THE VOLUME OF A PYRAMID THROUGH THE AGES

TO SLICE OR NOT TO SLICE, THAT'S THE QUESTION!

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Abstract

The volume of a pyramid equals one third of the area of the base multiplied by the height. Many teachers convince their pupils of this fact by pouring water, but this experiment does not explain why it is 'one third'. Do the pupils have to wait for an explanation until they study integrals? In this paper we will move through the history of the volume of prisms and pyramids in order to find elementary proofs of this factor 'one third'. We will distinguish two opposite tendencies in this history: on the one hand the recourse to infinitely thin slices and on the other hand the efforts to avoid the limit process and to make proofs by merely cutting the solids into a finite number of pieces and by reassembling these pieces.

1 INTRODUCTION

1.1 TO POUR WATER

The content of three equal hollow pyramids completely fills a prism with the same base and height. This experiment is carried out in many classes of primary school (or later) in order to show that the volume of the pyramid is one third of the volume of the prism. Because the volume of a prism equals the area of the base times the height, this gives the 'formula' for the volume of any pyramid:

$$\text{volume pyramid} = \frac{(\text{area of the base}) \cdot \text{height}}{3}.$$

This experiment is convincing and it is important that it is performed, but it is not a mathematical *proof* of the formula and it does not show *why* the ratio of the volumes is exactly $\frac{1}{3}$.

In the plane, the area of a triangle can be introduced by another experiment: two identical (congruent) cardboard triangles can be put together to form a parallelogram. Therefore, the area of the triangle is half the area of the parallelogram, thus half the length of the base times the height. The big difference with the water pouring experiment is that this one *does* contain a (pre-formal) proof, reducing the area of the triangle to the area of a parallelogram. This experiment explains *why* one has to divide by 2 in order to find the area of the triangle (supposing one already knows the area formula for the parallelogram).

Can we just do the same thing with a pyramid as with the triangle: to put together three congruent copies of the pyramid and form a prism? We will come back to this in paragraph 3. We have to study the volume of a prism first (in paragraph 2).

1.2 History of elementary mathematical concepts

The elementary mathematical concepts such as 'number', 'function', 'area', 'volume'... seem to be universal and unvarying. However, these concepts have changed radically through history: the numbers of the Ancient Greeks are not our (rational, real, complex) numbers; the functions Newton had in mind were not the general functions of the 20th century; areas and volumes were treated in a different way by the ancient Greeks as by my pupils today. According to my pupils, an area or a volume is essentially a *number* found by substituting the sizes of the figure into a 'formula'. The Greeks always compared two areas or two volumes with each other. The figures themselves *were* the quantities; they did not say 'the area of...' or 'the volume of...'. They stated, for instance: "two pyramids with equal bases are proportional to their heights". This difference is related to the different number concepts. Greek numbers were positive integers. They compared proportions of quantities and proportions of numbers, but these proportions were not considered as numbers. It is a huge anachronism to tell that the Pythagoreans proved that $\sqrt{2}$ is an irrational number...

Should we always treat history in an authentic way and confront the pupils with the original texts? I don't think this is always possible nor necessary. An anachronistic approach, with modern means and notations (computer, algebraic formulae) can make things much more accessible. But it is important that the mathematics teacher is aware of the differences between the actual concepts and their historical version, and that he talks about it to his pupils.

2 The volume of a parallelepiped and a prism

2.1 To reduce a prism to a parallelepiped

In order to prove that the volume of a prism is the area of the base multiplied by the height, it is sufficient to deal with a triangular prism (a prism with a triangular base), because each prism can be divided into triangular prisms. The volume of a triangular prism is half the volume of a parallelepiped (see figure 1: the triangular prism $ABC.\ DEF$ can be completed with a congruent triangular prism $FB'D.\ CE'A$ in order to form a parallelepiped. The second prism is the image of the first one by a central symmetry, so both prisms are congruent but not equally oriented.). In order to prove that the volume of the triangular prism is the area of the base multiplied by the height, it suffices to show this for the parallelepiped.

