DIFFERENTIAL TOPOLOGY - EXERCISES
LIST 5. Transversality - continued

Preliminary exercises.

1. Let Q be any countable subset in R^2. Show that any smooth curve in R^2 can be approximated (arbitrarily close in C^∞ topology) by smooth curves whose images are disjoint with Q. Generalize this observation to the case of smooth maps between other manifolds.
2. Consider the set of smooth maps $C^\infty(X, Y)$, and its subset E consisting of those maps which have at most double self-intersections. Find some sufficient condition on the dimensions of the manifolds X and Y under which Y is residual. Can you justify that this condition is also necessary?
3. Show that the subset of $C^\infty(S^1, R^3)$ consisting of those curves which have non-zero curvature at every point $t \in S^1$ is open and dense. (Compare Exercise 10 from List 4.)
4. A *point of flattening* of a smooth curve γ in R^3 is any t for which $\det(\gamma'(t), \gamma''(t), \gamma'''(t)) = 0$. Check that the set of those smooth curves in R^3 which are regular and have non-zero curvature at each point of flattening is residual.

Exercises.

5. (stability of a transversal intersection) Let $W \subset J^k(X, Y)$ be a submanifold, and let $f : X \to Y$ be a smooth map such that $j^k f(x_0) \in W$ and $j^k f$ is transversal to W at x_0. Show that for any smooth deformation f_t of f (such that $f_0 = f$) a point of transversal intersection of the map $j^k f_t$ with the submanifold W appears arbitrarily close to x_0 for all t sufficiently close to 0.
6. (A) Show that if $2 \cdot \dim X \leq \dim Y$ then the set $\mathrm{Imm}(X, Y)$ of smooth immersions is open and dense in $C^\infty(X, Y)$.
   (B) Show that if $2 \cdot \dim X + 1 \leq \dim Y$ then the set of injective immersions $X \to Y$ is a residual subset of $C^\infty(X, Y)$.
   (C) Show that if $2 \cdot \dim X + 1 \leq \dim Y$ then the set of all smooth proper embeddings $X \to Y$ is a residual subset of the set of all proper maps from X to Y. Hint: show that a proper injective immersion is an embedding.
7. Show that the result from Exercise 6.(A) can be strengthened as follows: if $2 \cdot \dim X \leq \dim Y$ then the set of smooth immersions with at most double self-intersections is a residual subset of $C^\infty(X, Y)$.
8. (A) Show that *generically* (i.e. for a residual subset in $C^\infty(M, N)$) a smooth map $f : M^m \to N^{2m-1}$ has rank $\geq m - 1$ at each point, and the set $\{x \in M : \mathrm{rank}(f, x) = m - 1\}$ is a closed 0-dimensional submanifold of M.
   (B) Give an example of a smooth map $f : R^2 \to R^3$ which has such a point of rank 1 which cannot be removed by an arbitrarily small smooth deformation of f.
9. (A) Show that any smooth manifold X admits a Morse function whose values at critical points are pairwise distinct.
   (B) Prove the following stronger property: any smooth manifold X admits a Morse function whose values at critical points are pairwise distinct, the set of critical values is discrete and has a minimal element (which corresponds to a global minimum of this function).

10. A *knot* is a smooth embedding $S^1 \to R^3$. A *class of a knot* is an equivalence class in the set of smooth embedded curves $S^1 \to R^3$ with respect to smooth deformations through such curves.
    (A) Show that each class of a knot is an open subset in $C^\infty(S^1, R^3)$. HINT: Use as an ingredient the fact that for a knot γ, any knot $\hat{\gamma}$ sufficiently close to γ (in C^∞ Whitney topology) can be obtained from γ by a smooth deformation (this fact is not very hard, so you can try to prove it as well).
    (B) Show that in each class of a knot a generic (i.e. residual) subset consists of knots whose orthogonal projections on a fixed plane $R^2 \subset R^3$ is a regular curve with at most double and transversal self-intersections. (Such a projection is called a *knot diagram.*)

Plane curves and curvature

Recall that for a regular curve γ in the plane R^2:

- its curvature at a point t is given by the formula $k(t) = \det(\gamma'(t), \gamma''(t))/||\gamma'(t)||^3$ (this is an oriented version of curvature, as it can be also a negative number);
- a *proper inflection point* of a regular curve is any point t at which $k(t) = 0$ and $k'(t) \neq 0$;
- a *proper vertex* of a regular curve is any point t at which $k'(t) = 0$ and $k''(t) \neq 0$ (i.e. a point at which the curvature function $k(t)$ has a proper local extremum).

11. Prove that for regular curves on the plane the following properties are generic:
    (A) all points with vanishing curvature are proper inflection points; HINT: show first and use the fact that a point t is a proper inflection point of a curve γ if and only if $\det(\gamma'(t), \gamma''(t)) = 0$ and $d/ds|_{s=t} \det(\gamma'(s), \gamma''(s)) \neq 0$;
    (B) each point with vanishing derivative of the curvature is a proper vertex with non-zero curvature.