Exercises - Combinatorial Group Theory
List 3. Presentations of groups.

Presentations

1. Prove that the "multiplicative table" of a finite group G yields a finite presentation of this group. More precisely, take as a set of generators the set of all elements of G, and as a set of relations all eqations of form $gh = k$, over all triples g, h, k of elements of G for which the equality $gh = k$ actually holds in G.
2. (a) Prove that $\langle a, b \mid [a, b] \rangle$ is a presentation of the group Z^2.
   (b) More generally, let $G_1 = \langle S_1 \mid R_1 \rangle$ and $G_2 = \langle S_2 \mid R_2 \rangle$. Prove that the product $G_1 \times G_2$ has a presentation $\langle S_1 \sqcup S_2 \mid R_1 \cup R_2 \cup \{[s_1, s_2] : s_1 \in S_1, s_2 \in S_2\} \rangle$
   (c) Find a similar presentation as in part (b) for the semidirect product of the groups G_1 i G_2 induced by a homomorphism $\alpha : G_1 \to \mathrm{Aut}(G_2)$.
3. Show that the presentation $\langle S \mid \{w^2 : w \text{ is any word over } S \cup S^{-1}\} \rangle$ describes the direct sum of $|S|$ copies of the group Z_2, i.e. the group of all functions $S \to Z_2$ with finite support and with the pointwise multiplication.
4. Show that the presentation $\langle a, b \mid a^{-1}ba = b^2, b^{-1}ab = a^2 \rangle$ describes the trivial group.

Applications of the universality of a group given by a presentation

5. Consider the group $G = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, [a, b], [b, c], [c, d], [d, a] \rangle$. Prove that the element ac has infinite order in G, and the order of ab is 2.
6. Prove that the group $\langle a, b \mid a^2b^2, (ab)^3 \rangle$ is nonabelian. Hint: find a homomorphism of this group onto the group of all symmetries of the equilateral triangle (equivalently, onto the symmetriic group $Sym(3)$).
7. Consider the group $G = \langle a, t \mid t^{-1}at = a^2 \rangle$, called the *Baumslag-Solitar group*.
   (a) Show that in this group the elemnts a, t have infinite order and do not commute.
   (b) Show thta the function given by $\psi(t) = t, \psi(a) = a^{-2}$ extends to a homomorphism $\psi : G \to G$.
   (c) Show thta the homomorphism ψ from part (b) is an automorphism of G.
   Hint: guess the form of a potential inverse homomorphism ψ^{-1}, by deriving the possible value of $\psi^{-1}(a)$.
8. Determine the ranks of the groups G given by the presentations below, calculating the number of homomorphisms $G \to Z_2$.
   (a) $G = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdot [a_2, b_2] \cdot \ldots \cdot [a_g, b_g] \rangle, \quad g \geq 1.$
   (b) $\langle a_1, \ldots, a_k \mid a_1^2 \cdot a_2^2 \cdot \ldots \cdot a_k^2 \rangle, \quad k \geq 1.$
   (These groups are in fact the fundamantal groups of (a) oriented and (b) non-orientable closed surfaces; they are briefly called the *surface groups*.)
9. Let G be a group having finite presentation with more generators than relations. Show that G has then a quotient isomorphic to the infinite cyclic group. Find an example of an infinite finitely generated group which does not have such a quotient.
10. Prove that the group $G = \langle b, t \mid t^{-1}b^2t = b^3 \rangle$ is non-hopfian (ti.e. it admits a surjective homomorphism $\theta : G \to G$ with nontrivial kernel). To do it, show that:
    (a) the assignments $\theta(t) = t$ and $\theta(b) = b^2$ induce a homomorphism $G \to G$;
    (b) this homomorphism is surjective;
    (c) commutator $[t^{-1}bt, b]$ is a nontrivial element of G,and its image under θ is trivial.

Tietze transformations

11. Apply carefully Tietze transformations in order to modify the presentation $\langle a, b, c \mid c^{-1}ac = b, c^{-1}bc = a, c^2 = 1\rangle$ into the presentation $\langle a, c \mid c^2 = 1\rangle$.
12. Show that the group $G = \langle a, b \mid ababa = 1\rangle$ is the infinite cyclic group.
13. Let $G = \langle s_1, \ldots, s_m \mid r_1, \ldots, r_n\rangle$ be a finitely presented group.
    (a) Let $\langle s_1, \ldots, s_m \mid \rho_1, \ldots\rangle$ be some another presentation of G, with the same generating set, and with infinite set of relations. Prove that then, for N sufficiently large, $\langle s_1, \ldots, s_m \mid \rho_1, \ldots, \rho_N\rangle$ is a finite presentation of G.
    (b) Let $\langle t_1, \ldots, t_k \mid p_1, \ldots\rangle$ be some another p[resentation of G, with infinitely many relations. Show thtat there is N such that $\langle t_1, \ldots, t_k \mid p_1, \ldots, p_N\rangle$ is a finite presentation of this group. Hint: use Tietze transformations and part (a).

Quotients of groups given by a presentation

14. Determine the groups obtained as abelianizations of dihedral groups, of the Baumslag-Solitar group from exercise 7, and of surface groups from exercise 8.