Chapter 1

Stochastic comparison of queueing networks

Ryszard Szekli, *University of Wrocław* [1]

[1]Mathematical Institute, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.

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1.1 Introduction

Classical network theory. A. K. Erlang developed the basic foundations of the teletraffic theory long before probability theory was popularized or even well developed. He established many of principal results which we still use today. The 1920's were basically devoted to the application of Erlang's results (Molina [1927], Thornton Fry [1928]). Felix Pollaczek [1930] did further pioneering work, followed by Khintchine [1932] and Palm [1938]. It was until the mid 1930's, when Feller introduced the birth-death process, that queueing was recognized by the world of mathematics as an object of serious interest. During and following World War II this theory played an important role in the development of the new field of operations research, which seemed to hold so much promise in the post war years. The frontiers of this research proceeded into far reaches of deep and complex mathematics. Not all of these developments proved to be useful. The fact that one of the few tools available for analyzing the performance of computer network systems is queueing theory largely stimulated development of it. Important contributions in 1950's and 60's are among others due to V. E. Benes , D. G. Kendall , D. R. Lindley , S. Karlin and J. L. McGregor, R. M. Loynes , J. F. C. Kingman, L. Takacs, R. Syski, N. U. Prabhu and J. W. Cohen. The literature grew from "solutions looking for a problem" rather than from "problems looking for a solution", which remains true in some sense nowadays. The practical world of queues abounds with problems that cannot be solved elegantly but which must be analyzed. The literature on queues abounds with "exact solutions", "exact bounds", simulation models, etc., with almost everything but little common sense methods of "engineering judgment". It is very often that engineers resort to using formulas which they know they are using incorrectly, or run to the computer even if they need only to know something to within a factor of two. There is a need for approximations, bounds, heuristic reasoning and crude estimates in modelling. The present chapter is an overview of methods based on stochastic ordering which are useful in obtaining comparisons and bounds. Early other efforts following the line of finding estimates are formulated in Newell [1971], and Gross, Harris [1974] where fluid and diffusion approximations were introduced. The theory of weak convergence has been a strong impetus for a systematic development of limit theorems for queueing processes (Whitt [2002]). Point processes have played an important role in the description of input and output processes. Palm measures and Palm-martingale calculus (see e.g. Baccelli and Bremaud [2003]) still play active role in stochastic network modelling not only because they are indispensable as a tool for solving stability questions but also because the Palm theory proved to be an appropriate tool to formalize arguments while proving dependence properties of queueing characteristics and showing bounds on them, as it will be presented in this chapter. In a more recent literature, martingale calculus influences modelling of fluid flow queues but this is another topic not touched in this chapter.

Traffic processes. Traffic is a key ingredient of queueing systems. While traditional analytical models of traffic were often devised and selected for the analytical tractability they induced in the corresponding queueing systems, this selection criterion is largely absent from recent (internet) traffic models. In particular queueing systems with offered traffic consisting of autoregressive type processes, self-similar processes are difficult to solve analytically. Consequently these are only used to derive simulation models. On the other hand some fluid models are analytically tractable, but only subject to considerable restrictions. Thus the most significant traffic research problem is to solve analytically induced systems or in the absence of a satisfactory solution to devise approximate traffic models which lead to analytically tractable systems. Comparison of complex systems with simpler ones or finding simple bounds on sojourn times or throughput seems to be important. We shall stress this point in the present chapter.

Traditional traffic models (renewal, Markov, autoregressive, fluid) have served well in advancing traffic engineering and understanding performance issues, primarily in traditional telephony. The advent of modern high speed communications networks results in a highly heterogeneous traffic mix. The inherent burstiness of several important services is bringing to the fore some serious modelling inadequacies of traditional models, particularly in regard to temporal dependence. This situation has brought about renewed interest in traffic modelling and has driven the development of new models. Statisticians are now aware that ignoring long range dependence can have drastic consequences for many statistical methods. However traffic engineers and network managers will only be convinced of the practical relevance of fractal traffic models by direct arguments, concerning the impact of fractal properties on network performance. Thus fractal traffic (stochastic modelling, statistical inference) has been a new task for researchers. While non-fractal models have inherently short-range dependence, it is known that adding parameters can lead to models with approximate fractal features. A judicious choice of a traffic model could lead to tractable models capable of approximating their intractable counterparts (and may work for some performance aspects). Therefore there is still a need to study traditional classical queueing network models. It is worth mentioning that long range dependence properties of traffic processes can be basically different when viewed under the continuous time stationary regime versus the Palm stationary regime therefore it is once again important to use the Palm theory.

Classical Networks. Classical networks described by Kelly [1979], and by Jackson [1957] or Gordon-Newell [1971] still remain in the range of interest of many researchers as basic tractable models, because of many interesting features such as product form, insensitivity, Poisson flows: Burke's [1956], product form for sojourn times (see Serfozo [1999] where Palm measures, stochastic intensities and time reversal are utilized). Large scale networks are interesting from topological point of view. Internet seen as a random graph has its vertex distribution following a power law. This is a surprising fact stimulating researches to use random graph theory, spectral graph theory and other methods to build new models, however researching classical models with "large" parameters remains to be important. One of the most important features of classical networks is a widespread property of being in some sense stochastically monotone. Various monotonicity and stochastic ordering results for queues are scattered in many books and very numerous papers in the existing literature, see for example parts of books by Baccelli and Bremaud [2003], Chen and Yao [2001], Glasserman and Yao [1994], Last and Brandt [1995], Müller and Stoyan [2002], Ridder [1987], Shaked and Shanthikumar [2007], Szekli [1995], Van Doorn [1981] among others.

The number of articles on various aspects of stochastic ordering for queueing systems is so large that a task of over-viewing them does not seem to be a reasonable one. Therefore this chapter concentrates only on results which are essentially for multi-node networks, excluding pure single systems results. Even with this restriction this text is certainly not complete in any sense. Formal definitions of classical networks models are recalled in order to unify notation. Networks with breakdowns are less known and the product formula for them is rather new.

