SIMPLICIAL NONPOSITIVE CURVATURE (SNPC)

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notes downloadable from
`http://www.math.uni.wroc.pl/~swiatkow/montreal.html`

What is SNPC?

It is a *purely combinatorial* condition for simplicial complexes that

- resembles metric nonpositive curvature (NPC)
- does not reduce to NPC, nor to small cancellation
- has many similar consequences as classical NPC
- provides examples different from classical ones,
  with various new and exotic properties

Terminology

- *systolic complex* = SNPC + simply connected
- *systolic group* = acting geometrically on a systolic simplicial complex

References

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`http://www.math.uni.wroc.pl/~swiatkow/montreal.html`

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BASIC DEFINITIONS

cycles

X - simplicial complex

- *cycle* γ in X - a subcomplex $\cong S^1$
  $|\gamma|$ - *length* of γ = number of edges in γ
- *diagonal* in γ - an edge of X connecting **nonconsecutive** vertices of γ

k-largeness

given a natural number $k \geq 4$

X is *k-large* if $\begin{cases} X \text{ is } \textbf{flag} \text{ and} \\ \text{every cycle in } X \text{ of length} < k \text{ has a diagonal} \end{cases}$

Recall that:

X is *flag* iff
any finite set of vertices in X that are pairwise connected by edges
spans a simplex of X

- 4-large ⇔ flag
- 5-large ⇔ "no empty square" (known as Siebenmann's condition)
- 6-large ⇔ "no empty square and pentagon"

Remarks

- if $4 \leq m < k$ then k-large $\Longrightarrow$ m-large
- k-largeness will be applied to **links** of a simplicial complex
- it will serve as a local **curvature-like** bound

simplicial curvature

- *link* of X at its simplex σ

  $X_\sigma := \{\tau \subset X \,|\, \tau \cap \sigma = \emptyset,\ \tau * \sigma \text{ is a simplex of } X\}$

  (link X_σ describes how X looks like locally around σ)

- X is *locally k-large* iff links of X at all simplices are k-large

  (local 6-largeness =: SNPC [simplicial nonpositive curvature])

- *k-systolic* := locally k-large, connected and simply connected

  (for $k = 6$ – simplicial analogue of $CAT(0)$ or Hadamard space)

- *k-systolic group* – acts properly discontinuously and cocomactly, by simplicial automorphisms, on a k-systolic simplicial complex

- we often abbreviate 6-systolic to *systolic*

k-largeness is easy to check for $k \geq 6$

- *homotopical systole* $\text{sys}_h(X)$ –
  – length of shortest homotopically nontrivial cycle in X

- if $k \geq 6$ then
  X is k-large iff links of X are k-large & $\text{sys}_h(X) \geq k$

  (proof will be sketched later)

- the following key feature of the above:

  [local] + [global related to topology] $\Longrightarrow$ [global]

  allows induction w.r.t. dimension in checking k-largeness and in constructing k-large complexes

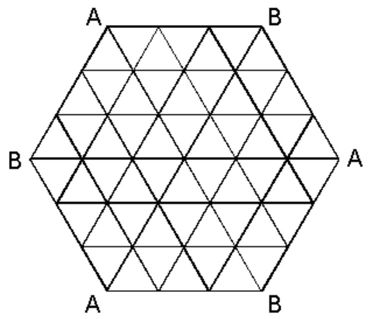
Examples and nonexamples of k-large and k-systolic complexes

- if $\dim X = 1$, $\quad X$ is k-large $\iff$ $\mathrm{sys}_h(X) \geq k$
  ($\iff$ X contains no cycles of length $< k$)

- a tree is k-large for arbitrary k $\qquad$ (∞-*large*)

- 6-large torus

  - links are 6-large
  - $\mathrm{sys}_h = 6$



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- regular triangulations of E^2 or H^2 (by equilateral triangles with angles π/k) are k-systolic

- ideal triangulation of H^2 is k-systolic for arbitrary k $\qquad$ (∞-*systolic*)

- regular ideal triangulation of H^3 is 6-systolic
  - links of edges are 6-cycles
  - links of vertices are regular triangulations of E^2

- tree $\times$ line $\quad$ has a 6-systolic triangulation

- tree $\times$ tree $\quad$ has not $\quad$ [D. Wise]

- no triangulation of 2-sphere S^2 is 6-large
  [by combinatorial Gauss-Bonnet]

  and hence no triangulation of a manifold with dim ≥ 3 is SNPC
  [because it has 2-spherical links]

- $\forall k \geq 6$ $\forall n$ there are n-dimensional k-systolic pseudomanifolds
  [construction uses simplices of groups]

DIAGRAMMATICS

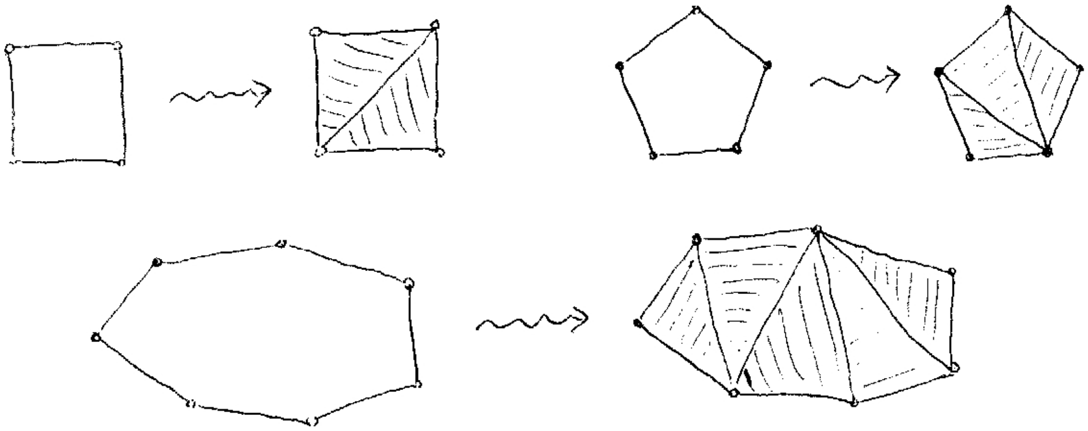
filling short cycles

Lemma. Let

- X be a k-large simplicial complex and
- let γ be a cycle in X of length m.

