Correlation inequalities for Gibbs point processes

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Abstract: We study Gibbs point processes on Borel spaces defined by Papangelou intensities and provide a formula for covariance of pairs of functionals on this process. We show that if the functionals are co-monotone then covariances are non-negative.

Keywords: Poisson Process; Gibbs point process; Papangelou intensity; FKG inequality; stochastic ordering;

1 Introduction

1.1 Correlation inequalities for random measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathbb{X}, \mathcal{X}, \lambda)$ a measurable state space with a σ-finite measure λ Let $\mathbf{M} = \mathbf{M}(\mathbb{X})$ be the space of Radon (i.e., locally finite) measures on a locally compact second countable Hausdorff space $\mathbb{X}$. Let $\mathcal{B}(\mathbb{X})$ be the class of Borel sets generated by the topology of $\mathbb{X}$, and let $\mathcal{F}_c$ be the class of nonnegative continuous functions $\mathbb{X} \to \mathbb{R}$ with compact supports. The space $\mathbf{M}$ can be endowed with the vague topology, for which the class of all finite intersections of sets of the form $\{\mu \in \mathbf{M} : s < \int_{\mathbb{X}} f d\mu < t\}$ for $s, t \in \mathbb{R}$ and $f \in \mathcal{F}_c$, may serve as a base. The space $\mathbf{M}$ with the vague topology is metrizable as a Polish space [Kallenberg [1], 15.7.7]. We call any $\mathbf{M}$- valued random element M a random measure on $\mathbb{X}$. The distributions of vectors $(M(B_1), \dots, M(B_n))$, $n \geq 1$, for arbitrary bounded sets $B_1, \dots, B_n \in \mathcal{B}$, entirely determine the distribution of a random measure M. We define a point process Ψ on the space $\mathbb{X}$ as a random measure confined with probability 1 to the subset $\mathbf{N}$ consisting of all integer Radon measures on the space $\mathbb{X}$. Elements of $\mathbf{M}$ will be denoted by μ, ν, with indices if necessary. For $\mu, \nu \in \mathbf{M}$ we write $\mu \prec \nu$ if $\mu(B) \leq \nu(B)$ for all $B \in \mathcal{B}(\mathbb{X})$. We say that a real function F defined on $\mathbf{M}$ is *isotone* (increasing) if $F(\mu) \leq F(\nu)$ for all $\mu \prec \nu$. For two random measures we write $M_1 \prec_{st} M_2$ if $E(F(M_1)) \leq E(F(M_2))$ for all F isotone (increasing). We say that a random measure M is associated if

$$E(F(M)G(M)) \geq E(F(M))E(G(M)) \text{ for all } F, G \text{ isotone.} \tag{1.1}$$

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It is known that general properties of stochastic ordering and association for random measures on topological spaces can be reduced to finite dimensional setting. We recall and state together Theorem 1 from [6] and Theorem 3.2 from Kwiecinski and Szekli [3].

Theorem 1.1.

$M_1 \prec_{st} M_2$ if and only if

$$E[F((M_1(B_1),\ldots,M_1(B_n)))] \le E[F((M_2(B_1),\ldots,M_2(B_n)))]$$

for all $n \ge 1$, bounded sets $B_1,\ldots,B_n \in \mathcal{B}$, and coordinatewise increasing F.

A random measure M is associated if and only if

$$E[F((M(B_1),\ldots,M(B_n)))G((M(B_1),\ldots,M(B_n)))] \ge$$
$$E[F((M(B_1),\ldots,M(B_n)))]E[G((M(B_1),\ldots,M(B_n)))]$$

for all $n \ge 1$, bounded sets $B_1,\ldots,B_n \in \mathcal{B}$, and all F, G coordinatewise increasing.

From the above theorem we have immediately the following result on association.

Corollary 1.2. *If $M \in \mathbf{M}$ is a random measure with independent increaments, i.e. the random variables $M(B_1),\ldots,M(B_n)$, are independent for all $n \ge 1$, and pairwise disjoint $B_1,\ldots,B_n \in \mathcal{B}$, then M is associated.*

The Poisson process $\Pi_\lambda \in \mathbf{M}$ is associated as random measure.