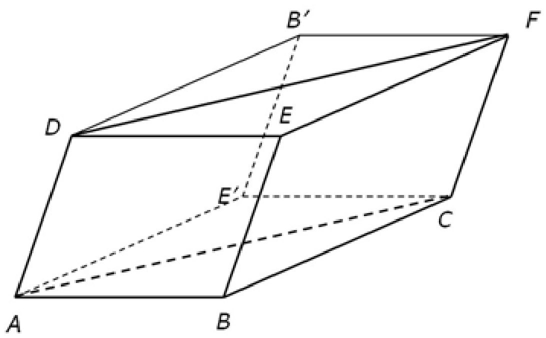

Figure 1

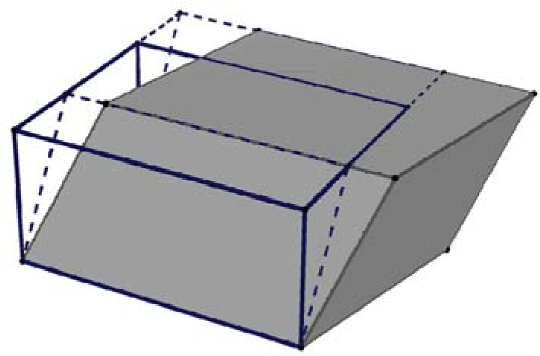

Figure 2

2.2 To reduce a parallelepiped to a rectangular box: by cutting and pasting

The parallelepiped of figure 2 can be transformed into a rectangular box by cutting and pasting. For instance, first cut the part that exceeds the box at the right side and shift it to the left side (dotted lines) and then cut the part that exceeds behind the box in shift it to the front (solid lines). This parallelepiped has the property that the orthogonal projection of the base onto the plane determined by the upper face has an non-empty intersection with the upper face. If this is not the case, the cutting and pasting is a little bit more complicated.

Euclid almost did the same thing, but he treated the case of a parallelepiped with a rectangular base first (figure 3, from the website of D. E. Joyce).

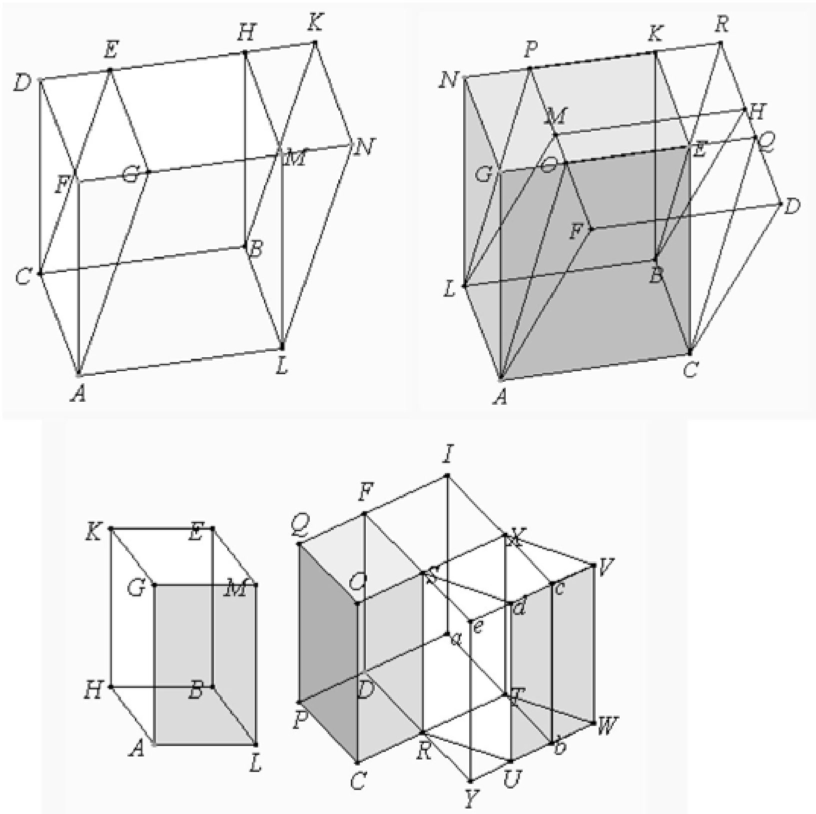

Figure 3

2.3 To reduce a parallelepiped to a rectangular box: by using the air

An alternative proof consists of transforming the parallelepiped in two steps as we did in figure 2, but without letting the solids overlap each other. In figure 4, the edges of the AB direction of the arbitrary parallelepiped (on the left in the foreground) have been extended and cut by a perpendicular plane. This gives rise to a parallelogram perpendicular to AB.