If $m < k$ then there is a simplicial map $D \to X$ such that

- D is a simplicial 2-disc,
- D has no interior vertices, and
- $f|_{\partial D}$ maps ∂D isomorphically onto γ.



filling diagram of arbitrary cycle γ in X

is a **nondegenerate** simplicial map $f : \Delta \to X$ such that

- Δ is a simplicial 2-disc,
- $f|_{\partial\Delta} : \partial\Delta \to X$ is an isomorphism onto γ.

Note that
every homotopically trivial cycle in any simplicial complex
has a filling diagram.

minimal filling diagrams in locally k-large complexes

Proposition.

Every homotopically trivial cycle in a locally k-large simplicial complex X has a filling diagram $f : \Delta \to X$ which is locally k-large (i.e. every interior vertex of Δ is contained in $\geq k$ triangles).

In fact, any **minimal area** filling diagram has this property.

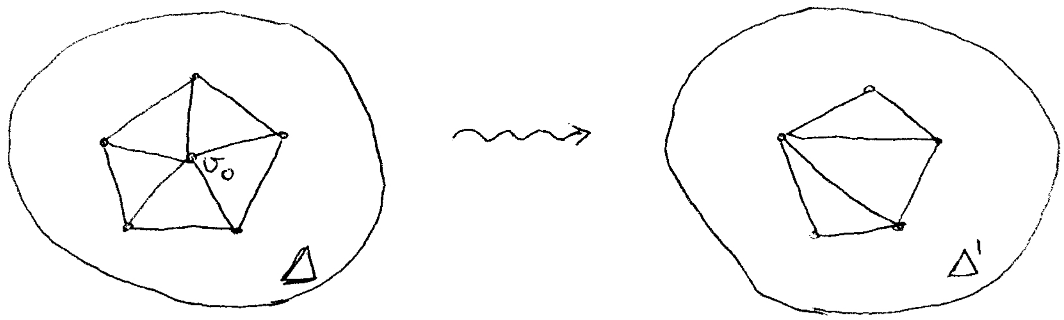
Sketch of proof:

If a minimal area $f : \Delta \to X$ is **not** locally k-large.
there is an interior vertex v_0 in Δ contained in m triangles, with $m < k$.

Then $f(\Delta_{v_0})$ is a polygonal loop of length m in the link $X_{f(v_0)}$.

If $f(\Delta_{v_0})$ is a cycle, it bounds a 2-disc D as in Filling-Short-Loops Lemma.

Replacing the subdisc in Δ bounded by Δ_{v_0} with D
we get a **less area** filling diagram, a contradiction.



If $f|\Delta_{v_0}$ is not injective,
similar arguments produce filling diagram with less area.

Comparison with $(3,k)$-small cancellation

$(3,k)$-small cancellation diagrammatics is based on the following:

> each filling diagram can be made locally k-large by **reductions of cancellable pairs** only

locally k-large diagrammatics:

> to get locally k-large filling diagrams, bigger class of reductions is allowed/necessary.

- if $\dim X = 2$ then

  X is locally k-large iff X is a $(3,k)$-complex

  (in particular: every $C(3)-T(k)$ small cancellation group is k-systolic)

- on the other hand

  2-skeleton of a locally k-large complex is in general **not** $(3,k)$

systolic is essentially more than small cancellation:

- [D. Wise] If $k \geq 6$ then any $C(k)$ small cancellation group is k-systolic.

- [T. Januszkiewicz – J. Ś] For each $k \geq 6$ there exist k-systolic groups of arbitrary cohomological dimension.

  On the other hand,

  small cancellation groups have cohomological dimension ≤ 2.

Inductive criterion for k-largeness

Proposition. For $k \geq 6$,
if X is locally k-large and $sys_h(X) \geq k$ then X is k-large.

Proof:

Let γ be a cycle in X **with no diagonal**. We need $|\gamma| \geq k$.

If γ homotopically nontrivial, this follows from $sys_h(X) \geq k$.

If γ homotopically trivial, let $f : \Delta \to X$ be a minimal area filling diagram for γ. Then

- Δ has at least one interior vertex,
- interior vertices of Δ are contained in $\geq k$ triangles,
- boundary vertices of Δ are contained in ≥ 2 triangles.

Euler characteristic argument (combinatorial Gauss-Bonnet) shows then that $|\gamma| \geq k$.

More precisely:

$$1 = \chi(\Delta) = \frac{1}{6} \cdot [\sum_{v \in \partial\Delta} [3 - \chi(v)] + \sum_{v \in int\Delta} [6 - \chi(v)]]$$

where $\chi(v)$ is the number of triangles in Δ containing v. Equivalently

$$6 = \sum_{v \in \partial\Delta} [3 - \chi(v)] + \sum_{v \in int\Delta} [6 - \chi(v)].$$

Since $3 - \chi(v) \leq 1$ for $v \in \partial\Delta$ and $6 - \chi(v) \leq 6 - k \leq 0$ for $v \in int\Delta$, we get

$$6 \leq |\gamma| + [6 - k], \qquad \text{and hence} \qquad k \leq |\gamma|.$$

7-systolic $\Longrightarrow$ Gromov hyperbolic

Proposition.

