Algebraic Topology 2. Exercises.
List 1 continued.

1. Check that for continuous maps of pairs $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$ we have $g_* f_* = (gf)_*$ for induced homomorphisms of relative homologies.
2. Show that for $B \subset A \subset X$ and for any fixed n the sequence of homomorphisms

$$0 \to C_n(A, B) \to C_n(X, B) \to C_n(X, A) \to 0$$

induced by inclusions is exact.
3. Justify that the quotient homomorhisms $j : C_n X \to C_n(X, A) = C_n X/C_n A$ form a chain map between the corresponding singular chain complexes (verify that they commute with boundary homomorphisms).
4. Let X be a topological space and let $\mathcal{U}$ be any family of its subsets.
   (0) Show that the subset $C_n^{\mathcal{U}} X \subset C_n X$ given by
   $$C_n^{\mathcal{U}} X := \{\sum n_i \sigma_i | \forall i \, \exists U \in \mathcal{U} : \mathrm{im}(\sigma_i) \subset U\}$$
   is a subgroup in $C_n X$.
   (1) Show that the boundary homomorphism restricted to $C_n^{\mathcal{U}} X$ has its image in $C_{n-1}^{\mathcal{U}} X$.
   (2) Show that for the inclusion homomorphisms $\iota : C_n^{\mathcal{U}} X \to C_n X$ form a chain map of chain complexes.
   (3) Show that if $\mathcal{U}$ is a covering of X by not necessarily open sets then the induced homomorphisms of homology groups $\iota_* : H_n^{\mathcal{U}} X \to H_n X$ need not be isomorphisms.
5. Let X be a contractible space. Motivated by the cone operator $b : LC_n E \to LC_{n+1} E$ from the proof of excision theorem, for each $n \geq 1$ describe some homomorphism $\Sigma : C_n X \to C_{n+1} X$ which is a chain homotopy between the identity and the zero homomorphisms $C_n X \to C_n X$. Deduce that $H_n X = 0$ for $n \geq 1$. Why this argument does not work for $n = 0$, and what happens instead?
6. Check in detail that the following subsequence of the long exact sequence for pairs is indeed exact
$$H_n A \to H_n X \to H_n(X, A).$$
7. Motivated by the prism operator, for any two topological spaces X, Y describe some nontrivial homomorphism $p : C_1 X \times C_1 Y \to C_2(X \times Y)$ and show that it maps pairs of cycles into cycles. Check also that if one of the cycles in the argument is a boundary then the image is a boundary too.