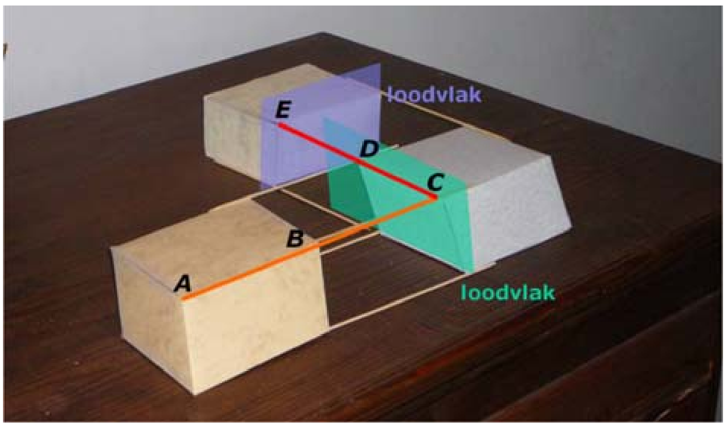

Figure 4

By shifting this parallelogram over the vector $\overrightarrow{AB}$, we construct a second parallelepiped, four faces of which are rectangles. Both parallelepipeds have the same base area and height. Furthermore, they have the same volume, because the first parallelepiped together with the 'air' between the first and the second parallelepiped is mapped, by the translation over the vector $\overrightarrow{AB}$, to the same 'air' together with the second parallelepiped. By repeating this procedure in the CD-direction, we can construct a third parallelepiped which is a rectangular box and has the same height, base area and volume as the other two. We conclude that a parallelepiped has the same volume as a rectangular box with the same base area and height.

2.4 To reduce a parallelepiped to a rectangular box: by using thin slices

A pile of paper sheets form a rectangular box. We can push and convert it into a parallelepiped (still with a rectangular base). It seems plausible that the volume does not change, because the pile is still composed of the same sheets of paper.

Bonaventura Cavalieri (1598–1647) generalized this idea: it is sufficient that the paper sheets have the same area; they do not need to be congruent. The sheets in one pile may also be different. The first principle of Cavalieri states about two solids resting on a horizontal plane (e.g. a table): *If the areas of the intersections with any horizontal plane are equal, then the solids have the same volume.* His second principle is even more general: *If the areas of the intersections with any horizontal plane are in a fixed proportion, then the volumes are in the same proportion.* These principles have already been used by Archimedes, but it was Cavalieri who formulated them explicitly.

Using the first principle of Cavalieri, it is easy to show that the volume of a parallelepiped equals the volume of a rectangular box with the same base area and volume.

There is a difference between Cavalieri's principle and the sheets of paper. The sheets of paper have a nonzero thickness and are, in fact, rectangular boxes. Together, they form an approximation of a parallelepiped, and the thinner they are, the better this approximation. On the other hand, Cavalieri's plane sections are two-dimensional figures. A sum of areas can never be a volume: this is a problematic 'dimension jump' which has been solved only in the integral calculus (17th century), where the solid is seen as a *limit* of thin slices the thickness of which tends to zero. The limit concept itself has been defined (and stripped of its mystery) by Augustin Louis Cauchy (1789–1857) and others.

This problem with the thin slices in an elementary approach is a good motivation to look for proofs by cutting and pasting with a finite number of pieces. This is possible for a parallelepiped (see paragraph 2.2) but it will be more problematic for a pyramid, as will be shown in paragraph 3.

2.5 The volume of a rectangular box

The volume of a rectangular box is the area of the base multiplied by the height. This is obvious when the length, the depth and the height are integer numbers of length units: the volume is simply the number of unit cubes filling the box (figure 5). This is the most elementary volume idea. If they are rational numbers of units, the box can be filled with smaller cubes, and this leads to the same result. However, this is impossible if the proportion between two or more of the sizes of the box is irrational. In this case, the volume of the rectangular box requires a limit process.