If X is a 7-systolic simplicial complex then the 1-skeleton $X^{(1)}$ is Gromov hyperbolic.

Corollary. Any 7-systolic group is word-hyperbolic.

Proof of Proposition (sketch):

We need to show that
geodesic triangles in $X^{(1)}$ are δ-thin for some universal δ.

- $\diamond$ Geodesic bigons are 1-thin (exercise);
- $\diamond$ thus we may restrict to **embedded** geodesic triangles

Let γ be the boundary of an embedded geodesic triangle T,
and let $f : \Delta \to X$ be a filling diagram for γ with minimal area.

- $\diamond$ $3 - \chi(v) \leq 2$ at vertices of T;
- $\diamond$ there are no vertices v inside sides of T with $3 - \chi(v) = 2$
- $\diamond$ any two vertices with $3 - \chi(v) = 1$ inside one side of T;
are separated by a vertex with $3 - \chi(v) \leq -1$.

Thus, total curvature $\sum[3 - \chi(v)]$ inside each side of T is ≤ 1, and hence

$$\sum_{v \in \partial\Delta} [3 - \chi(v)] \leq 2 + 2 + 2 + 1 + 1 + 1 = 9.$$

- $\diamond$ By minimality, for any $v \in int\Delta$ we have $6 - \chi(v) \leq -1$;
- $\diamond$ thus by Gauss-Bonnet, there are at most 3 interior vertices in Δ;
- $\diamond$ hence $\delta = 4$ works.

CONVEXITY

3-convexity in 6-large complexes

A subcomplex Q in a 6-large simplicial complex X is *3-convex* if

- Q is full in X, and
- for every geodesic (v_0, v_1, v_2) in X with $v_0, v_2 \in Q$ we have $v_1 \in Q$.

Equivalently, polygonal paths with no diagonals intersecting Q only at end-points have length ≥ 3. [this explains "3" in the term]

Examples. • Every X is 3-convex in itself.

- Any simplex is 3-convex (trivially)
- The *residue* (or *star*) of a simplex σ in X is the subcomplex
  $$Res(\sigma, X) := \cup\{\tau \mid \sigma \subset \tau\} = \sigma * X_\sigma.$$
  Exercise: the residue of any simplex is 3-convex.

Diameter Criterion. Let Q be a subcomplex in a 6-large X. If

- Q is connected with diam$Q \leq 3$, and
- $\forall \sigma \subset X$ either $Q_\sigma = X_\sigma$ or Q_σ is connected with diam$Q_\sigma \leq 3$

then Q is 3-convex in X. [proof uses diagrammatics]

Convexity in systolic complexes

A subcomplex Q in a systolic simplicial complex X is *convex* if

- Q is connected, and
- Q is locally 3-convex, i.e. $\forall \sigma \subset Q \quad Q_\sigma$ is 3-convex in X_σ.

Examples. • A subcomplex of the equilaterally triangulated E^2 is convex iff it is convex in the ordinary sense.

- The same is **not true** in equilaterally triangulated H^2.

Proposition. A subcomplex Q in a systolic complex X is convex iff it is geodesically convex. [proof will be sketched later]

ASPHERICITY and π_1-INJECTIVITY

Asphericity Theorem. Let X be a locally 6-large (i.e. SNPC) connected simplicial complex. Then

- X is aspherical (i.e. its universal cover $\widetilde{X}$ is contractible),
- if Q is a connected locally 3-convex subcomplex of X then $\pi_1 Q$ injects in $\pi_1 X$.

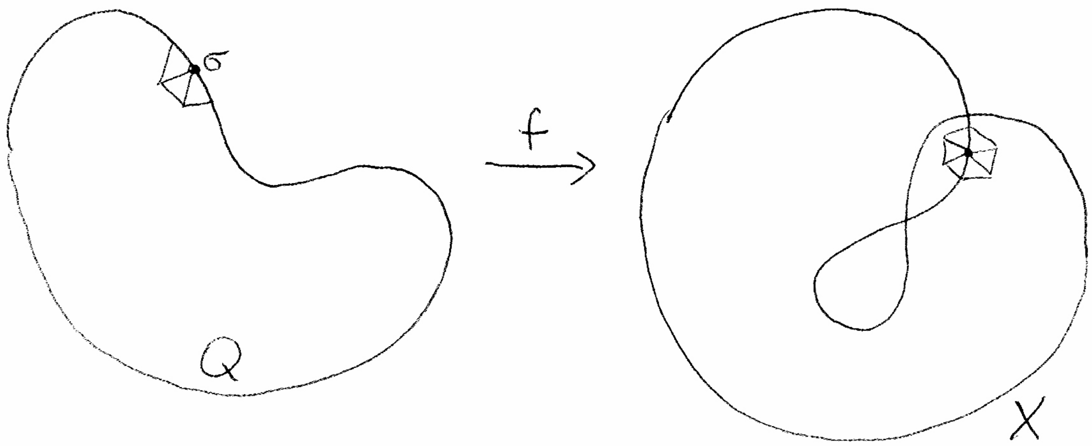
Corollary [Cartan-Hadamard for SNPC].
Any systolic simplicial complex is contractible.

Sketch of proof of Asphericity Theorem:

locally convex maps

A simplicial map $f : Q \to X$ to an SNPC simplicial complex X is *locally convex* if

- f is nondegenerate,
- f is locally injective [i.e $\forall \sigma \subset Q$ $f_\sigma : Q_\sigma \to X_{f(\sigma)}$ is injective]
- $\forall \sigma \subset Q$ the image $f_\sigma(Q_\sigma)$ is 3-convex in $X_{f(\sigma)}$.



