Correlation inequalities for Gibbs point processes

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Abstract: We study Gibbs point processes on Borel spaces defined by Papangelou intensities and provide a formula for covariance of pairs of functionals on this process. We show that if the functionals are co-monotone then covariances are non-negative.

Keywords: Poisson Process; Gibbs point process; Papangelou intensity; FKG inequality; stochastic ordering;

1 Introduction

1.1 Correlation inequalities for random measures

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \((\mathcal{X}, \mathcal{X}, \lambda)\) a measurable state space with a σ-finite measure \(\lambda\). Let \(M = M(\mathcal{X})\) be the space of Radon (i.e., locally finite) measures on a locally compact second countable Hausdorff space \(\mathcal{X}\). Let \(\mathcal{B}(\mathcal{X})\) be the class of Borel sets generated by the topology of \(\mathcal{X}\), and let \(\mathcal{F}_c\) be the class of nonnegative continuous functions \(\mathcal{X} \to \mathbb{R}\) with compact supports. The space \(M\) can be endowed with the vague topology, for which the class of all finite intersections of sets of the form \(\{\mu \in M : s < \int_X f d\mu < t\}\) for \(s, t \in \mathbb{R}\) and \(f \in \mathcal{F}_c\), may serve as a base. The space \(M\) with the vague topology is metrizable as a Polish space [Kallenberg [1], 15.7.7]. We call any \(M\)-valued random element \(M\) a random measure on \(\mathcal{X}\). The distributions of vectors \((M(B_1), \ldots, M(B_n))\), \(n \geq 1\), for arbitrary bounded sets \(B_1, \ldots, B_n \in \mathcal{B}\), entirely determine the distribution of a random measure \(M\). We define a point process \(\Psi\) on the space \(\mathcal{X}\) as a random measure confined with probability 1 to the subset \(\mathcal{N}\) consisting of all integer Radon measures on the space \(\mathcal{X}\). Elements of \(M\) will be denoted by \(\mu, \nu\), with indices if necessary. For \(\mu, \nu \in M\) we write \(\mu \prec \nu\) if \(\mu(B) \leq \nu(B)\) for all \(B \in \mathcal{B}(\mathcal{X})\). We say that a real function \(F\) defined on \(M\) is isotone (increasing) if \(F(\mu) \leq F(\nu)\) for all \(\mu \prec \nu\). For two random measures we write \(M_1 \prec_{st} M_2\) if \(E(F(M_1)) \leq E(F(M_2))\) for all \(F\) isotone (increasing). We say that a random measure \(M\) is associated if

\[
E(F(M)G(M)) \geq E(F(M))E(G(M)) \quad \text{for all } F, G \text{ isotone.} \tag{1.1}
\]

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It is known that general properties of stochastic ordering and association for random measures on topological spaces can be reduced to finite dimensional setting. We recall and state together Theorem 1 from [6] and Theorem 3.2 from Kwiecinski and Szekli [3].

**Theorem 1.1.**

\[ M_1 \prec_{st} M_2 \ 	ext{if and only if} \ 
E[F((M_1(B_1), \ldots, M_1(B_n))))] \leq E[F((M_2(B_1), \ldots, M_2(B_n))))] \]

for all \( n \geq 1 \), bounded sets \( B_1, \ldots, B_n \in \mathcal{B} \), and coordinatewise increasing \( F \).

A random measure \( M \) is associated if and only if

\[ E[F((M(B_1), \ldots, M(B_n))))G((M(B_1), \ldots, M(B_n))))] \geq E[F((M(B_1), \ldots, M(B_n))))]E[G((M(B_1), \ldots, M(B_n))))] \]

for all \( n \geq 1 \), bounded sets \( B_1, \ldots, B_n \in \mathcal{B} \), and all \( F, G \) coordinatewise increasing.

From the above theorem we have immediately the following result on association.

**Corollary 1.2.** If \( M \in \mathcal{M} \) is a random measure with independent increments, i.e. the random variables \( M(B_1), \ldots, M(B_n) \), are independent for all \( n \geq 1 \), and pairwise disjoint \( B_1, \ldots, B_n \in \mathcal{B} \), then \( M \) is associated.

The Poisson process \( \Pi_\lambda \in \mathcal{M} \) is associated as random measure.