It is very often that for simple models even elementary questions are not easy to answer. In order to illustrate this point consider a simple example of an open queueing network which is the Simon–Foley [1979] network of single server queues, see Figure 1. A customer traversing path $(1, 2, 3)$ can be overtaken by customers proceeding directly to node 3 when departing from node 1. This is one of the reasons why the traffic structure in a network can be very complicated and not easy to analyze. Simon and Foley [1979] proved that the vector (ξ_1, ξ_2, ξ_3) of the successive sojourn times for a customer traversing path $(1, 2, 3)$ has positively correlated components ξ_1 and ξ_3, .

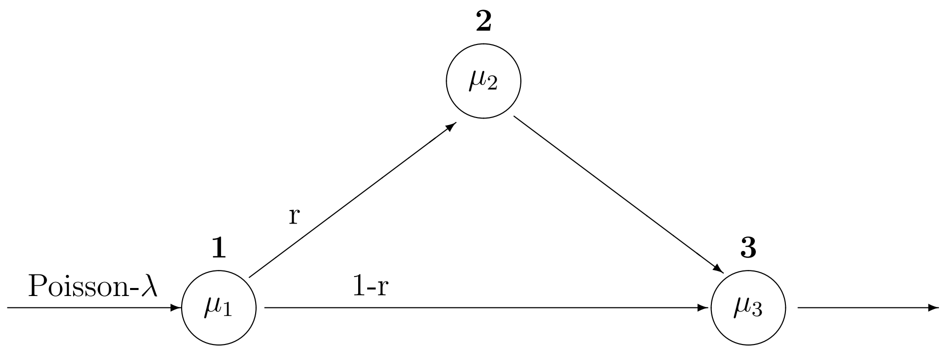

Figure 1: The Simon–Foley network with overtaking due to the network topology

While the Simon–Foley network provides us with an example where overtaking is due to the topological structure of the network, an early example of Burke [1969] (see Figure 2) shows that overtaking due to the internal node structure prevents sojourn times on a linear path from independence as well: a three–station path $(1, 2, 3)$ with a multiserver node 2 ($m_2 > 1$) has dependent components ξ_1 and ξ_3.

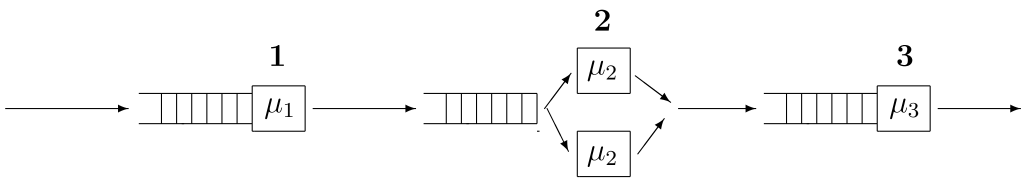

Figure 2: The tandem network with overtaking due to the internal node structure

The question whether on the three–station path of the Simon–Foley network the complete sojourn time vector (ξ_1, ξ_2, ξ_3) is associated remains unanswered. We shall give some related results on sojourn times later in this chapter, also for closed networks. Before doing this we shall recall a general description of classical queueing networks, and shall discuss in a detail the topic of stochastic monotonicity of networks which is a basic property connected with stochastic comparison of networks.

1.1.1 Jackson networks

Consider a **Jackson network** which consists of J numbered nodes, denoted by $\mathrm{J} = \{1, \ldots, J\}$. Station $j \in \mathrm{J}$, is a single server queue with infinite waiting room under FCFS (First Come First Served) regime. Customers in the network are indistinguishable. There is an external Poisson arrival stream with intensity $\lambda > 0$ and arriving customers are sent to node j with probability $\tilde{r}_{0j}$, $\sum_{j=1}^{J} \tilde{r}_{0j} = r \leq 1$. The quantity $\tilde{r}_{00} := 1 - r$ is then the rejection probability with that customers immediately leave the network again. Customers arriving at node j from the outside or from other nodes request a service which is exponentially distributed with mean 1. Service at node j is provided with intensity $\mu_j(n_j) > 0$ $(\mu_j(0) := 0)$, where n_j is the number of customers at node j including the one being served. All service times and arrival processes are assumed to be independent.

A customer departing from node i immediately proceeds to node i with probability $\tilde{r}_{ij} \geq 0$ or departs from the network with probability $\tilde{r}_{i0}$. The routing is independent of the past of the system given the momentary node where the customer is. Let $\mathrm{J}_0 := \mathrm{J} \cup \{0\}$. We assume that the matrix $\tilde{R} := (\tilde{r}_{ij}, i, j \in \mathrm{J}_0)$ is irreducible.

Let $\tilde{X}_j(t)$ be the number of customers present at node j at time $t \geq 0$. Then $\tilde{X}(t) = (\tilde{X}_1(t), \ldots, \tilde{X}_J(t))$ is the joint queue length vector at time instant $t \geq 0$ and $\tilde{\mathbf{X}} := (\tilde{X}(t), t \geq 0)$ is the joint queue length process with state space $(\mathbb{E}, \prec) := (\mathbb{N}^J, \leq^J)$ (where $\leq^J$ denotes the standard coordinate-wise ordering, $\mathbb{N} = \{0, 1, 2, \ldots\}$).

The following theorem is classical (Jackson [1957]).

Theorem 1.1.1 *Under the above assumptions the queueing process $\tilde{\mathbf{X}}$ is a Markov process with infinitesimal operator $Q^{\tilde{X}} = (q^{\tilde{X}}(x, y) : x, y \in E)$ given by*

$$q^{\tilde{X}}(n_1, \ldots, n_i, \ldots, n_J; n_1, \ldots, n_i + 1, \ldots, n_J) = \lambda \tilde{r}_{0i}$$

and for $n_i > 0$

$$q^{\tilde{X}}(n_1, \ldots, n_i, \ldots, n_J; n_1, \ldots, n_i - 1, \ldots, n_J) = \mu_i(n_i) \tilde{r}_{i0},$$

$$q^{\tilde{X}}(n_1, \ldots, n_i, \ldots, n_j, \ldots, n_J; n_1, \ldots, n_i - 1, \ldots, n_j + 1, \ldots, n_J) = \mu_i(n_i) \tilde{r}_{ij}\,.$$

Furthermore

$$q^{\tilde{X}}(x, x) = -\sum_{y \in E \setminus \{x\}} q^{\tilde{X}}(x, y) \text{ and } q^{\tilde{X}}(x, y) = 0 \text{ otherwise.}$$

The parameters of a Jackson network are: the arrival intensity λ, the routing matrix $\tilde{R}$ (with its routing vector $\tilde{\boldsymbol{\eta}}$), the vector of service rates $\boldsymbol{\mu} = (\mu_1(\cdot), \ldots, \mu_J(\cdot))$, and the number of nodes J. We shall use the symbol $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$ to denote such a Jackson network.