3 The volume of a pyramid

3.1 Two easy cases

In figure 6, you see three congruent pyramids pasted together to form a cube. The base of this special pyramid is a square; the apex is upright above one of the vertices of this square

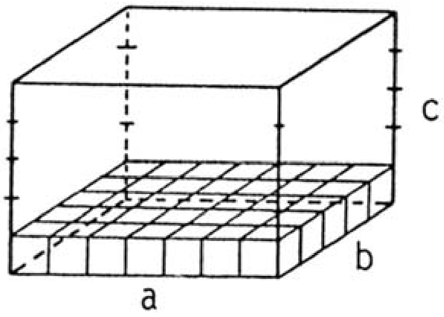

Figure 5

Figure 6

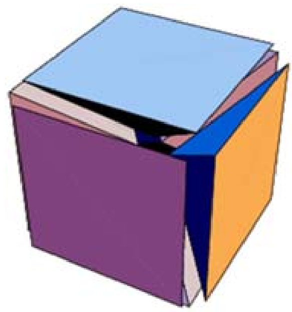

Figure 7

and the height equals the side of the square. The volume is one third of the volume of the cube, so the volume is one third of area of the base times the height.

A second special case (figure 7) has also a square base, but the apex is upright above the centre of the square and the height equals half the side of the square. Three congruent copies can be assembled to make a cube. So the volume is one sixth of the volume of the cube, or — again — one third of the area of the base times the height.

3.2 The volume of an 'Egyptian' pyramid by using thin slices

It is known that the Egyptians of the pharaohs' period were able to compute the volume of a pyramid. However, we do not know how they discovered it; they did not provide 'proofs' of their mathematical results. The Egyptian pyramids had a square base and the apex was upright above the centre of this square, but the height was not equal to half the side of the square, so an Egyptian pyramid is not the special case of figure 7. It is likely that they discovered how to calculate the volume by reasoning with slices. The Egyptian pyramids are build as 'stair pyramids'; the slices are not the infinitely thin ones of Cavalieri, but rough layers of stones (figure 8).

Figure 8

Take an 'Egyptian' pyramid. We can approximate it by starting with a pyramid of the type of figure 7 (one sixth of a cube), the base of which has the same size as the real one. We only have to adapt the height. We can approximate this special pyramid by a 'stair pyramid' with layers of height Δh. The volume (one third of the area of the base times the height, see 3.1) is approximated by the sum of the volumes of these layers. Now adapt the height by stretching vertically with an appropriate factor k so that the height grows to be the real one. Then each layer is stretched with factor k, so the volume of each layer is multiplied by k and so is the volume of the whole step pyramid. This does not change if we take more, thinner layers, and 'in the limit' the volume of the Egyptian pyramid equals one third of the area of the base times the height.

As in 2.4, this limit idea has always been seen as problematic before the integral calculus. Democritos of Abdera (460–370 b.C.) used a similar reasoning and he added the following comment (Lloyd, 1996): "What must we think of the surfaces forming the sections? If they are unequal, they will make the pyramid irregular with many indentations, like steps. If they are equal, the pyramid will appear to have the property of the prism and be of equal squares, which is very absurd."

3.3 Liu Hui and the volume of the *yangma*

The legend says that in 213 b.C. the emperor Qin Shi Huang commanded to burn all books and that 40 years later Zhang Cang wrote what he remembered from his mathematics education. This engendered the *Jiuzhang Suanshu* (Nine Chapters of the Mathematical Art), a text with 246 problems and their solution, written for engineers, architects and merchants. The Nine Chapters contained only results and methods, no proofs. Centuries later, in the 3rd century, Liu Hui (about 220–280) wrote the *Commentaries to the Nine Chapters*, in which he explained *why* the results of Nine Chapters are true. However, these proofs were not organized as an axiomatic theory, formulating explicitly the 'rules of the game' as in Euclid's Elements.

The *yangma* studied by Liu Hui is a pyramid with a rectangular base, the apex of which is upright above one of the vertices. It is more general than the special case of figure 6 because the three dimensions are not necessarily equal. With three congruent yangmas it is not possible to build one rectangular box, so it is not as easy as in 3.1.

In order to show that the volume of the yangma is one third of the volume of a rectangular box with same base and height, Hui adds a *bienao* (an adapted tetrahedron) so that the union of both forms a *qiandu* (a prism with a rectangular triangle as base, see figure 9). Because it is clear that the *qiandu* is half the rectangular box, the only thing he has to prove is that the volume of the *yangma* is twice the volume of the *bienao*.