Extension Lemma. Any locally convex map $f : Q \to X$ extends to a map $\overline{f} : \overline{Q} \to X$ so that

- $\overline{f}$ is a covering map, and
- $Q \subset \overline{Q}$ is a deformation retract.

Proof of Extension Lemma:

An *elementary extension* of a locally convex map $f : Q \to X$ is a map $Ef : EQ \to X$ such that

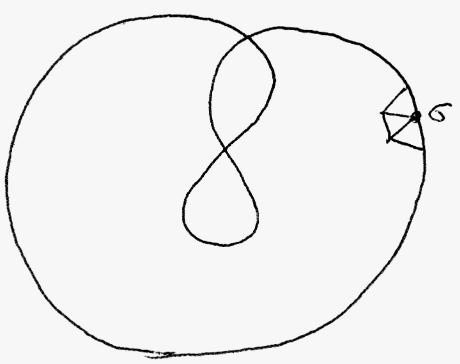
(E1) $Q \subset EQ$ is a deformation retract,

(E2) every simplex of EQ is a face of a simplex intersecting Q,

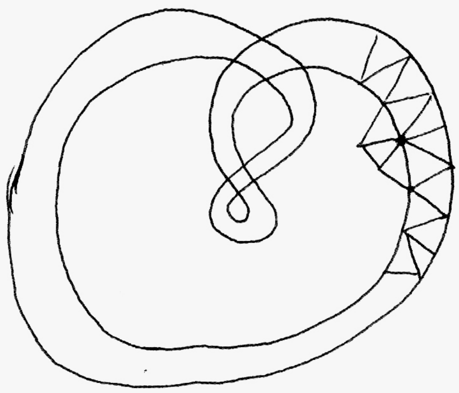
(E3) $\forall \sigma \subset Q$ the induced map $(Ef)_\sigma : (EQ)_\sigma \to X_{f(\sigma)}$ is an isomorphism.

Fact. Every locally convex map has an elementary extension which is also locally convex.

(proof is difficult)



$f : Q \to X$



$Ef : EQ \to X$

Construction of extension

Put recursively: $E_1 f = Ef$, $E_1 Q = EQ$ and

$$E_{n+1} f = E(E_n f),\ E_{n+1} Q = E(E_n Q).$$

Then put

$$\overline{Q} = \cup_{n=1}^{\infty} E_n Q, \quad \overline{f} = \cup_{n=1}^{\infty} E_n f \quad \text{where} \quad \overline{f} : \overline{Q} \to X.$$

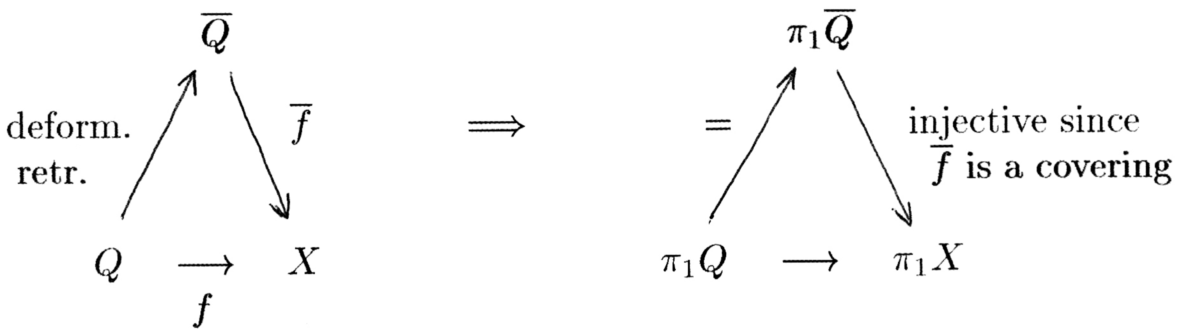
By (E3), $\overline{f}$ is a covering map.

By (E1), $Q \subset \overline{Q}$ is a deformation retract. ◊

back to the proof of Asphericity Theorem

- Inclusion map $f : |\sigma| \to X$, for any simplex Δ of X, is locally convex.
- f extends to a covering map $\overline{f} : Y \to X$ so that $|\sigma| \subset Y$ is a deformation retract, i.e. Y is contractible.
- Thus $\widetilde{X} = Y$, and hence $\widetilde{X}$ is contractible.

to prove π_1-injectivity part:



Thus $\pi_1 Q \to \pi_1 X$ is injective. ◊

DEVELOPABILITY

(remarks for those who know complexes of groups)

Definition. A complex of groups $\mathcal{G}$ is *locally 6-large*, or *SNPC*, if all local developments of $\mathcal{G}$ are 6-large simplicial complexes.

Developability Theorem.

Any locally 6-large (i.e. SNPC) complex of groups is developable.

Sketch of proof:

A sequence of elementary extensions yields the universal covering map to $\mathcal{G}$, hence developability.

[We will discuss later, in greater deteail, the case of simplices of groups.]

BALLS AND SPHERES

For a subcomplex $A \subset X$ define *balls* (or *neighbourhoods*)

$$B_1(A, X) = \cup\{\tau \subset X \mid \tau \cap A \neq \emptyset\}, \quad B_n(A, X) = B_1(B_{n-1}(A, X), X)$$

and *spheres*

$$S_n(A, X) = \cup\{\tau \subset B_n(A, X) \mid \tau \cap B_{n-1}(A, X) = \emptyset\}.$$

balls and spheres in systolic complexes

Let Q be a convex subcomplex of a systolic simplicial complex X.

Elementary extension techniqes yield the following results.