1.1.2 Gordon-Newell networks

By a **Gordon-Newell** network we mean a closed network with $N \geq 1$ customers cycling. The node structure is the same as in a Jackson network, the routing of the customers is

Markovian, governed by an irreducible stochastic matrix $R = (r_{ij}, 1 \le i, j \le J)$. The Gordon-Newell network process $\mathbf{X}$ with state space $\mathbb{E}_N = \{\mathbf{n} = (n_1, \ldots, n_J) : n_j \in \{0, 1, \ldots\}, j = 1, \ldots, J, n_1 + \ldots + n_J = N\}$ is a generalized migration process, describing the joint queue length vector with the following transition rates:

$$q^{\mathbf{X}}(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = r_{ij}\mu_i(n_i), \qquad n_i \ge 1,$$

and $q^{\mathbf{X}}(\mathbf{n}, \mathbf{n}') = 0$ for all other states, where $\mathbf{e}_j$ is the j-th base vector in $\mathbb{R}^J$.

We assume that every node can be reached from any other node in a finite number of steps with positive probability. This ensures that the set of routing (traffic) equations

$$\eta_j = \sum_{i=1}^{J} \eta_i r_{ij}, \qquad j = 1, \ldots, J, \tag{1.1.1}$$

has a unique probability solution which we denote by $\boldsymbol{\eta} = (\eta_j : j = 1, \ldots, J)$.

If at node $j \in \{1, \ldots, J\}$, n_j customers are present (including the one in service, if any) the service rate is $\mu_j(n_j) \ge 0$; we set $\mu_j(0) = 0$. Service and routing processes are independent.

Let $\mathbf{X} = (X(t) : t \ge 0)$ denote the vector process recording the joint queue lengths in the network for time t. For $t \in \mathbb{R}_+$, $X(t) = (X_1(t), \ldots, X_J(t))$ reads: at time t there are $X_j(t)$ customers present at node j, either in service or waiting. The assumptions put on the system imply that $\mathbf{X}$ is a strong Markov process with infinitesimal operator $Q^{\mathbf{X}} = (q^{\mathbf{X}}(x, y) : x, y \in \mathbb{E}_N)$.

The parameters of a Gordon-Newell network are: the routing matrix R, the vector of service rates $\boldsymbol{\mu} = (\mu_1(\cdot), \ldots, \mu_J(\cdot))$, the number of nodes J, and the number of customers N. We shall use the symbol $(R/\boldsymbol{\mu}/J + N)$ to denote such a network.

1.1.3 Ergodicity of classical networks

For Jackson networks by the product formula for stationary distribution we mean the next formula appearing in the following theorem.

Theorem 1.1.2 *The unique invariant and limiting distribution $\tilde{\pi}^J$ of the Jackson network state process $\tilde{\mathbf{X}}$ is given by*

$$\tilde{\pi}^J(n_1, \ldots, n_J) = K(J)^{-1} \prod_{j=1}^{J} \prod_{k=1}^{n_j} \frac{\tilde{\eta}_j}{\mu_j(k)}, \quad (n_1, \ldots, n_J) \in \mathbb{N}^J \tag{1.1.2}$$

with the normalization constant $K(J) = \prod_{j=1}^{J} \left(1 + \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{\tilde{\eta}_j}{\mu_j(k)}\right)$, and with $\tilde{\boldsymbol{\eta}} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_J)$ the unique solution of the routing (or traffic) equation of the network:

$$\tilde{\eta}_j = \tilde{r}_{0j}\lambda + \sum_{i=1}^{J} \tilde{\eta}_i \tilde{r}_{ij} \quad , \ j \in \mathrm{J}. \tag{1.1.3}$$

We have therefore that $\tilde{\pi}^J(n_1,\ldots,n_J)=\prod_{j=1}^J \tilde{\pi}_j^J(n_j)$, for the marginal distributions $\tilde{\pi}_j^J(n)=\tilde{\pi}_j^J(0)\prod_{k=1}^n \frac{\tilde{\eta}_j}{\mu_j(k)}$, for $n\geq 1$, and $\tilde{\pi}_j^J(0)=\left(1+\sum_{n=1}^\infty \prod_{k=1}^n \frac{\tilde{\eta}_j}{\mu_j(k)}\right)^{-1}$, $j=1,\ldots,J$.

$\tilde{\eta}$ is usually not a stochastic vector and we define the unique stochastic solution of (1.1.3) by

$$\boldsymbol{\xi}=(\xi_j : j=0,1,\ldots,J). \tag{1.1.4}$$

Regarding ergodicity of closed networks the following theorem is classical (Gordon, Newell [1967]).

Theorem 1.1.3 *The process* $\mathbf{X}$ *is ergodic and its unique steady–state and limiting distribution is given by*

$$\pi^{(N,J)}(\boldsymbol{n})=G(N,J)^{-1}\prod_{j=1}^J\prod_{k=1}^{n_j}\frac{\eta_j}{\mu_j(k)}, \tag{1.1.5}$$

for $\boldsymbol{n}\in\mathbb{E}_N$*, (for products with upper limit* $n_j=0$ *we set value 1) where*

$$G(N,J)=\sum_{n_1+\ldots+n_J=N}\prod_{j=1}^J\prod_{k=1}^{n_j}\frac{\eta_j}{\mu_j(k)}$$

is the norming constant.

Define independent random variables Y_j, $j=1,\ldots,J$ such that

$$\Pr(Y_j=0)=\left(1+\sum_{n=1}^\infty\prod_{k=1}^n\frac{\eta_j}{\mu_j(k)}\right)^{-1},\ \Pr(Y_j=n)=\Pr(Y_j=0)\prod_{k=1}^n\frac{\eta_j}{\mu_j(k)}.$$

Note that we have then

$$\pi^{(N,J)}(\boldsymbol{n})=\prod_{j=1}^J\Pr(Y_j=n_j)/\Pr(Y_1+\ldots+Y_J=N)=\Pr(Y_1=n_1,\ldots,Y_J=n_J\mid Y_1+\ldots+Y_J=N).$$

Therefore the stationary distribution in a closed network can be interpreted as the conditional distribution of an open network, given the number of customers present.