### 2 Correlation inequalities for Gibbs point processes

We denote by \( \mathcal{N} \) the space of integer valued \( \sigma \)-finite measures on \( (\mathcal{X}, \mathcal{X}) \), and by \( \Pi_\lambda : \Omega \to \mathcal{N} \) the Poisson process with intensity \( \lambda \). \( \mathcal{N} \) is equipped with the \( \sigma \)-algebra \( \mathcal{N} \) generated by the counting variables \( N(A) : \mathcal{N} \to \mathbb{R} \), \( N(A)(\mu) = \mu(A) \) for \( \mu \in \mathcal{N} \), \( A \in \mathcal{X} \). The distribution of \( \Pi_\lambda \) on \( \mathcal{N} \) we denote by \( \mathbb{P}_\lambda \). For \( \mu, \nu \in \mathcal{N} \) we write \( \mu \prec \nu \) if \( \mu(A) \leq \nu(A) \) for all \( A \in \mathcal{X} \). We say that a real function \( \Phi \) defined on \( \mathcal{N} \) is isotone if \( \Phi(\mu) \leq \Phi(\nu) \) for all \( \mu \prec \nu \). \( (\mathcal{N}, \prec) \) is a lattice ordered set. For \( \mu, \nu \in \mathcal{N} \) we denote the corresponding maximum and minimum elements by \( \mu \lor \nu \) and \( \mu \land \nu \), respectively.

We consider two point processes \( \Psi : \Omega \to \mathcal{N} \) and \( \Phi : \Omega \to \mathcal{N} \) such that

\[ \mathbb{P}(\Psi \in d\mu) = f(\mu)\mathbb{P}(\Pi_\lambda \in d\mu) \]
\[ \mathbb{P}(\Phi \in d\mu) = g(\mu)\mathbb{P}(\Pi_\lambda \in d\mu), \]

which means that the distributions \( \mathbb{P}_\Psi \) and \( \mathbb{P}_\Phi \) of \( \Psi \) and \( \Phi \), respectively are absolutely continuous with respect to \( \mathbb{P}_\lambda \) (with Radon-Nikodym densities \( f \) and \( g \), respectively). Our aim is to find sufficient conditions on \( f \) and \( g \) in order to obtain stochastic ordering between \( \Psi \) and \( \Phi \) defined as

\[ \Psi \prec_{st} \Phi \text{ if } E(F(\Psi)) \leq E(F(\Phi)) \text{ for all } F \text{ isotone.} \]
We also search for conditions on $f$ implying

$$E(F(\Psi)G(\Psi)) \geq E(F(\Psi))E(G(\Psi)) \quad \text{for all } F, G \text{ isotone.} \quad (2.1)$$

In particular we shall check that $f(\mu)f(\nu) \leq f(\mu \lor \nu)f(\mu \land \nu)$ for all $\mu, \nu \in \mathbb{N}$ is sufficient for (2.1), which is known as FKG inequality or association of $\Psi$.

The fact that (1.1) holds for $M := \Pi_\lambda$ is known in a quite general setting, see Last and Penrose [4]. We shall recall now their result on association. For any measurable $f : \mathbb{N} \rightarrow \mathbb{R}$ we define difference operator $D_x$ by

$$D_x f(\mu) := f(\mu + \delta_x) - f(\mu), \quad \mu \in \mathbb{N},$$

where $\delta_x$ denotes the Dirac measure located at $x \in \mathbb{X}$. Iterating for $n \geq 2$ and $x_1, \ldots, x_n \in \mathbb{X}$ we set

$$D_{x_1, \ldots, x_n} F := D_{x_1} \cdots D_{x_n} F,$$

and use $D^1 = D$, $D^0 F = F$. We denote by $L^2(P_\lambda)$ the space of functions $F$ on $\mathbb{N}$ such that $\int_\mathbb{N} F^2(\mu)P_\lambda(d\mu) = E[F^2(\Pi_\lambda)] < \infty$. A function $f$ is increasing on $B \subset \mathbb{N}$ if $f(\mu) \leq f(\mu + \delta_x)$ for all $\mu \in \mathbb{N}$ and all $x \in \mathbb{X}$. $F$ is decreasing on $B$ if $-f$ is increasing on $B$.