- Balls $B_n(Q, X)$ are convex in X
  (in particular, they are 3-convex and thus full).

- $B_n(Q, X)$ is the simplicial span of the vertex set
  $$\{v \in X \mid dist(v, Q) \leq n\}.$$

- $S_n(Q, X)$ is the simplicial span of the vertex set
  $$\{v \in X \mid dist(v, Q) = n\}.$$

- **Projection Lemma.** For any $\tau \subset S_1(Q, X)$ the intersection
  $$Res(\tau, X) \cap Q$$
  is a single (nonempty) simplex.

  Definition. We call this simplex *the projection of* τ *on* Q.

- **Link Lemma.** Let $\tau \subset S_1(Q, X)$ and let σ be the projection of τ on Q. Then
  $$(S_1(Q, X))_\tau = B_1(\sigma, X_\tau).$$
  [exercise; use Projection Lemma]

STRONG CONVEXITY

Link Lemma motivates the following

Definition.
A connected subcomplex Q of a systolic complex X is *strongly convex* if $\forall \tau \subset Q$

- either $Q_\tau = X_\tau$ or
- $Q_\tau = B_1(\sigma, X_\tau)$ for some $\sigma \subset Q_\tau$.

Example.
For any convex subcomplex Q balls $B_n(Q, X)$ are strongly convex.

Strong convexity is stronger than convexity

Proof: apply Diameter Criterion of 3-convexity to links. $\diamondsuit$

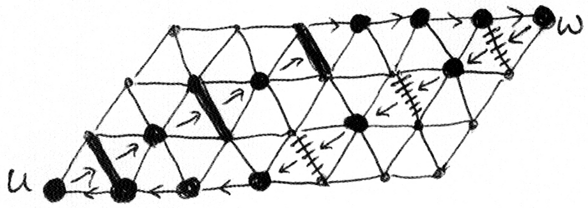
Remark. Strong convexity will play crucial role in the construction of high dimensional systolic spaces, as described later.

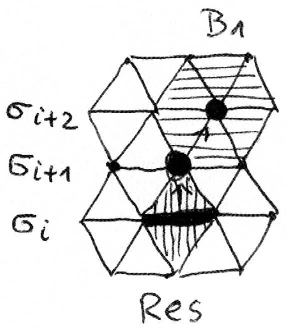
DIRECTED GEODESICS

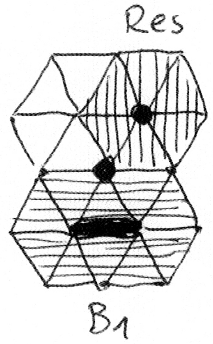
A sequence (σ_i) of simplices in a systolic complex X is a *directed geodesic* if

$$Res(\sigma_i, X) \cap B_1(\sigma_{i+2}, X) = \sigma_{i+1}$$

for all appropriate i.







Examples: projection rays

A sequence $\sigma_0, \sigma_1, \ldots, \sigma_n$ of simplices is a *projection ray* on the final simplex σ_n if $\diamond$ $\sigma_0 \subset S_n(\sigma_n, X)$, and

$\diamond$ σ_{i+1} is the projection of σ_i on $B_{n-i-1}(Q, X)$, for $0 \leq i \leq n-1$.

Note that: • projection ray from σ_0 to σ_n, if exists, is **unique**;

- for any two vertices $u, w \in X$ **there is** a projection ray from u to v;
- each projection ray is a directed geodesic.

[follows, essentially, from Projection Lemma]

Characterization of directed geodesics

A finite sequence of simplices is a directed geodesic iff it is a projection ray on its final simplex.

Properties of directed geodesics

(1) If $\sigma_0, \ldots, \sigma_n$ is a directed geodesic then any sequence $v_0, \ldots, v_n$ of vertices with $v_i \in \sigma_i$ is a geodesic in $X^{(1)}$.

(2) Any two vertices are connected with unique directed geodesic.

(3) All simlices of a directed geodesic connecting two simplices from a convex ubcomplex Q are contained in Q.

(4) Directed geodesics satisfy the 2-sided fellow traveller property.

[(1) and (2) follow from Characterization; (3) and (4) are difficult]

CONVEXITY $\iff$ GEODESIC CONVEXITY

Proposition. Any convex subcomplex Q is geodesically convex.

Remark. The converse implication is easier.

Proof: Let u, w be any two vertices of Q.

- Any two geodesics from u to w can be modified to one another by a sequence of *rhomb modifications.*
  [induction on lekgth of geodesics, using Projection Lemma]
- There is a geodesic from u to w entirely contained in Q.
  [use existence of a directed geodesic from u to w]
- Rhomb modifications turn geodesics contained in Q to geodesics contained in Q.

Thus, any geodesic from u to w is contained in Q. $\diamondsuit$

BIAUTOMATICITY

We omit definition of biautomaticity, mention consequences,
and present a geometric criterion with which we prove it for systolic group.

Consequences. Biautomatic groups are **semihyperbolic**. In particular, for a biautomatic group

- each abelian subgroup is undistorted,
- each solvable subgroup is virtually abelian, and
- quadratic isoperimetric inequality holds.

Geometric Criterion

Let Γ be a graph and G a group acting on Γ by automorphisms, properly discontinuously and cocompactly. Let $\mathcal{P}$ be a set of finite polygonal paths in Γ such that:

(1) for some vertex v_0 of Γ, any two vertices in the orbit $G \cdot v_0$ are connected with a path from $\mathcal{P}$ ($\mathcal{P}$ is *transitive on* $G \cdot v_0$),

(2) $\mathcal{P}$ satisfies 2-sided fellow traveller property, and

(3) $\mathcal{P}$ is d-locally recognized for some d.