A natural measure of performance for a network in stationary conditions is for each j, $E(\mu_j(X_j(t)))$. It is well known that

$$\begin{aligned}E(\mu_j(X_j(t)))&=\eta_j\Pr(Y_1+\ldots+Y_J=N-1)/\Pr(Y_1+\ldots+Y_J=N)\\&=\eta_jG(N-1,J)/G(N,J).\end{aligned}$$

Therefore $E(\mu_j(X_j(t)))/\eta_j$ does not depend on j and is called the **throughput** of this network. We denote the throughput $G(N-1,J)/G(N,J)$ of a Gordon-Newell network by

$$TH(R/\boldsymbol{\mu}/J+N).$$

It is interesting to compare throughput for two structured networks with different routing and/or service properties. We shall present such results later in this chapter.

1.2 Stochastic monotonicity and related properties for classical networks

1.2.1 Stochastic orders and monotonicity

From a general point of view we shall consider probability measures on a partially ordered Polish space $\mathbb{E}$ endowed with a closed partial order $\prec$ and the Borel $\sigma-$algebra $\mathcal{E}$ denoted by $(\mathbb{E},\mathcal{E},\prec)$ along with random elements $X:(\Omega,\mathcal{F},P)\to(\mathbb{E},\mathcal{E},\prec)$. We denote by $\mathcal{I}^*(\mathbb{E})$ ($\mathcal{I}^*_+(\mathbb{E})$) the set of all real valued increasing measurable bounded (non-negative) functions on $\mathbb{E}$ (f increasing means: for all x,y, $x\prec y$ implies $f(x)\le f(y)$), and $\mathcal{I}(\mathbb{E})$ the set of all increasing sets (i.e. sets for which indicator functions are increasing). The decreasing analogues are denoted by $\mathcal{D}^*(\mathbb{E})$, ($\mathcal{D}^*_+(\mathbb{E})$) and $\mathcal{D}(\mathbb{E})$, respectively. For $A\subseteq\mathbb{E}$ we denote $A^{\uparrow}:=\{y\in\mathbb{E}:y\succ x\ for\ some\ x\in A\}$, and $A^{\downarrow}:=\{y\in\mathbb{E}:y\prec x\ for\ some\ x\in A\}$. Further we define $\mathcal{I}_p(\mathbb{E})=\{\{x\}^{\uparrow}:x\in\mathbb{E}\}$ and $\mathcal{D}_p(\mathbb{E})=\{\{x\}^{\downarrow}:x\in\mathbb{E}\}$, the classes of one-point generated increasing, resp. decreasing, sets.

For product spaces we shall use the following notation, $\mathbb{E}^{(n)}=\mathbb{E}_1\times...\times\mathbb{E}_n$, for $\mathbb{E}_i$ partially ordered Polish spaces ($i=1,\dots,n$). If $\mathbb{E}_i=\mathbb{E}$ for all i then we write $\mathbb{E}^n$ instead of $\mathbb{E}^{(n)}$. Analogously we write $\mathbb{E}^{(\infty)}$ and $\mathbb{E}^{\infty}$ for infinite products. Product spaces will be considered with the product topology. Elements of $\mathbb{E}^{(n)}$ will be denoted by $x^{(n)}=(x_1,\dots,x_n)$, of $\mathbb{E}^{(\infty)}$ by $x^{(\infty)}$. For random elements we use capital letters in this notation. We denote the coordinatewise ordering on $\mathbb{E}^{(n)}$ by $\prec^{(n)}$.

The theory of dependence order via integral orders for finite dimensional vectors is well established, surveys can be found in Mueller and Stoyan [2002], Joe [1997], and Szekli [1995]. In recent years this theory and its applications were extended to dependence order of stochastic processes, see for examples with state spaces $\mathbb{R}^n$ or subsets thereof the work of Hu and Pan [2000] and Li and Xu [2000], and for a more general approach to Markov processes in discrete and continuous time with general partially ordered state space Daduna and Szekli [2006].

Definition 1.2.1 *We say that two random elements* $\mathbf{X},\mathbf{Y}$ *of* $(\mathbb{E},\mathcal{E},\prec)$ *are stochastically ordered (and write* $\mathbf{X}\prec_{st}\mathbf{Y}$ *or* $\mathbf{Y}\succ_{st}\mathbf{X}$*) if* $Ef(\mathbf{X})\le Ef(\mathbf{Y})$ *for all* $f\in\mathcal{I}^*(\mathbb{E})$, *for which the expectations exist.*

In the theory of stochastic orders and especially in specific applications a well established procedure is to tailor suitable classes of functions, which via integrals over these functions extract the required properties of the models under consideration. The most well known example is the class of integrals over convex functions which describes the volatility of processes and therefore the risks connected with the process.

Similar ideas will guide our investigations of network processes $\mathbf{X}=(X_t:t\ge0)$ and $\mathbf{Y}=(Y_t:t\ge0)$. These are comparable in the concordance ordering, $\mathbf{X}\prec_{cc}\mathbf{Y}$, if for each pair $(X_{t_1},\dots,X_{t_n})$ and $(Y_{t_1},\dots,Y_{t_n})$ it holds

$$E\left[\prod_{i=1}^{n}f_i(X_{t_i})\right]\le E\left[\prod_{i=1}^{n}f_i(Y_{t_i})\right], \tag{1.2.1}$$

for all increasing functions f_i, and for all decreasing functions as well (i.e. for all comonotone functions). It is our task to identify subclasses $\mathcal{F}$ of functions such that (1.2.1) holds for all comonotone functions which are additionally in $\mathcal{F}$ and that additionally $\mathbf{X}$ and $\mathbf{Y}$ fulfill the corresponding stochastic monotonicity properties with respect to the integral order defined via intersecting the class of monotone functions with $\mathcal{F}$.

In the general theory of concordance order the set (1.2.1) of inequalities implies that $\mathbf{X}$ and $\mathbf{Y}$ have the same marginals and that standard covariances $cov(f(X_s), g(X_t)) \leq cov(f(Y_s), g(Y_t))$ are ordered for comonotone f, g. If $\mathcal{F}$ is sufficiently rich, these properties will be maintained.