**Theorem 2.1** (Theorem 1.4 in Last and Penrose [4]). Suppose $B \in \mathcal{X}$. Let $f, g \in L^2(P_\lambda)$ be increasing on $B$ and decreasing on $\mathbb{X} \setminus B$. Then

$$E[f(\Pi_\lambda)G(\Pi_\lambda)] \geq E[f(\Pi_\lambda)]E[g(\Pi_\lambda)]. \quad (2.4)$$

The above result follows immediately from a covariance formula obtained for a marked Poisson process. To be more precise, consider marked Poisson process $\tilde{\Pi}_\lambda$ defined on $\tilde{\mathbb{X}} := [0, 1] \times \mathbb{X}$ with the intensity measure $\tilde{\lambda}$ being the product measure of Lebesgue measure on $[0, 1]$ with $\lambda$. Denote the corresponding restriction by $\Pi_\lambda^t(A) := \tilde{\Pi}_\lambda(A \times [0, t]), \ t \in [0, 1]$.

From Theorem 5.1 in Last and Penrose [4] we know that for $f, g \in L^2(P_\lambda)$

$$E[f(\Pi_\lambda)f(\Pi_\lambda)] - E[f(\Pi_\lambda)]E[g(\Pi_\lambda)] =$$

$$E\{\int_\mathbb{R} \int_0^1 E[D_x F(\Pi_\lambda)|\Pi_\lambda^t]E[D_x G(\Pi_\lambda)|\Pi_\lambda^t]dt \lambda(dx)\}.$$  

We rewrite the above formula as a covariance formula

$$\text{Cov}(F(\Pi_\lambda), G(\Pi_\lambda)) = E\{\int_\mathbb{R} E[D_x F(\Pi_\lambda)|\Pi_\lambda^t]E[D_x G(\Pi_\lambda)|\Pi_\lambda^t]dt \lambda(dx)\},$$

where we identify $\Pi_\lambda^t$ with $\Pi_\lambda$ and apply $F$ (and then $D_x$) marginally to the first coordinate.
2.0.1 Positive correlations via operator calculus for Gibbs point processes

In this section we assume that \( \Phi \) is a Gibbs process on a Borel space \( X \) in the following (abstract) way. We assume that there is a measurable function \( \kappa : X \times N \to [0, \infty) \) (the Papangelou intensity of \( \Phi \)) such that

\[
\mathbb{E} \int h(x, \Phi) \Phi(dx) = \mathbb{E} \int h(x, \Phi + \delta_x) \kappa(x, \Phi) \lambda(dx)
\]  
(2.5)

for all measurable \( h : X \times N \to [0, \infty) \).

By \( L^p_\Phi, p \geq 0 \), we denote the space of all random variables \( F \in L^p(\mathbb{P}) \) such that

\( F = f(\Phi) \mathbb{P}\text{-almost surely with a measurable function } f : N \to \mathbb{R} \). We call such a function \( f \) a representative of \( F \). In this case we define \( D_x F := D_x f(\Phi) \) for \( x \in X \) and interpret \( DF \) as the function \( (x, \omega) \mapsto D_x f(\Phi(\omega)) \). It follows from (2.5) that \( D_x f(\mu) = D_x \tilde{f}(\mu) \) for \( C^1_\Phi \)-a.e. \((x, \mu)\) if \( f, \tilde{f} \) are two representatives of \( F \). Here \( C^1_\Phi \) is the reduced Campbell measure on \( X \times N \) defined by

\[
C^1_\Phi := \mathbb{E} \int 1\{(x, \Phi - \delta_x) \in \cdot\} \Phi(dx) = \mathbb{E} \int 1\{(x, \Phi) \in \cdot\} \kappa(x, \Phi) \lambda(dx),
\]  
(2.6)

where the second identity is due to (2.5).

In the remainder of this section we fix two measurable functions \( b, d : X \times N \to [0, \infty) \) satisfying

\[
b(x, \mu) = d(x, \mu + \delta_x) \kappa(x, \mu), \quad (x, \mu) \in X \times N. \quad (2.7)
\]

One possible choice is \( d \equiv 1 \). We let \( \text{dom} L \) denote the set of all \( F \in L^0_\Phi \) such that

\[
\mathbb{E} \int |D_x F| b(x, \Phi) \lambda(dx) < \infty.
\]  
(2.8)

For \( F \in \text{dom} L \) we define a random variable \( LF \) by

\[
LF := \int (f(\Phi - \delta_x) - f(\Phi)) d(x, \Phi) \Phi(dx) + \int (D_x F) b(x, \Phi) \lambda(dx),
\]  
(2.9)

where \( f \) is a representative of \( F \). We interpret the operator \( L \) as the generator of a birth-death process with birth rate \( b \) and death rate \( d \).

