Geometric constructions and elements of Galois' theory
List 9
Groups of permutations and solvable groups

Groups of permutations

1. Prove carefully that the map $h : S_n \to C_2$ given by

$$h(\sigma) = \begin{cases} id & \text{gdy } \sigma \text{ jest permutacj/a parzyst/a} \\ (12) & \text{gdy } \sigma \text{ jest permutacj/a nieparzyst/a} \end{cases}$$

is a homomorphism of groups.

2. Justify that the following groups (a) S_3, (b) S_n for $n > 3$, (c) A_4, (d) A_n for $n > 4$, are not abelian.
3. Prove that a cycle of length k in a permutation group S_n is an even permutatioin if and only if k is odd.
4. Describe the group A_4 as a group of symmetries of the regular 3-simplex (i.e. describe the symmetries of the 3-simplex corresponding to all permutations from A_4). Do the same for the group A_3 and the symmetries of a regular triangle.
5. Describe the group of all symmetries of a rhombus, viewed as a subgroup in the group S_4 of all permutations of its vertex set.
6. Viewing the symmetry group of a regular hexagon as a subgroup in the permutation group S_6 (via permutations of the vertices), decompose into combinations of cycles all the symmetries from this group.
7. Verify that the cyclic subgroup $C_n < S_n$, consisting of the powers

$$\sigma^k : k = 0, 1, \ldots, n-1$$

of the cycle $\sigma = (12 \ldots n)$, acts transitively on the set $1, \ldots, n$.

8. Find a nontrivial subgroup $H < S_5$ such that its order $|H|$ is not divisible by 5. Check that H does not act transitively on the set $1, 2, 3, 4, , 5$. Find as many of such subgroups as possible, of pairwise distinct orders not divisible by 5.
9. Show that for any $n \geq 3$ the subgroup $A_n < S_n$ consisting of all even permutations of the set $1, 2, \ldots, n$ acts transitively on this set. Verify directly (not by referring to a lemma discussed during a lecture) that the order of A_n is divisible by n.

Solvability of groups

10. Show that the group of all symmetries of a regular triangular prism (which has a regular triangle as its base) is nonabelian, and yet solvable. HINT: consider a natural map from this group to the group of permutations of the set of two triangular faces of this prism; check that this map is a homomorphism; find the kernel of this homomorphism; use this homomorphism to show that the group is solvable.
11. Consider the group Q, called the *quaternion group.* consisting of 8 elements

$$1, -1, i, -i, j, -j, k, -k.$$

The operation of product is given by the following rules, together with assosiativity:

- 1 is the neutral element of this group,
- $(-1) \cdot x = x \cdot (-1) = -x$ for any $x \in Q$, where we use the convention that $-(-a) = a$ for $a = 1, i, j, k$,
- $i^2 = j^2 = k^2 = -1$ and $ij = k$, $jk = i$, $ki = j$.

(1) Calculate $(-1)^2$, ji, $i \cdot (-j)$, $j \cdot (-i)$, $i \cdot (-i)$, $(-j) \cdot (-j)$ and $(-j) \cdot (-i)$.
(2) Verify that the group Q is not abelian.
(3) Check that the map $h : Q \to C_2 = (\{-1, 1\}, \cdot)$ given by $h(1) = h(-1) = h(i) = h(-i) = 1$ oraz $h(j) = h(-j) = h(k) = h(-k) = -1$ is a group homomorphism.
(4) Verify that the kernel of the homomorphism h from (3) is an abelian group, and conclude that Q is solvable.
(5) [special exercise] Show that the group Q is not isomorphic to the group of all symmetries of the square, even though both these groups have the same order 8, and are both non-abelian and solvable. HINT: compare the numbers of elements of order 2 in these groups.

12. Prove in the following steps that the group K of all 48 symmetries of the cube is solvable.
    (a) Consider the map $h_1 : K \to C_2 = (\{-1, 1\}, \cdot)$, which associates the element $1 \in C_2$ to all those symmetries of the cube which preserve the orientation of the space (i.e. to all rotations and to the trivial symmetry from K), and which associates the element $-1 \in C_2$ to all other symmetries of the cube (i.e. to mirror symmetries with respect to various planes, to the central symmetry with respect to the center of the cube, and to all remaining symmetries). Check that this map is a homomorphism.
    (b) Consider the colouring of the vertices in the cube into white and black so that any two vertices contained in an edge of this cube have distinct colours. Show that each symmetry of the cube either preserves all colours of the vertices, or reverses them all. Consider then the map $h_2 : \ker(h_1) \to C_2$ which associates the number 1 to those symmetries which preserve the colours, and which associates -1 to those symmetries which reverse colours. Justify that h_2 is a homomorphism. Make a list of all 12 symmetries which belong to the kernel $\ker(h_2)$ of this homomorphism.
    (c) Consider the homomorphism $h_3 : \ker(h_2) \to S_3$, which to any symmetry of the cube associates the permutation of the set of 3 directions parallel to the edges of the cube, according to how this symmetry changes the directions of the edges. Check that the image of this homomorphism in fact coincides with the subgroup A_3 in S_3, so that it is a homomorphism into an abelian group A_3. List all 4 symmetries belonging to the kernel $\ker(h_3)$, and check that this kernel is an abelian group.
    (d) Summarize the steps (a), (b) and (c) into a final argument cioncluding that K is solvable.