Geometric constructions and elements of Galois' theory
List 2

Number fields and quadratic extensions.

1. (a) Let $p = (2/3) + \sqrt{3/5}$ and $q = 1 - (1/2)\sqrt{3/5}$. Express the numbers $p+q$, $p-q$, pq and p/q in the form $a + b\sqrt{3/5}$, where $a, b \in Q$.
   (b) Prove that $\sqrt{3/5}$ is not a rational number, and that the set $\mathbf{Q}(3/5)$ consisting of all numbers of form $a + b\sqrt{3/5}$, where a and b are rational, is a field.
   (c) Show that the expression of any number x from the field $\mathbf{Q}(3/5)$ in the form $a + b\sqrt{3/5}$ for some rational a and b is unique.
2. Show that the square roots of the nuumbers 5, $\sqrt{3}$ and $2 + \sqrt{3}$ do not belong to the field $Q(\sqrt{3})$.
3. Denote by F the field $Q(\sqrt{3})$, and by k the number $2 + \sqrt{3}$.
   (a) Let $p = 1 - \sqrt{3} - (1 + 2\sqrt{3})\sqrt{k}$ and $q = 2 + \sqrt{3} + (1 + \sqrt{3})\sqrt{k}$, so that we have $p, q \in F(\sqrt{k})$. Express their sum, difference, product and quotient in the form i $a + b\sqrt{k}$, where a and b belong to F.
   (b) Does the number $\sqrt{3} + \sqrt{6 + 3\sqrt{3}}$ belong to the field $F(\sqrt{k})$?
4. Let $F_1 = \mathbf{Q}(\sqrt{3})$ and $F_2 = F_1(\sqrt{2 + \sqrt{3}})$ be quadratic extensions. Find the general form for numbers from the field F_2. Do the same for $F_1 = \mathbf{Q}(\sqrt{3})$ and $F_2 = F_1(\sqrt{\sqrt{3}})$.
5. Let F be a field consisting of all numbers $p + q\sqrt[4]{2}$, where p and q both have forms $a + b\sqrt{2}$ for some rational a and b. Express in the same manner the number
$$\frac{1}{\sqrt{2} + \sqrt[4]{2} + \sqrt{2}\sqrt[4]{2}}$$
which belongs to this field.
6. Finf sequences of quadratic extensions for the following constructible numbers (so that the sequence starts with the field of rational numbers and terminates with a field which contains the given number):
$$\frac{1}{\sqrt{2/3} + 1}, \frac{\sqrt{2} + \sqrt{3}}{1 + \sqrt{6}}, \sqrt{2} - \sqrt{2 + \sqrt{2}}, 2 - \sqrt{3} + \sqrt[4]{3}.$$
7. Prove that the field $\mathbf{K}$ of all constructible numbers does not have any quadratic extension.
8. Prove that $\sqrt[3]{2}$ is an irrational number. What about roots of higher degree?
9. Give a proof of the following theorem of an ancient Greek mathematitian Theaetetus (pol. Teajtet) from 4th century BC: *For a natural number n, its square root $\sqrt{n}$ is rational if and only if n is a square of a natural number.* Is something similar true for roots of higher degree?
10. Check if one can always construct a segment whose length is a. an arithmetic, b. a geometric, c. a harmonic, d. a quadratic mean of the lenghts a, b, c of three given segments.
11. We are given a segment a and a twice longer segment d. Is it possible to construct geometrically segments b and c such that the lengths a, b, c, d form a geometric progression?