For \( h \in L^1(C^1_\Phi) \) we define the Kabanov-Skorohod integral \( \delta(h) \) as the random variable

\[
\delta(h) := \int h(x, \Phi - \delta_x) \Phi(dx) - \int h(x, \Phi) b(x, \Phi) \lambda(dx).
\]  
(2.10)

This integral can also be defined for a \( \mathcal{X} \otimes \sigma(\Phi) \)-measurable function \( H : X \times \Omega \to \mathbb{R} \) satisfying

\[
\mathbb{E} \int |H(x)| b(x, \Phi) \lambda(dx) < \infty,
\]  
(2.11)

where, as usual, \( H(x) \) denotes the random variable \( H(x, \cdot) \). The space of such functions is denoted by \( \text{dom} \delta \). To this end we take a representative \( h \) of \( H \) that is, a measurable function \( h : X \times N \to \mathbb{R} \) such that \( H(x, \omega) = h(x, \Phi(\omega)) \) for all \( \lambda \otimes \mathbb{P} \)-a.e. \((x, \omega) \in \{ b > 0 \} \). Then we define \( \delta(H) := \delta(h) \). If \( \tilde{h} \) is another representative of \( H \), then \( \delta(h) = \delta(\tilde{h}) \).
Lemma 2.2. For any $F \in \text{dom } L$ we have a.s. that $LF = -\delta(DF)$.

Proof. This is just a simple matter of computation using (2.5) and the detailed balance equation (2.7).

The next result shows that $L$ is injective.

Lemma 2.3. Assume that $LF = LG$ a.s. for $F, G \in \text{dom } L$. Then $F - EF = G - EG$ a.s.

Proof. In view of Lemma 2.3 we define for $G \in \text{im } L$ (the image of $L$) $L^{-1}G$ as the a.s. unique element of $\text{dom } L$ satisfying $L(F) = G$ and $EF = 0$.

We proceed with a duality formula (partial integration).

Proposition 2.4. Suppose that $F \in L^1_\Phi$ is bounded and that $H \in \text{dom } \delta$. Then

$$
\mathbb{E}F\delta(H) = \mathbb{E}\int (D_x F) H(x) b(x, \Phi) \lambda(dx).
$$

(2.12)

Proof. Again this just a simple computation.

The following covariance identity is our main technical tool for handling Gibbs processes.

Theorem 2.5. Suppose that $F \in L^1_\Phi$ satisfies $\bar{F} := F - EF \in \text{im } L$ and $DL^{-1}\bar{F} \in \text{dom } \delta$. Let $G \in L^0_\Phi$ be bounded. Then

$$
\text{Cov}(F, G) = \mathbb{E}\int (D_x G)(-D_x L^{-1}\bar{F}) b(x, \Phi) \lambda(dx).
$$

(2.13)

Proof. By Lemma 2.3 we have a.s. that $\bar{F} = LL^{-1}\bar{F}$. Therefore we obtain from Lemma 2.2 that

$$
\mathbb{E}(F - EF)G = \mathbb{E}[-\delta(DL^{-1}\bar{F})G].
$$

Proposition 2.4 yields the assertion.

Corollary 2.6. Let the assumptions of Theorem 2.5 be satisfied. Assume that $G$ and $-L^{-1}\bar{F}$ have increasing representatives. Then $\text{Cov}(F, G) \geq 0$.

If we introduce the semigroup $T_t := e^{tL}$, $t \geq 0$, associated with the operator $L$ then we can establish the identity

$$
-L^{-1}F = \int_0^\infty T_tFdt, \quad F \in \text{im } L.
$$

(2.14)

Then (2.13) takes the form

$$
\text{Cov}(F, G) = \mathbb{E}\int_X \int_0^\infty (D_x G)(D_x T_tF)b(x, \Phi) dt \lambda(dx).
$$

(2.15)

Such covariance identities are well-known in stochastic analysis, see for instance [5] and [4].
References