2 Correlation inequalities for Gibbs point processes

We denote by $\mathbf{N}$ the space of integer valued σ- finite measures on $(\mathbb{X},\mathcal{X})$, and by $\Pi_\lambda : \Omega \to \mathbf{N}$ the Poisson process with intensity λ. $\mathbf{N}$ is equipped with the σ algebra $\mathcal{N}$ generated by the counting variables $N(A) : \mathbf{N} \to \mathbb{R}$, $N(A)(\mu) = \mu(A)$ for $\mu \in \mathbf{N}$, $A \in \mathcal{X}$. The distribution of Π_λ on $\mathbf{N}$ we denote by $\mathbf{P}_\lambda$. For $\mu, \nu \in \mathbf{N}$ we write $\mu \prec \nu$ if $\mu(A) \le \nu(A)$ for all $A \in \mathcal{X}$. We say that a real function F defined on $\mathbf{N}$ is *isotone* if $F(\mu) \le F(\nu)$ for all $\mu \prec \nu$. $(\mathbf{N}, \prec)$ is a lattice ordered set. For $\mu, \nu \in \mathbf{N}$ we denote the corresponding maximum and minimum elements by $\mu \vee \nu$ and $\mu \wedge \nu$, respectively.

We consider two point processes $\Psi : \Omega \to \mathbf{N}$ and $\Phi : \Omega \to \mathbf{N}$ such that

$$\mathbb{P}(\Psi \in d\mu) = f(\mu)\mathbb{P}(\Pi_\lambda \in d\mu)$$
$$\mathbb{P}(\Phi \in d\mu) = g(\mu)\mathbb{P}(\Pi_\lambda \in d\mu),$$

which means that the distributions $\mathbf{P}_{\mathbf{\Psi}}$ and $\mathbf{P}_{\mathbf{\Phi}}$ of Ψ and Φ, respectively are absolutely continuous with respect to $\mathbf{P}_\lambda$ (with Radon-Nikodym densities f and g, respectively). Our aim is to find sufficient conditions on f and g in order to obtain stochastic ordering between Ψ and Φ defined as

$$\Psi \prec_{st} \Phi \text{ if } E(F(\Psi)) \le E(F(\Phi)) \text{ for all } F \text{ isotone.}$$

We also search for conditions on f implying

$$E(F(\Psi)G(\Psi)) \geq E(F(\Psi))E(G(\Psi)) \text{ for all } F,G \text{ isotone.} \tag{2.1}$$

In particular we shall check that $f(\mu)f(\nu) \leq f(\mu\vee\nu)f(\mu\wedge\nu)$ for all $\mu,\nu \in \mathbf{N}$ is sufficient for (2.1), which is known as FKG inequality or association of Ψ.

The fact that (1.1) holds for $M := \Pi_\lambda$ is known in a quite general setting, see Last and Penrose [4]. We shall recall now their result on association. For any measurable $f : \mathbf{N} \to \mathbb{R}$ we define difference operator D_x by

$$D_x f(\mu) := f(\mu+\delta_x) - f(\mu), \quad \mu \in \mathbf{N}, \tag{2.2}$$

where δ_x denotes the Dirac measure located at $x \in \mathbb{X}$. Iterating for $n \geq 2$ and $x_1,\dots,x_n \in \mathbb{X}$ we set

$$D_{x_1,\dots,x_n} F := D_{x_1} D^{n-1}_{x_2,\dots,x_n} F, \tag{2.3}$$

and use $D^1 = D$, $D^0F = F$. We denote by $L^2(P_\lambda)$ the space of functions F on $\mathbf{N}$ such that $\int_{\mathbf{N}} F^2(\mu)P_\lambda(d\mu) = E[F^2(\Pi_\lambda)] < \infty$. A function f is increasing on $B \subset \mathbf{N}$ if $f(\mu) \leq f(\mu+\delta_x)$ for all $\mu \in \mathbf{N}$ and all $x \in \mathbb{X}$. F is decreasing on B if $-f$ is increasing on B.

Theorem 2.1 (Theorem 1.4 in Last and Penrose [4])**.** *Suppose $B \in \mathcal{X}$. Let $f,g \in L^2(P_\lambda)$ be increasing on B and decreasing on $\mathbb{X}\setminus B$. Then*

$$E[f(\Pi_\lambda)G(\Pi_\lambda)] \geq E[f(\Pi_\lambda)]E[g(\Pi_\lambda)]. \tag{2.4}$$