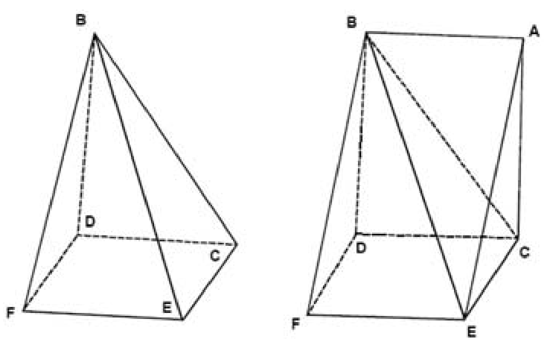

Figure 9

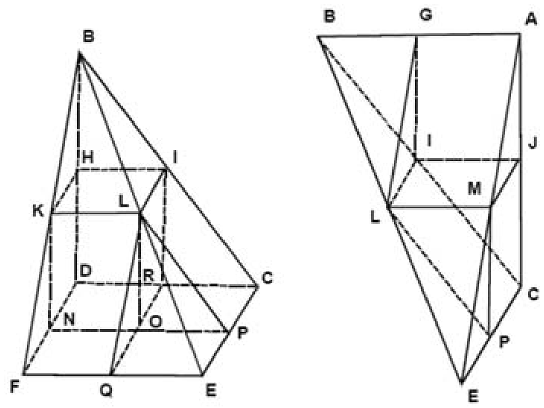

Figure 10

Hui divides both solids as in figure 10. The yangma is divided into a smaller rectangular box, two smaller qiandus and two smaller yangmas, the sizes of which are half the original ones. The bienao is divided into two smaller qiandus and two smaller bienaos (figure 10).

Because the volume of a smaller qiandus is half the volume of a smaller rectangular box, we can say that the yangma contains two rectangular boxes and the bienao one rectangular box (of the same size). The small yangmas and bienaos can be divided again. Again the number of rectangular boxes in the yangmas are twice the number of rectangular boxes in the bienao. And so on: the remaining yangmas and bienaos can always be broken up into smaller parts, until, as stated by Liu Hui, “they are so small that they do not have a volume any more”. Here it appears that Liu Hui thinks of a ‘physical limit’ (as in ‘smaller than one molecule’...) instead of our Archimedes-Cauchy limit concept (whatever epsilon, by going on long enough, the total volume of the remaining yangmas and bienaos can be made smaller than epsilon, so that the proportion of the volumes equals the proportion of the rectangular boxes). This different limit concept is confirmed by other phrases by Hui: “The ultimately small has nothing inside it.”; “If one cuts and further cuts, until one reaches what one can no longer cut, then it coincides and there is no error.”

3.4 Volume of an arbitrary pyramid, using pieces and...

Until now we only looked into proofs for the volume of *special cases* of pyramids. For the general case, it is enough to give a proof for an arbitrary pyramid with a triangular base (in other words: for an arbitrary tetrahedron). Indeed, every pyramid can be split up into pyramids with triangular bases and the volumes can be added... We found a comic strip proof (Kindt, 1999) in which the volume formula for an arbitrary tetrahedron seems to be proven by using only cutting and pasting (figure 11).