Then G is biautomatic.

[J.Ś, *Regular path systems and (bi)automatic groups*, Geometriae Dedicata 118 (2006), 23–48]

d-locally recognized path system

- $\mathcal{R}$ - set of G-congruence classes of length d polygonal paths in Γ
- $\mathcal{P}(\mathcal{R})$ - the set of all finite paths γ in Γ such that
  - ⋄ if $|\gamma| < d$ then $[\gamma \cdot \gamma'] \in \mathcal{R}$ for some γ'
  - ⋄ if $|\gamma| \geq d$ then for any subpath $\eta \subset \gamma$ of length d we have $[\eta] \in \mathcal{R}$
- $\mathcal{P}$ is *d-locally recognized* if $\mathcal{P} = \mathcal{P}(\mathcal{R})$ for some $\mathcal{R}$ as above

Biautomaticity Theorem. Every systolic group G is biautomatic.

Proof: Let G act geometrically on a systolic complex X.

Put $\Gamma = (X')^{(1)}$ - the 1-skeleton of the 1st barycentric subdivision.

Then G acts on Γ, geometrically.

Let $\mathcal{P}$ be the set of all paths in Γ of form

$$(b(\sigma_0), b(\sigma_0 * \sigma_1), b(\sigma_1), \ldots, b(\sigma_{n-1}), b(\sigma_{n-1} * \sigma_n), b(\sigma_n))$$

where $\sigma_0, \ldots, \sigma_n$ is a directed geodesic, $b(\sigma)$ is the barycenter of σ.

- $\mathcal{P}$ is transitive on $G \cdot b(v)$ for any vertex $v \in X$,
  since any two vertices in X are joined with a directed geodesic.
- $\mathcal{P}$ satisfies the 2-sided fellow traveller
  since directed geodesics do so.
- $\mathcal{P}$ is 2-locally recognized, by definition of directed geodesics.

By Geometric Criterion, G is biautomatic. $\diamondsuit$

RELATIONSHIP TO $CAT(0)$

- in dim = 2: 6-systolic = $CAT(0)$ (for standard PE metric)

- in general: 6-systolic is not $CAT(0)$ with standard PE metric

  Example 1. Fix any $k \geq 6$ and consider X equal to the simplicial cone over $\cup_{i \in Z/kZ}\ \sigma_i * \sigma_{i+1}$, with $\dim \sigma_i = n$. X is clearly k-systolic, but if n is sufficiently large, X is not $CAT(0)$.

- conversely, there are simplicial complexes X which are $CAT(0)$ for standard PE metric, and are not systolic.

  Example 2. $X =$ (pentagon)$*\sigma$, with $\dim \sigma = n$, is clearly not systolic ($X_\sigma =$ (pentagon)), and for sufficiently large n X is $CAT(0)$ (dihedral angle between codimension 1 faces of a regular simplex converges to $\pi/2$ as the dimension grows).

- $CAT(0)$**-Lemma.**
  For $\mathcal{S}$ denoting a finite set of shapes of Euclidean simplices we have:
  $\forall \mathcal{S}\, \exists k \quad X$ is k-systolic & Shapes$(X) \subset \mathcal{S} \quad \Longrightarrow \quad X$ is $CAT(0)$

- in particular, for standard PE metric:
  $\forall n\, \exists k \quad X$ is k-systolic & $\dim X \leq n \Longrightarrow X$ is $CAT(0)$
  (if $n \to \infty$ then necessarily $k \to \infty$, due to Example 1 above)

  Some explicit estimate. If

$$k \geq \frac{7\pi\sqrt{2}}{2} \cdot n + 2$$

  then any k-systolic simplicial complex X with $\dim X \leq n$ is $CAT(0)$ for standard PE metric.

- similar results hold for $CAT(-1)$

PROOF OF $CAT(0)$-LEMMA (SKETCH):

Since PE simply connected complex is $CAT(0)$ iff its spherical links are $CAT(1)$, it is sufficient to prove

$CAT(1)$**-Lemma.** Let Π be a finite set of shapes of spherical simplices.
Then there is $k \geq 6$, depending only on Π, such that
if X is a PS k-large simplicial complex with $\text{Shapes}(X) \subset \Pi$
then X is $CAT(1)$.

Preparations:

- for a closed geodesic γ in PS complex X, *size* of γ is the number of maximal nontrivial subsegments in γ contained in a single simplex of X
- if $\text{Shapes}(X)$ is finite then size of γ is finite [Bridson]
- given a finite set $\mathcal{S}$ of shapes of spherical simplices, there is N such that if $|\gamma| < 2\pi$ for a closed geodesic γ in a PS complex X with $\text{Shapes}(X) \subset \mathcal{S}$, then $\text{size}(\gamma) < N$ [Bridson]
- if X is PS and ∞-large simplicial complex, then X contains no closed local geodesic

Steps of argument:

- take $\mathcal{S}$ = link completion of Π; the $\mathcal{S}$ is finite
- consider all closed geodesics γ, $|\gamma| < 2\pi$, in all PS flag simplicial complexes with Shapes$(X) \subset \mathcal{S}$
- for each such X, let K_γ be the full subcomplex of X spanned by the union of simplices whose interiors are intersected by γ
- there are finitely many K_γ, up to simplicial isomorphisms, since the number of their vertices is universally bounded
- complexes K_γ are not ∞-large, since they contain closed geodesics
- put

  $$k = \max\{sys(K_\gamma) \,|\, K_\gamma \text{ as above}\} + 1$$

  where sys = length of the shortes cycle without diagonals
- if X is k-large, with Shapes$(X) \subset \mathcal{S}$, then X has no closed geodesic γ with $|\gamma| < 2\pi$

  (otherwise K_γ for this X, as full subcomplex of a k-large complex, is k-large, a contradiction)
- thus, if X is k-large and Shapes$(X) \subset \Pi$, then neither X nor any of its spherical links contains a closed gedesic γ with $|\gamma| < 2\pi$
- hence X is $CAT(1)$

CONSTRUCTION

We present a construction of systolic complexes and groups of **arbitrary dimension**.