The above result follows immediately from a covariance formula obtained for a marked Poisson process. To be more precise, consider marked Poisoon process $\tilde{\Pi}_\lambda$ defined on $\tilde{\mathbb{X}} := [0,1]\times\mathbb{X}$ with the intensity measure $\tilde{\lambda}$ being the product measure of Lebesgue measure on $[0,1]$ with λ. Denote the corresponding restriction by

$$\Pi^t_\lambda(A) := \tilde{\Pi}_\lambda(A\times[0,t]),\ t\in[0,1].$$

From Theorem 5.1 in Last and Penrose [4] we know that for $f,g \in L^2(P_\lambda)$

$$E[f(\Pi_\lambda)f(\Pi_\lambda)] - E[f(\Pi_\lambda)]E[g(\Pi_\lambda)] =$$
$$E\{\int_{\mathbb{X}}\int_0^1 E[D_xF(\Pi_\lambda)|\Pi^t_\lambda]E[D_xG(\Pi_\lambda)|\Pi^t_\lambda]dt\lambda(dx)\}.$$

We rewrite the above formula as a covariance formula

$$Cov(F(\Pi_\lambda),G(\Pi_\lambda)) = E\{\int_{\tilde{\mathbb{X}}} E[D_xF(\Pi^1_\lambda)|\Pi^t_\lambda]E[D_xG(\Pi^1_\lambda)|\Pi^t_\lambda]\tilde{\lambda}(dt\times dx)\},$$

where we identify Π^1_λ with Π_λ and apply F (and then D_x) marginally to the first coordinate.

2.0.1 Positive correlations via operator calculus for Gibbs point processes

In this section we assume that Φ is a Gibbs process on a Borel space $\mathbb{X}$ in the following (abstract) way. We assume that there is a measurable function $\kappa : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ (the *Papangelou intensity* of Φ) such that

$$\mathbb{E} \int h(x, \Phi)\Phi(dx) = \mathbb{E} \int h(x, \Phi + \delta_x)\kappa(x, \Phi)\lambda(dx) \tag{2.5}$$

for all measurable $h : \mathbb{X} \times \mathbf{N} \to [0, \infty)$.

By L^p_Φ, $p \geq 0$, we denote the space of all random variables $F \in L^p(\mathbb{P})$ such that $F = f(\Phi)$ $\mathbb{P}$-almost surely with a measurable function $f : \mathbf{N} \to \mathbb{R}$. We call such a function f a *representative* of F. In this case we define $D_x F := D_x f(\Phi)$ for $x \in \mathbb{X}$ and interpret DF as the function $(x, \omega) \mapsto D_x f(\Phi(\omega))$. It follows from (2.5) that $D_x f(\mu) = D_x \tilde{f}(\mu)$ for $C^!_\Phi$-a.e. (x, μ) if $f, \tilde{f}$ are two representatives of F. Here $C^!_\Phi$ is the *reduced Campbell measure* on $\mathbb{X} \times \mathbf{N}$ defined by

$$C^!_\Phi := \mathbb{E} \int \mathbf{1}\{(x, \Phi - \delta_x) \in \cdot\}\Phi(dx) = \mathbb{E} \int \mathbf{1}\{(x, \Phi) \in \cdot\}\kappa(x, \Phi)\lambda(dx), \tag{2.6}$$

where the second identity is due to (2.5).

In the remainder of this section we fix two measurable functions $b, d : \mathbb{X} \times \mathbf{N} \to [0, \infty)$ satisfying

$$b(x, \mu) = d(x, \mu + \delta_x)\kappa(x, \mu), \quad (x, \mu) \in \mathbb{X} \times \mathbf{N}. \tag{2.7}$$

One possible choice is $d \equiv 1$. We let $\operatorname{dom} L$ denote the set of all $F \in L^0_\Phi$ such that

$$\mathbb{E} \int |D_x F| b(x, \Phi)\lambda(dx) < \infty. \tag{2.8}$$

For $F \in \operatorname{dom} L$ we define a random variable LF by

$$LF := \int (f(\Phi - \delta_x) - f(\Phi))d(x, \Phi)\Phi(dx) + \int (D_x F)b(x, \Phi)\lambda(dx), \tag{2.9}$$

where f is a representative of F. We interpret the operator L as the *generator* of a *birth-death process* with birth rate b and death rate d.

For $h \in L^1(C^!_\Phi)$ we define the Kabanov-Skorohod integral $\delta(h)$ as the random variable

$$\delta(h) := \int h(x, \Phi - \delta_x)\Phi(dx) - \int h(x, \Phi)b(x, \Phi)\lambda(dx). \tag{2.10}$$

This integral can also be defined for a $\mathcal{X} \otimes \sigma(\Phi)$-measurable function $H : \mathbb{X} \times \Omega \to \mathbb{R}$ satisfying

$$\mathbb{E} \int |H(x)| b(x, \Phi)\lambda(dx) < \infty, \tag{2.11}$$

where, as usual, $H(x)$ denotes the random variable $H(x, \cdot)$. The space of such functions is denoted by $\operatorname{dom} \delta$. To this end we take a representative h of H that is, a measurable function $h : \mathbb{X} \times \mathbf{N} \to \mathbb{R}$ such that $H(x, \omega) = h(x, \Phi(\omega))$ for all $\lambda \otimes \mathbb{P}$-a.e. $(x, \omega) \in \{b > 0\}$. Then we define $\delta(H) := \delta(h)$. If $\tilde{h}$ is another representative of H, then $\delta(h) = \delta(\tilde{h})$.