The introduced above dependence ordering can be generalized to more general spaces. For $(\mathbb{E}, \mathcal{E}, \prec)$ which is a lattice (i.e. for any $x, y \in \mathbb{E}$ there exist a largest lower bound $x \wedge y \in \mathbb{E}$ and a smallest upper bound $x \vee y \in \mathbb{E}$ uniquely determined) we denote by $\mathcal{L}_{sm}(\mathbb{E})$ the set of all real valued bounded measurable supermodular functions on $\mathbb{E}$, i.e., functions which fulfill for all $x, y \in \mathbb{E}$

$$f(x \wedge y) + f(x \vee y) \geq f(x) + f(y).$$

Definition 1.2.2 *We say that two random elements* $\mathbf{X}, \mathbf{Y}$ *of* $(\mathbb{E}, \mathcal{E}, \prec)$ *are supermodular stochastically ordered (and write* $\mathbf{X} \prec_{sm} \mathbf{Y}$ *or* $\mathbf{Y} \succ_{sm} \mathbf{X}$*) if* $Ef(\mathbf{X}) \leq Ef(\mathbf{Y})$ *for all* $f \in \mathcal{L}_{sm}(\mathbb{E})$, *for which the expectations exist.*

A weaker than $\prec_{sm}$ can be defined on product spaces. A function $f : \mathbb{E}^{(2)} \to \mathbb{R}$ has *isotone differences* if for $x_1 \prec_1 x_1'$, $x_2 \prec_2 x_2'$ we have

$$f(x_1', x_2') - f(x_1, x_2') \geq f(x_1', x_2) - f(x_1, x_2). \tag{1.2.2}$$

A function $f : \mathbb{E}^{(n)} \to \mathbb{R}$ has *isotone differences* if (1.2.2) is satisfied for any pair i, j of coordinates, whereas the remaining variables are fixed. If $\mathbb{E}_i$, $i = 1, \ldots, n$ are totally ordered then both definitions are equivalent. The class of functions with isotone differences, defined by (1.2.2) we denote by $\mathcal{L}_{\text{idif}}(\mathbb{E}^{(n)})$. Note that the definition of a function with isotone differences does not require that $\mathbb{E}_i$ are lattices. If, additionally, f is taken to be increasing we shall write $f \in \mathcal{L}_{\text{i-idif}}(E^{(n)})$. The following lemma is due to Heyman and Sobel [1984].

Lemma 1.2.3 .

(i) *Let* $\mathbb{E}_1, \mathbb{E}_2, \ldots, \mathbb{E}_n$ *be lattices. If* f *is supermodular on* $(\mathbb{E}^{(n)}, \prec^{(n)})$ *then it has also isotone differences.*

(ii) *Let* $\mathbb{E}_1, \ldots, \mathbb{E}_n$ *be totally ordered. If* f *has isotone differences on* $(\mathbb{E}^{(n)}, \prec^{(n)})$ *then it is also supermodular.*

The above lemma implies that in case of totally ordered spaces both notions are equivalent. This is not the case when $\mathbb{E}_i$, $i = 1 \ldots, n$, are partially (but not linearly) ordered.

Definition 1.2.4 *Let* $\mathbf{X} = (X_1, \ldots, X_n)$, $\mathbf{Y} = (Y_1, \ldots, Y_n)$ *be random vectors with values in* $\mathbb{E}^{(n)}$.

(i) $\mathbf{X}$ *is smaller than* $\mathbf{Y}$ *in the isotone differences ordering (*$\mathbf{X} \prec_{idif} \mathbf{Y}$*) if*

$$E\left[f(X_1 \ldots, X_n))\right] \leq E\left[f(Y_1, \ldots, Y_n)\right]$$

for all $f \in \mathcal{L}_{\text{idif}}(E^{(n)})$.

Before going back to networks let us summarize basic definitions for Markov processes.

Discrete time

Let $\mathbf{X} = (X_t : t \in \mathbb{Z})$ and $\mathbf{Y} = (Y_t : t \in \mathbb{Z})$, $X_t, Y_t : (\Omega,\mathcal{F},P) \to (\mathbb{E}, \mathcal{E}, \prec)$, be discrete time, stationary, homogeneous Markov processes. Assume that π is an invariant (stationary) one–dimensional marginal distribution the same for both $\mathbf{X}$ and $\mathbf{Y}$, and denote the 1–step transition kernels for $\mathbf{X}$ and $\mathbf{Y}$, by $K^X : \mathbb{E} \times \mathcal{E} \to [0, 1]$, and $K^Y : \mathbb{E} \times \mathcal{E} \to [0, 1]$, respectively. Denote the respective transition kernels for the time reversed processes $\overleftarrow{\mathbf{X}}$, $\overleftarrow{\mathbf{Y}}$ by $\overleftarrow{K}^X$, $\overleftarrow{K}^Y$. We say that a stochastic kernel $K : \mathbb{E} \times \mathcal{E} \to [0, 1]$ is **stochastically monotone** if $\int f(x)K(s, dx)$ is increasing in s for each $f \in \mathcal{I}^*(\mathbb{E})$. It is known (see e.g. Müller and Stoyan [2002], section 5.2) that a stochastic kernel K is stochastically monotone iff $K(x, \cdot) \prec_{st} K(y, \cdot)$ for all $x \prec y$. Another equivalent condition for this property is that $\mu K \prec_{st} \nu K$ for all $\mu \prec_{st} \nu$, where μK denotes the measure defined by $\mu K(A) = \int K(s, A)\mu(ds)$, $A \in \mathcal{E}$. It is worth mentioning that for $\mathbb{E} = \mathbb{N}$, using traditional notation $P^X = [p^X(i, j)]_{i,j\in\mathbb{N}}$ for the transition matrix of $\mathbf{X}$ (that is $p^X(i, j) := K^X(i, \{j\})$), stochastic monotonicity can be expressed in a very simple form, namely (see Keilson and Kester [1977]), we say that P^X is stochastically monotone if

$$T^{-1}P^XT(i, j) \geq 0, \; i, j \in \mathbb{N}, \tag{1.2.3}$$

where T is the lower triangular matrix with zeros above the main diagonal and ones elsewhere.

