Exercises - Combinatorial Group Theory
List 4. Free products.

Basic simple properties of free products

1. Show that the free product of a family of at least two nontrivial groups contains an element of infinite order..
2. Show that the groups G_α naturally embed as subgroups in the free product $*_\alpha G_\alpha$, and that the intersection of any two such subgroups is trivial. Present two separate solutions of this exercise: (1) refering only to the universal property appearing in the definition of free products (or using the canonical presentation for free product); (2) using the description of free product as consisting of appropriate reduced words (normal form of elements).
3. Prove that the rank k freee group is (isomorphic to) the free product of k copies of the infinite cyclic group.
4. Show that the free products of groups is a free group if and only if the factor groups are all free.
5. Suppose we are given a collections of subgroups $H_i < G_{\alpha_i} < *_\alpha G_\alpha$ such that $\alpha_i \neq \alpha_j$ for $i \neq j$. Prove that the subgroup $H < *_\alpha G_\alpha$ generated by $\cup_i H_i$ is canopnically isomorphic with the free product $*_i H_i$.
6. Find, state and verify a condition for two subgroups H, K of a group G under which G is canonically isomorphic with the free product $H * K$.
7. Let Γ be the group of isometries of the line generated by some two reflections (with respect to distinct points of this line). Show that Γ is isomorphic to the free product of two copies of the cyclic group Z_2.
8. Show that the free product of more than two nontrivial groups is isomorphic tp the free product of precisely two nontrivial groups.

Some further and more advanced properties of free groups

9. Show that a nontrivial free product has trivial center.
10. Investigate the conjugacy problem in a free product. Use appropriately adapted concept of cyclically reduced expressions.
11. Use the conjugacy criterion derived in the previos exercise to show that each element of finite order in a free product is conjugate with some finite order element in one of the factors (actually, in precisely one of the factors).
12. Show that the groups $Z_2 * Z_2$, $Z_2 * Z_2 * Z_2$ oraz $Z * Z_2$ are pairwiese non-isomorphic.
13. Use the two hints (a) and (b) below to show that the free product of two nontrivial groups is never isomorphic to the direct product of two nontrivial groups.
    (a) Given $g \in G \setminus \{1\}$ and $h \in H \setminus \{1\}$, show that the centralizer of the product $gh \in G * H$ (i.e. the subgroup consisting of all elements in $G * H$ which commute with gh) is siomorphic to the infinite cyclic group.
    (b) Verify that the centralizer of any element in the direct product $G \oplus H$ of any two nontrivial groups is also (isomorphic to) the direct product of two nontrivial groups.
14. Prove that the free product of two nontrivial groups at least one of which is distinct from Z_2 contains the rank 2 gree group as subgroup.

Some application of free products

15. Let U be a finitely presented group with undecidable word problem. Using U as an ingredient, and using the operation of free product, construct a finitely presented group, for which the following problem is undecidable:
    (a) determination whether an element (expressed in terms of generators) belongs to the center of G;
    (b) detremination whether an element $g \in G$ commutes with some fixed element $g_0 \in G$;
    (c) detremination whether an element $g \in G$ is the n-th power of some element of G, for fixed $n > 1$;
    (d) determination whether the conjugacy class of an element $g \in G$ is finite;
    (e) detremination whether an element $g \in G$ has finite order > 1;
    (f) detremination whether an element $g \in G$ is a commutator of some elements of G.