SNPC simplices of groups:

X simplicial complex, G acts on X simplicially,
$G\backslash X \cong \Delta$ top simplex in X

- *simplex of groups* associated to such action

  $G\backslash\backslash X := (\Delta, \{G_\sigma\}, \{\varphi_{\sigma\tau}\})$

  ◇ for a face $\sigma \subset \Delta$ $G_\sigma := \mathrm{Stab}(\sigma, G)$ (*local groups*)

  ◇ for $\sigma \supset \tau$ $\varphi_{\sigma\tau} : G_\sigma \hookrightarrow G_\tau$ (*structure monomorphisms*)

  ◇ if $\sigma \supset \tau \supset \rho$ then $\varphi_{\tau\rho} \circ \varphi_{\sigma\tau} = \varphi_{\sigma\rho}$ (*compatibility*)

- **abstract** simplex of groups $\mathcal{G} = (\Delta, \{G_\sigma\}, \{\varphi_{\sigma\tau}\})$

  ◇ *developable* – $\mathcal{G} = G\backslash\backslash X$ for some G, X

  ◇ if developable, there are also $\widetilde{G}, \widetilde{X}$ (**uniquely** determined) s.t. $\mathcal{G} = \widetilde{G}\backslash\backslash\widetilde{X}$ and $\widetilde{X}$ simply connected

- then $\widetilde{G} =: \pi_1(\mathcal{G})$, $\widetilde{X} =: D(\mathcal{G}) = \widetilde{\mathcal{G}}$ – *universal development* or *universal covering* of $\mathcal{G}$

- $\mathcal{G}$ contains information about **links** in potential $D(\mathcal{G})$ called *local developments*

- $\mathcal{G}$ *is SNPC* – local developments are 6-large

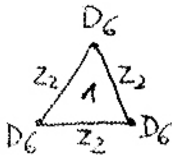
- generalizes to locally k-large

- if $\mathcal{G}$ is a locally k-large simplex of finite groups, with $k \geq 6$, then $\mathcal{G}$ is developable, $\widetilde{\mathcal{G}}$ and $\pi_1(\mathcal{G})$ are k-systolic

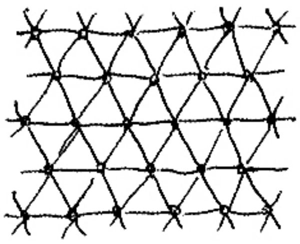
- if moreover $G_\Delta = 1$, $G_\sigma = Z_2$ for codimension 1 faces σ, then $\widetilde{G}$ is a pseudomanifold, $\mathrm{vcd}(\pi_1\mathcal{G}) = \dim\Delta$

Examples of SNPC simplices of groups:

in dimension 2

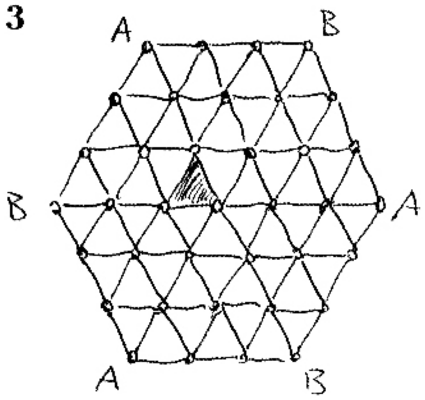
$\mathcal{G} =$ 

local developments at vertices = 

$D(\mathcal{G}) =$ (3, 6)-plane 

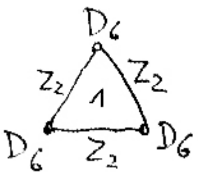
$\pi_1(\mathcal{G}) =$ reflection group of type $\tilde{A}_2$

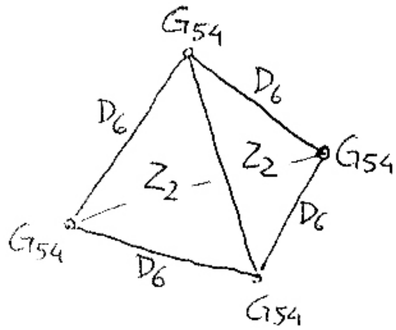
in dimension 3



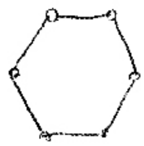
T^2 6-large torus

G_{54} generated by reflections

$G_{54}\backslash\backslash T^2 =$ 

$\mathcal{G} =$ 

local developments:

at edges = 

at vertices = T^2

- $\mathcal{G}$ is SNPC (local developments are 6-large), hence $\mathcal{G}$ is developable
- $D(\mathcal{G})$ is a 6-systolic 3-dimensional pseudomanifold

Inductive construction of simplices of groups:

- **Lemma.** $\forall k \geq 6\ \forall n\ \exists\ n$-dimensional simplex of groups $\mathcal{G}$ s.t.
  - ◇ $\mathcal{G}$ is locally k-large ($\Rightarrow$ developable),
  - ◇ local groups G_σ are finite, $\quad G_\Delta = 1$,
  - ◇ $\pi_1(\mathcal{G})$ is **residually finite.**

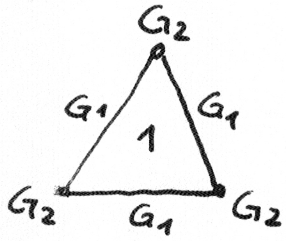
  Moreover, such $\mathcal{G}$ exists for any choice of finite groups G_σ at codim 1 faces σ of Δ.