Continuous time

Let $\mathbf{X} = (X_t : t \in \mathbb{R})$ and $\mathbf{Y} = (Y_t : t \in \mathbb{R})$, $X_t, Y_t : (\Omega,\mathcal{F},P) \to (\mathbb{E}, \mathcal{E}, \prec)$, be stationary homogeneous Markov processes. Denote the corresponding families of transition kernels of $\mathbf{X}$, and $\mathbf{Y}$, by $\mathbb{K}^X = (K_t^X : \mathbb{E} \times \mathcal{E} \to [0, 1] : t \geq 0)$, and $\mathbb{K}^Y = (K_t^Y : \mathbb{E} \times \mathcal{E} \to [0, 1] : t \geq 0)$, respectively, and the respective transition kernels for the stationary time reversed processes $\overleftarrow{\mathbf{X}}$, $\overleftarrow{\mathbf{Y}}$ by $\overleftarrow{\mathbb{K}}^X = (\overleftarrow{K}_t^X : \mathbb{E} \times \mathcal{E} \to [0, 1] : t \geq 0)$, and $\overleftarrow{\mathbb{K}}^Y = (\overleftarrow{K}_t^Y : \mathbb{E} \times \mathcal{E} \to [0, 1] : t \geq 0)$, respectively. Assume that π is an invariant distribution common for both $\mathbb{K}^X$ and $\mathbb{K}^Y$, that is $\int K_t^X(x, dy)\pi(dx) = \int K_t^Y(x, dy)\pi(dx) = \pi(dy)$, for all $t > 0$. We say that $\mathbb{K}^X$ ($\mathbb{K}^Y$) is **stochastically monotone** if for each $t > 0$, K_t^X (K_t^Y) is stochastically monotone as defined previously. If $\mathbb{E}$ is countable and $Q^{\mathbf{X}} = [q^{\mathbf{X}}(x, y)]$ and $Q^{\mathbf{Y}} = [q^{\mathbf{Y}}(x, y)]$ denote intensity matrices (infinitesimal generators) of the corresponding chains $\mathbf{X}$ and $\mathbf{Y}$ then the following condition due to Massey [1987] is useful: if $Q^{\mathbf{X}}$ is bounded, conservative then $\mathbb{K}^X$ is stochastically monotone iff

$$\sum_{y\in F} q^{\mathbf{X}}(x_1, y) \leq \sum_{y\in F} q^{\mathbf{X}}(x_2, y),$$

for all $F \in \mathcal{I}(\mathbb{E})$, and $x_1 \prec x_2$ such that $x_1 \in F$ or $x_2 \notin F$. An analogous condition for arbitrary time continuous Markov jump processes (also for unbounded generators) is given by Mu-Fa Chen [2004], Theorem 5.47. It is worth mentioning that if $\mathbb{E} = \mathbb{N}$ then similarly to (1.2.3), we say that $Q^{\mathbf{X}} = [q^{\mathbf{X}}(i, j)]_{i,j\in\mathbb{N}}$ is stochastically monotone if $T^{-1}Q^{\mathbf{X}}T(i, j) \geq 0$ for all $i \neq j$.

1.2.2 Stochastic monotonicity and networks

The fundamental property of stochastic monotonicty of networks is presently a classical one. Massey [1987], Propostion 8.1, proved this property using analytical methods for Jackson networks with constant service rates, Daduna and Szekli [1995], Corollary 4.1, utilized a coupling argument combined with point processes description for variable rates and closed networks. Lindvall [1997], p. 7, used a coupling proof for Jackson networks.

Property 1.2.5 *Consider* $\tilde{\mathbf{X}} := (\tilde{X}(t), t \geq 0)$ *the joint queue length process in Jackson network* $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$ *as a Markov process with the partially ordered state space* $(\mathbb{E}, \prec) := (\mathbb{N}^J, \leq^J)$*, and* $\mathbf{X} = (X(t) : t \geq 0)$ *the process recording the joint queue lengths in the Gordon-Newell network* $(R/\boldsymbol{\mu}/J+N)$ *as a Markov process with state space* $\mathbb{E}_N$ *also ordered with* $\leq^J$ *(the standard coordinate-wise ordering). If* $\boldsymbol{\mu}$ *is increasing as a function of the number of customers then for both processes the corresponding families of transition kernels are stochastically monotone with respect to* $\leq^J$.

Remark 1.2.6 For a formulation of the above result in terms of marked point processes see Last and Brandt [1995], Theorem 9.3.18. For a version of the stochastic monotonicity property for Jackson networks with infinite denumerable number of nodes see Kelbert et al. [1988]. For a refined stochastic monotonicity property for partition separated orderings see Proposition 8.1 in Massey [1987]. For generalizations to Jackson type networks with batch movements see Economou [2003a] and [2003b].

Apart from the traditional, coordinatewise ordering on the state space it is possible and reasonable to consider other orderings and monotonicities which for example turned out to be useful to describe special properties of tandems.

For two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$, we define **partial sum** order by

$$\mathbf{x} \leq_* \mathbf{y} \text{ if } \sum_{i=1}^{j} x_i \leq \sum_{i=1}^{j} y_j, \qquad j = 1, \ldots, n.$$

The next property was first stated by Whitt [1981], and restated using other methods in Massey [1987], Theorem 8.3, and Daduna and Szekli [1995], Proposition 4.4.

Property 1.2.7 *Consider* $\tilde{\mathbf{X}} := (\tilde{X}(t), t \geq 0)$ *the joint queue length process in Jackson network* $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$ *as a Markov process with the partially ordered state space* $(\mathbb{E}, \prec) := (\mathbb{N}^J, \leq_*)$*. Assume that* $\boldsymbol{\mu}$ *is increasing as a function of the number of customers. Then the corresponding family of transition kernels of* $\tilde{\mathbf{X}}$ *is stochastically monotone with respect to* $\leq_*$ *if and only if* $i, j \in \mathrm{J}$ *and* $\tilde{r}(i,j) > 0$ *implies that* $j = i+1$ *or* $j = i-1$*, and* $\tilde{r}(i, 0) > 0$ *iff i=J.*

An interesting monotonicity property for increments of cumulative number of customers in Jackson networks starting empty was proved by Lindvall [1997] using coupling methods.