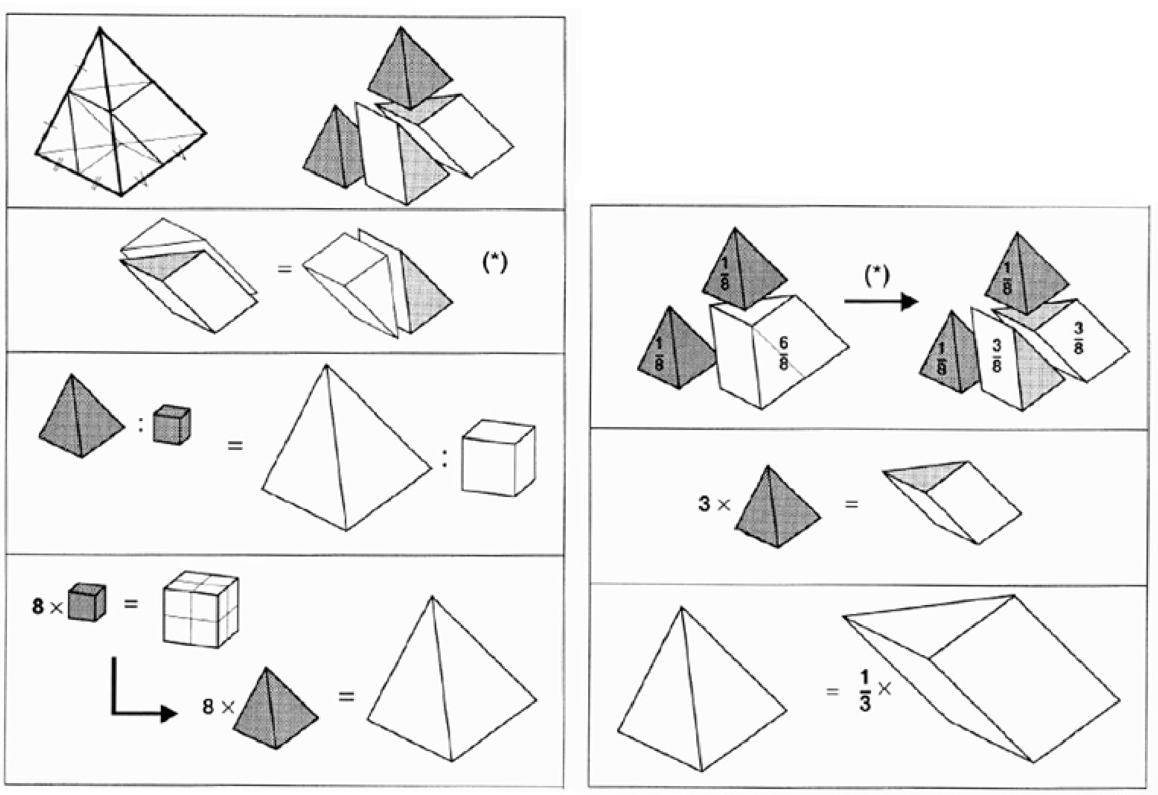

Figure 11

We invite the readers to analyse this comic strip for themselves. In fact, the proof does not use *only* cutting and pasting, but also the fact that by doubling all edges of a tetrahedron, the volume is multiplied by 8. Martin Kindt explains this by using the same effect for a cube (which is obvious) and the assumption that the proportion of two volumes does not change if the edges of both solids are doubled. This assumption, which seems very elementary, cannot be proven by mere cutting and pasting.

3.5 The impossibility of a proof by cutting and pasting only

In 1900, at the great International Congress of Mathematicians in Paris, David Hilbert (1862–1943) presented a list of 23 open problems that would determine mathematics research in the 20th century. Many of them have been solved; some of them are still open. Hilbert's third problem concerns the impossibility to prove the volume formula of an arbitrary tetrahedron by using only cutting and pasting. Hilbert had the intuition that this would be impossible, but a rigorous proof had not been found. In fact, the formulation of Hilbert was slightly different ("Prove that there exist two tetrahedra with same base and height which cannot be transformed into each other by cutting and pasting"), but this can be shown to be equivalent. In the plane, two polygons with the same area are always cut-and-paste-equivalent, as had been proven by János Bolyai. The question of Hilbert concerns the extension of this property to three-dimensional space. Some months after the congress, Max Dehn (1878–1952), a student of Hilbert, solved Hilbert's third problem. He proved that polyhedra that are cut-and-paste-equivalent have the same 'Dehn-invariant', and he exhibited two tetrahedra with equal base and height but with different Dehn-invariant. This implies that it is impossible to prove the volume formula for an arbitrary pyramid using only 'pieces'.

History includes many attempts, by mathematicians of different times and cultures, to avoid the use of infinitely thin slices and to make proofs by cutting and pasting with a finite number of pieces. Now, all these attempts come out to be doomed to failure. Meanwhile, with the integral calculus since the end of the 17th century, the problems with the slices have been solved. But the historical attempts, even if they do not prove the general case by cutting and pasting only, can still inspire teachers to answer the questions of younger pupils about the volume of a pyramid.

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