Exercises - Combinatorial Group Theory
List 5. Free products with amalgamation.

Amalgamated free products, their normal forms, universal properties, etc.

1. Let $G = \langle c, d \,|\, c^2 = d^3 \rangle$ be the amalgamated free product of the groups $H = \langle c \,|\, \emptyset \rangle$ and $K = \langle d \,|\, \emptyset \rangle$ with respect to the subgroups $A = \langle c^2 \rangle$, $B = \langle d^3 \rangle$ and the isomorphism $\varphi : A \to B$, $\varphi(c^2) = d^3$. Consider the sets of representative of cosets (1) $Y = \{1, c\}$ for cosets of A in H, and (2) $Z = \{1, d, d^{-1}\}$ for cosets of B in K. Determine the normal forms of elements $c^3 d^{-2} c^{-4} d^4$ and $c^{-3} d c^{-4} d^2 c d^2$ with respect to those representatices.
2. Show that the group $B = \langle x, y \,|\, xyx = yxy \rangle$ con be expressed, for appropriate choice of subgroups, as an amalgamated free product $Z *_Z Z$.
3. Find the center of the group $G = \langle c, d \,|\, c^2 = d^3 \rangle$. How does generally the center of the amalgamated free product look like?
4. Introduce the appropriate variant of the concept of a *cyclically reduced alternating word* (słowo naprzemienne cyklicznie zredukowane) for amalgamated free products, and show that every element in such a product is conjugate with an element of such a form (i.e. represented by such a word).
5. Verify that in any amalgamated free product every element of finite oprder is conjugate with some element from one of the factor groups. Deduce that any malgamated free product of torsion free groups is also torsion free.
6. The dihedral group $D_4 = \langle a, b \,|\, a^2, .b^2, (ab)^4 \rangle$ contains, up to conjugation, two distinct subgroups of order 2. This means that potentially we have three distinct amalgamated free products $D_4 *_{Z_2} D_4$. Prove that these three product groups are indeed pairwise non-isomorphic.
7. Check that any 3 from the generators of the surface group $\langle a, b, c, d \,|\, [a, b][c, d] \rangle$ freely generate a free subgroup of this group.
8. Niech $\varphi : G \to H *_A K$ będzie epimorfizmem (surjekcją). Uzasadnij, że wtedy
$$G = \varphi^{-1}(H) *_{\varphi^{-1}(A)} \varphi^{-1}(K).$$
9. Given $\Gamma = G *_K H$, suppose that K is a normal subgroup in both G and H.
   (a) Show that K is then a normal subgroup in Γ.
   (b) Show that the quotient group Γ/K is canonically isomorphic to the free product $(G/K) * (H/K)$.
10. Given $G = H *_{\varphi: A \to B} K$, let A_0 be a proper subgroup of A and let $B_0 = \varphi(A_0)$. Denote by $\varphi_0 : A_0 \to B_0$ the restriction of the isomorphism φ, and consider the group $G_0 = H *_{\varphi_0 : A_0 \to B_0} K$.
    (1) Show that the map $H \cup_{\varphi_0} K \to H \cup_\varphi K$ induced by the identities of H and K extends to a homomorphism $h : G_0 \to G$.
    (2) Verify that h is not an isomorphism.
11. Uzasadnij, że produkt wolny z amalgamacją $H *_M K$ skończenie prezentowalnych grup H, K jest skończenie prezentowalny wtedy i tylko wtedy gdy podgrupa amalgamowania M jest skończenie generowalna. Wskazówka: poprzednie zadanie może być pomocne.

Ping-Pong Lemma

12. Show thta the subgroup of $SL(2, Z)$ generated by the elements $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is (isomorphic to) the free product $Z * Z_2$.
13. Identify a subgroup in the group $SL(2, Z)$ isomorphic to the amalgamated free product $Z_4 *_{Z_2} Z_4$.
14. Consider the subgroups in the group $SL(3, Z)$ given by

$$G_1 = \{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \text{ are even}\}, \quad G_2 = \{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & z & 1 \end{pmatrix} : x, y, z \text{ are even}\}.$$

   (1) Check that $G_1 \cap G_2 \cong Z^2$.
   (2) Prove that the subset $G_1 \cup G_2$ generates in $SL(3, Z)$ a subgroup canonically isomorphic to the malgamated free product $G_1 *_{Z^2} G_2$. Hint: bot subgroups G_1 and G_2 preserve the subspace $V_1 < R^3$ spanned on the first versor; consider the induced actions of those subgroups (and of the subgroup generated by their union) on the quotient space $R^3/V_1 \cong R^2$.