Property 1.2.8 *Consider* $\tilde{\mathbf{X}}$ *the joint queue length process in Jackson network* $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$ *such that at time 0 the system is empty. Assume that* $\boldsymbol{\mu}$ *is increasing as a function of the number of customers. Then for each* $\epsilon > 0$, $\sum_{j=1}^{J} \tilde{X}_j(t+\epsilon) - \sum_{j=1}^{J} \tilde{X}_j(t)$ *is stochastically* $(\leq_{st})$ *decreasing as a function of* t.

1.2.3 Bounds in transient state

Analytical approach of Massey [1984], [1986] resulted in a triansient bound for Jackson networks which was generalized then by Tsoucas and Walrand [1984]. The joint distribution of the number of customers on an upper orthant can be bounded from above by the product of the corresponding state distributions of single systems at any time provided they start from the same state. This is a useful upper bound on the probability of overload in transient Jackson networks.

Property 1.2.9 *Consider* $\tilde{\mathbf{X}}$ *the joint queue length process in Jackson network* $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$ *such that* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_J)$ *is constant as a function of the number of customers. Independently, for each* $j \in \mathrm{J}$ *denote by* $\hat{X}_j(t)$ *the number of customers in* $M/M/1$ *FCFS classical system with the arrival rate*

$$\tilde{\lambda}_j = \tilde{r}_{0j}\lambda + \sum_{i=1}^{J} \mu_i \tilde{r}_{ij},$$

and service rate μ_j*. Then* $\tilde{X}(0) = (\hat{X}_1(0), \ldots, \hat{X}_J(0)$ *implies*

$$P(\tilde{X}(t) \geq \boldsymbol{a}) \leq P(\hat{X}_1(t) \geq a_1) \cdots P(\hat{X}_j(t) \geq a_j),$$

for each $t > 0$ *and* $\boldsymbol{a} = (a_1, \ldots, a_j) \in \mathbb{R}^J$*.*

1.2.4 Bounds in stationary state

Bounds for time stationary evolution of networks have a different nature than transient bounds. The next property can be found in Daduna and Szekli [1995], Corollary 5.1.

Property 1.2.10 *Consider* $\tilde{\mathbf{X}}$ *the joint queue length process in Jackson network* $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$ *such that* $\boldsymbol{\mu} = (\mu_1(\cdot), \ldots, \mu_J(\cdot))$ *is increasing as a function of the number of customers. Then in stationary conditions*

$$E(f[\tilde{X}(t_1), \ldots, \tilde{X}(t_i)]g[\tilde{X}(t_{i+1}), \ldots, \tilde{X}(t_k)]) \geq E(f[\tilde{X}(t_1), \ldots, \tilde{X}(t_i)])E(g[\tilde{X}(t_{i+1}), \ldots, \tilde{X}(t_k)])$$

for all non-decreasing real $f, g,$ *and* $0 \leq t_1 < \ldots < t_k, i < k, \quad i, k \in \mathbb{N}$*.*

This inequality can be written as

$$\mathrm{Cov}(f(\tilde{X}(t_i), i = 1, \ldots, k), g(\tilde{X}(t_i), i = k+1, \ldots, n)) \geq 0, \tag{1.2.4}$$

for all $f \in \mathcal{I}^*(\mathbb{R}^k), g \in \mathcal{I}^*(\mathbb{R}^{n-k}), k = 1, \ldots, n-1, t_1 < \ldots < t_n$. Note that the property (1.2.4) is a rather strong positive dependence property in the time evolution of $\tilde{\mathbf{X}}$. We shall recall now some definitions from the theory of positive dependence. A natural way to define positive dependence for a random vector (or alternatively for a distribution on a state space) $\mathbf{X} = (X_1, \ldots, X_n)$ is to use a dependency ordering in order to compare it with its iid version, i.e. with $\mathbf{X}^{\perp} = (X_1^{\perp}, \ldots, X_n^{\perp})$, where $X_i =^d X_i^{\perp}$, and $(X_1^{\perp}, \ldots, X_n^{\perp})$ being independent. For example, if $\mathbb{E} = \mathbb{R}$, $\mathbf{X}^{\perp} \leq_{cc} \mathbf{X}$ is equivalent to the fact that $\mathbf{X}$ is positively orthant dependent (POD) (for definitions of this and other related concepts see e.g. Szekli [1995]). POD is weaker than association of $\mathbf{X}$ defined by the condition that $\mathrm{Cov}(f(\mathbf{X}), g(\mathbf{X})) \geq 0$ for all $f, g \in \mathcal{I}^*(\mathbb{R}^n)$. However, it is not possible to characterize association in terms of some ordering, that is by stating that $\mathbf{X}$ is greater than $\mathbf{X}^{\perp}$ for some ordering. But Christofides and Veggelatou [2004] show that association implies that $\mathbf{X}^{\perp} \leq_{sm} \mathbf{X}$ (positive supermodular dependence - PSMD). In fact they show that the weak association (defined by $\mathrm{Cov}(f(X_i, i \in A), g(X_i, i \in A^c)) \geq 0$ for all real, increasing f, g of appropriate dimension, and all $A \subset \{1, \ldots, n\}$) implies PSMD. Rüschendorf [2004] defined a weaker than weak association positive dependence by $\mathrm{Cov}(I_{(X_i > t)}, g(X_{i+1}, \ldots, X_n)) \geq 0$ for all increasing g, all $t \in \mathbb{R}$, and all $i = 1, \ldots, n-1$, which he called weak association in sequence (WAS). He showed that WAS implies PSMD. Hu et al. [2004] gave counterexamples showing that the mentioned positive dependence concepts are really different.

Note that the property (1.2.4) implies that $(f_1(\tilde{X}(t_1)), \ldots, f_n(\tilde{X}(t_n)))$ is weakly associated in sequence for all $f_i \in \mathcal{I}^*_+(\mathbb{R}^J)$, and therefore is also PSMD, which implies possibility to compare maxima, minima and other supermodular functionals of the time evolution of $\tilde{\mathbf{X}}$, $(f_1(\tilde{X}(t_1)), \ldots, f_n(\tilde{X}(t_n)))$ with the corresponding independent versions (separated single queue systems). This is in accordance with intuitions since the joint time evolution of a network should generate more correlations than independent single queue systems.

