link $L(X, \tilde{o})$ with the base graph L. Then the set of vertices adjacent to $\tilde{o}$ in X_o is identified with the set V_1. For each $w \in V_1$ detote by r_w the unique elements in Γ with $r_w(\tilde{o}) = w$. It follows then from Lemma 7.1 that the set $T = \{r_w : w \in V_1\}$ generates Γ.

Since for each $w \in V_1$ the element r_w corresponds to the unique nonzero element in the group G_w of Q, it is likewise a reflection and has order 2. It is then not difficult to note that $r_w : w \in V_1$ are the only relations of types 1 and 2 for Γ.

To get relations of type 3 note first that, since the stabilizer H is trivial, a set F

of representatives consists of all faces of X' adjacent to $\tilde{o}$. Keeping the identification $L(X, \tilde{o}) \equiv L$ in mind, note that the set F splits into disjoint subset F_v consisting of faces of this Cayley complex K_v which replaces the open star $st(v, X)$. Now, for each $v \in V_2$, the set S_v of the generators of the group H_v identifies canonically with the set $\{r_w : w \text{ is adjacent to } v \text{ in } L\}$, and this identification extends to the isomorphism of H_v with the subgroup in Γ generated by the corresponding reflections r_w. Denote by R_v the set of relations in Γ obtained by substituting the elements r_w for the elements $s \in S_v$ in the relations from the set P_v. It turns out that the relations in R_v are all of type 3, and correspond bijectively to the faces in F_v. Therefore we may take the set $\cup_{v \in V_2} R_v$ as a representative set for the relations of type 3. In view of Proposition 7.2, the group Γ has then a presentation

$$\Gamma = \langle T | \{r_w^2 : w \in V_1\} \cup \bigcup_{v \in V_2} R_v \rangle.$$

References

[BB] Ballmann, W., Brin, M.: Polygonal complexes and combinatorial group theory. Geom. Dedicata **50** (1994), 165–191.

[BH] M. Bridson, A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Springer, 1999.

[BS] Ballmann, W., Świątkowski, J.: On L^2-cohomology and property (T) for automorphism groups of polyhedral cell complexes. GAFA **7** (1997), 615–645.

[Bou] Bourdon, M.: Sur les immeubles fuchsienes et leur type de quasiisometrie. Erg. Theory and Dyn. Systems 20 (2000).

[BJS] Bożejko, M., Januszkiewicz, T., Spatzier, R.: Infinite Coxeter groups do not have Kazhdan's property. J. Operator Theory **19** (1988), 63–68.

[BCN] Brouwer, A., Cohen, A., Neumaier, A.: Distance-Regular Graphs. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 18, Springer Verlag, Berlin - Heidelberg - New York 1989.

[CDS] Cvetković, D., Doob, M., Sachs, H.: Spectra of Graphs. Academic Press, New York 1980.

[Dav] Davis, M.: Groups generated by reflections and aspherical manifolds not covered by Euclidean space. Ann. of Math. (2) **117** (1983), 293–324.

[F] Friedman, J.: Some graphs with small second eigenvalue, Combinatorica **15** (1) (1995), 31–42.

[Hae] Haefliger, A.: Complexes of groups and orbihedra. In: Group theory from a geometrical viewpoint (ed. by E. Ghys, A. Heafliger & A. Verjovsky), pp. 504–542, Proceedings ICTP Trieste, World Scientific, Singapore 1991.

[HV] P. de la Harpe & A. Valette, *La propriété (T) de Kazhdan pour les groupes localement compacts,* Astérisque 175, Soc. Math. France, 1989.

[Ron] M. Ronan, *Lectures on Buildings*, Academic Press, 1989.

[Swi] Świątkowski, J.: A class of automorphism groups of polygonal complexes with Property (T), to appear in Quarterly J. Math.

[Val] Valette, A.: Graphes de Ramanujan et applications. Seminaire N. Bourbaki, Vol. 1996-97, Mars 1997, Expose 829.

[ZVC] Zieschang, H., Vogt, E., Coldewey, H.: Surfaces and Planar Discontinuous groups. Lecture Notes in Mathematics 835, Springer-Verlag, Berlin - Heidelberg - New York 1980.

[Żuk] Żuk, A.: La propriete (T) de Kazhdan pour les groupes agissant sur les polyedres. C. R. Acad. Sci. Paris **323**, Serie I (1996), 453–458.

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