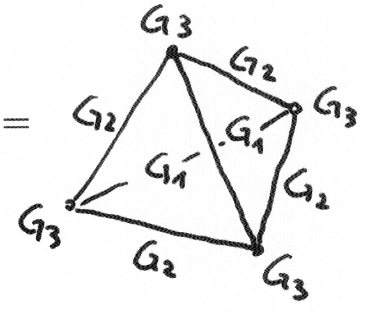
- for "sufficiently deep" finite index normal subgroup $H \triangleleft \pi_1(\mathcal{G})$

  quotient $H \backslash D(\mathcal{G})$ is compact k-large of dimension n

  ("sufficiently deep" includes torsion-free, so that H acts freely)
  - ◇ if codim 1 groups $= Z_2$ this gives pseudomanifolds

- inductive step in the proof of Lemma (sketch):

$\mathcal{G}_2 =$ 

$\rightsquigarrow$ $\mathcal{G}_3 :=$ 

where $G_3 = \pi_1(\mathcal{G}_2)/H, \quad H \triangleleft \pi_1(\mathcal{G}_2)$ sufficiently deep

(to get residual finiteness of $\pi_1(\mathcal{G}_3)$
some nontrivial **extra care** is necessary)

getting residual finiteness

residual finiteness

A G is *residually finite*, if $\forall g \in G$, $g \neq 1$, there is a subgroup A of finite index in G with $g \notin A$.

Remark. Note that, w.l.o.g., we may require that $A < G$ is additionally normal (just take the intersection of conjugates of A in G).

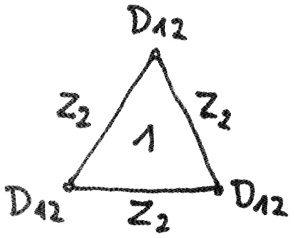
extra-tilability

A locally 6-large simplex of groups $\mathcal{G}$ over a simplex Δ is *locally extra-tilable* if

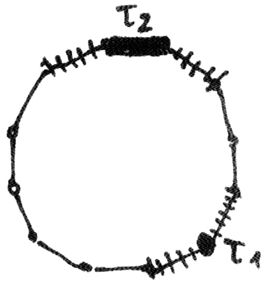
$\forall \sigma \subset \Delta \subset \widetilde{\mathcal{G}}\ \forall \tau \subset \widetilde{\mathcal{G}}_\sigma$ ball $B_1(\tau, \widetilde{\mathcal{G}}_\sigma)$ is the strict fundamental domain for the action of some subgroup $A_\tau < G_\sigma$ on $\widetilde{\mathcal{G}}_\sigma$.

Example: $D_{12} = \langle a, b \,|\, a^2, b^2, (ab)^6 \rangle$,

$\mathcal{G} =$



$\widetilde{\mathcal{G}}_v =$



Proposition. If $\mathcal{G}$ is a simplex of groups over Δ s.t.

- local groups G_σ are finite, $G_\Delta = 1$, and
- $\mathcal{G}$ is locally 6-large and locally extra-tilable,

then $\pi_1\mathcal{G}$ is residually finite.

(Note: proposition applies to $\mathcal{G}$ as in Example)

To prove Proposition

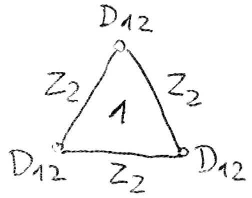
Tilability Lemma. Under assumptions of Proposition, for any strongly convex subcomplex $Q \subset \widetilde{\mathcal{G}}$,

Q is the strict fundamental domain for the action of some subgroup $A_Q < \pi_1\mathcal{G}$ on $\widetilde{\mathcal{G}}$.

Moreover, if Q is finite then A_Q has finite index in $\pi_1\mathcal{G}$.

Idea of proof (through an instructive example)

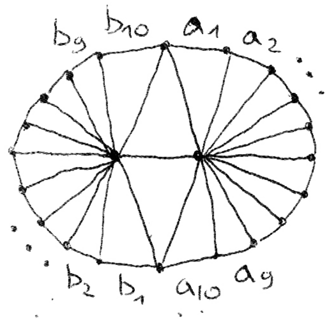
$\mathcal{G} =$



$\widetilde{\mathcal{G}} =$ tesselation of H^2 by equilateral triangles with angles $\pi/6$

$\pi_1\mathcal{G} = \langle s_1, s_2, s_3 \mid s_i^2, (s_i s_j)^6 \rangle$

$Q =$



$$A_Q = \langle a_1, \ldots, a_{10}, b_1, \ldots, b_{10} \mid a_i^2, b_i^2, (a_i a_{i+1})^3, (b_i b_{i+1})^3, (b_{10} a_1)^2, (a_{10} b_1)^2 \rangle$$

Essential: links Q_σ have the form $B_1(\tau, X_\sigma)$, which nicely matches with local extra-tilability.

Back to the proof of Proposition

Let $g \in \pi_1\mathcal{G}$, $g \neq 1$.

We need a finite index subgroup $A < \pi_1\mathcal{G}$ with $g \notin A$.

- Let $Q = B_N(v, \widetilde{\mathcal{G}})$ be a ball containing both Δ and $g \cdot \Delta$.
- Q is finite and strongly convex, hence is the strict fundamental domain for a subgroup $A_Q < \pi_1\mathcal{G}$ of finite index.
- Since then $g \notin A_Q$, $\quad A = A_Q$ does the job. $\diamondsuit$