Lemma 2.2. *For any $F \in \operatorname{dom} L$ we have a.s. that $LF = -\delta(DF)$.*

Proof. This is just a simple matter of computation using (2.5) and the detailed balance equation (2.7). □

The next result shows that L is *injective*.

Lemma 2.3. *Assume that $LF = LG$ a.s. for $F, G \in \operatorname{dom} L$. Then $F - \mathbb{E}F = G - \mathbb{E}G$ a.s.*

Proof.

In view of Lemma 2.3 we define for $G \in \operatorname{im} L$ (the image of L) $L^{-1}G$ as the a.s. unique element of $\operatorname{dom} L$ satisfying $L(F) = G$ and $\mathbb{E}F = 0$.

We proceed with a duality formula (partial integration).

Proposition 2.4. *Suppose that $F \in L^0_\Phi$ is bounded and that $H \in \operatorname{dom} \delta$. Then*

$$\mathbb{E}F\delta(H) = \mathbb{E} \int (D_x F) H(x) b(x, \Phi) \lambda(dx). \tag{2.12}$$

Proof. Again this just a simple computation. □

The following covariance identity is our main technical tool for handling Gibbs processes.

Theorem 2.5. *Suppose that $F \in L^1_\Phi$ satisfies $\bar{F} := F - \mathbb{E}F \in \operatorname{im} L$ and $DL^{-1}\bar{F} \in \operatorname{dom} \delta$. Let $G \in L^0_\Phi$ be bounded. Then*

$$\mathbb{C}\mathrm{ov}(F, G) = \mathbb{E} \int (D_x G)(-D_x L^{-1}\bar{F}) b(x, \Phi) \lambda(dx). \tag{2.13}$$

Proof. By Lemma 2.3 we have a.s. that $\bar{F} = LL^{-1}\bar{F}$. Therefore we obtain from Lemma 2.2 that

$$\mathbb{E}(F - \mathbb{E}F)G = \mathbb{E}[-\delta(DL^{-1}\bar{F})G].$$

Proposition 2.4 yields the assertion. □

Corollary 2.6. *Let the assumptions of Theorem 2.5 be satisfied. Assume that G and $-L^{-1}\bar{F}$ have increasing representatives. Then $\mathbb{C}\mathrm{ov}(F, G) \geq 0$.*

If we introduce the semigroup $T_t := e^{tL}$, $t \geq 0$, associated with the operator L then we can establish the identity

$$-L^{-1}F = \int_0^\infty T_t F dt, \quad F \in \operatorname{im} L. \tag{2.14}$$

Then (2.13) takes the form

$$\mathbb{C}\mathrm{ov}(F, G) = \mathbb{E} \int_{\mathbb{X}} \int_0^\infty (D_x G)(D_x T_t F) b(x, \Phi) dt \lambda(dx). \tag{2.15}$$

Such covariance identities are well-known in stochastic analysis, see for instance [5] and [4].

References

[1] Kallenberg, O. (1983). *Random Measures.* Akademie-Verlag, Berlin.

[2] Kondratiev, Y., Lytvinov, E. and Roeckner, M. (2007). Equilibrium kawasaki dynamics of continuous particle systems. *Infinite Dimensional Analysis, Quantum Probability and Related Topics* **10,** 185–209.

[3] Kwieciński, A. and Szekli, R. (1996). Some monotonicity and dependence properties of self exciting point processes. *Ann. Appl. Probab.* **6,** 1211–1231.

[4] Last, G. and Penrose, M. D. (2010). Poisson process Fock space representation, chaos expansion and covariance inequalities. *Probab. Theory Relat. Fields* **150,** 663–690.

[5] Privault, N. (2009). Stochastic analysis in discrete and continuous settings. *Lect. Notes Math.*.

[6] Rolski, T. and Szekli, R. (1991). Stochastic ordering and thinning of point processes. *Stoch. Process. Their Appl.* **37,** 299–312.

[7] Schreiber, T. and Yukich, J. (2013). Stabilization and limit theorems for geometric functionals of gibbs point processes. *arXiv0802.0647* 1–34.