It is worth mentioning that in order to obtain a joint space and time positive dependence for a Markov process $\mathbf{X}$ one requires additional assumptions. For example it is known (see e.g. Liggett [1985], Szekli [1995], Theorem A, section 3.7.) that if $I\!\!K^X$ is stochastically monotone, π associated on $\mathbb{E}$, and (so called up-down property) $Q^{\mathbf{X}}(fg) \geq fQ^{\mathbf{X}}g + gQ^{\mathbf{X}}f$, for all f, g increasing then $\mathbf{X}$ is space-time associated (i.e. $\mathrm{Cov}(\phi(X_{t_i}, i = 1, \ldots, n), \psi(X_{t_i}, i = 1, \ldots, n)) \geq 0$, for all ϕ, ψ increasing). Unfortunately networks in general do not fulfill this up-down requirement therefore the last property needed another argument strongly based on stochastic monotonicity.

The next property is a corollary from the previous one but it is interesting to know that it is possible to extend this property to networks of infinite channel queues with arbitrary service time distribution, see Kanter [1985], Daduna and Szekli [1995], Corollary 5.2. In contrast to the transient case these bounds are lower bounds and are formulated with respect to the time evolution in stationary conditions.

Property 1.2.11 *Consider $\tilde{\mathbf{X}}$ the joint queue length process in Jackson network $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$ such that $\boldsymbol{\mu} = (\mu_1(\cdot), \ldots, \mu_J(\cdot))$ is increasing as a function of the number of customers. Independently, for each $j \in$ J, denote by $\hat{X}_j(t)$ the number of customers in $M/M(n)/1$ FIFO classical system with arrival rate $\tilde{\lambda}_j = \tilde{\eta}_j$ and service rate $\mu_j(\cdot)$. Then for processes in stationary conditions*

$$P(\tilde{X}(t_1) \geq (\leq)\boldsymbol{a}^1, \ldots, \tilde{X}(t_k) \geq (\leq)\boldsymbol{a}^k) \geq \prod_{1 \leq i \leq k, 1 \leq j \leq J} P(\hat{X}_j(t_i) \geq (\leq)a^i_j),$$

for each $t_1 < \cdots < t_k$, and $\boldsymbol{a}^k = (a^k_1, \ldots, a^k_j) \in \mathbb{R}^J$, $k \in \mathbb{N}$.

For open networks in stationary state positive correlations are prevailing. For closed networks however it is natural to expect negative correlations for the state in closed networks, but negative association is perhaps a bit surprising at the first glance. The next property can be found in Daduna and Szekli [1995], Proposition 5.3.

Property 1.2.12 *Consider* $\mathbf{X} = (X(t) : t \geq 0)$ *the process recording the joint queue lengths in the Gordon-Newell network* $(R/\boldsymbol{\mu}/J+N)$ *as a Markov process with state space* $\mathbb{E}_N$ *ordered with* $\leq^J$. *If* $\boldsymbol{\mu}$ *is increasing as a function of the number of customers then for every* $t > 0$, $X(t)$ *is negatively associated with respect to* $\leq^J$, *i.e.*

$$E(f(X_i(t), i \in \mathrm{I})g(X_j(t), j \in \mathrm{I}^c)) \leq E(f(X_i(t), i \in \mathrm{I}))E(g(X_j(t), j \in \mathrm{I}^c)),$$

for all increasing f, g, *and all* $\mathrm{I} \subset \mathrm{J}$.

For analogous result for discrete time queueing networks see Pestien and Ramakrishnan [2002]. Negative association can be used to obtain upper bounds on the joint distribution of the state vector.

1.2.5 Sojourn times in networks

Dependence properties for sojourn times

A path of length M in the network $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$ is a finite sequence of nodes $\mathcal{P} = (j_1, j_2, \ldots, j_M)$, not necessarily distinct, which a customer can visit consecutively, i.e., $\tilde{r}_{j_k, j_{k+1})} > 0, k = 1, \ldots, M-1$. For a customer traversing path $\mathcal{P}$ we denote by $(\xi_1, \xi_2, \ldots, \xi_M)$ the vector of his successive sojourn times at the nodes of the path. Strong interest is focused on determining the joint distribution of the vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ in equilibrium. In general this is an unsolved problem, explicit expressions are rare.

The first results were obtained by Reich [1957], [1963], and Burke [1956], [1968]. For closed cycles the parallel results were developed by Chow [1980], Schassberger and Daduna [1983], and Boxma, Kelly, and Konheim [1984]. Clearly in this case independence was not found due to the negative correlation of the queue lengths in the network, but a product form structure emerged there as well. The research which followed the mentioned early results was also concentrated on proving that similar results hold for *overtake–free* paths as well. Extensions to single server overtake–free paths for networks with general topology were obtained for the open network case by Walrand and Varaiya [1980] and Melamed [1982], and for closed networks by Kelly and Pollett [1983]. The result for overtake–free paths with multiserver stations at the beginning and the end of the path was proved by Schassberger and Daduna [1987]. (For a review see Boxma and Daduna [1990].)

The most prominent example where overtaking appears is the Simon–Foley [1979] network of single server queues, see Figure 1. As we have already mentioned before, the question whether on the three–station path of the Simon–Foley network the complete sojourn time vector (ξ_1, ξ_2, ξ_3) is associated remains unanswered. The methods provided by the proof of Foley and Kiessler [1989] seemingly do not apply to that problem. However it is possible to prove a little bit weaker dependence results. Probability measure used in this statement is the Palm probability with respect to the point process of arrivals to the first station.

Property 1.2.13 *Consider Jackson network* $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$ *with constant* $\boldsymbol{\mu}$, *and a path* $\mathcal{P}$ *consisting of three nodes which we assume to be numbered* $\mathcal{P} = (1, 2, 3)$. *In equilibrium, the* *successive sojourn times* (ξ_1, ξ_2, ξ_3) *of a customer on a three node path of distinct nodes are positive upper orthant dependent, i.e.*

$$P(\xi_1 \geq a_1, \xi_2 \geq a_2, \xi_3 \geq a_3) \geq P(\xi_1 \geq a_1)P(\xi_2 \geq a_2)P(\xi_3 \geq a_3)$$

.

More generally the above result holds true in open product form networks with multi-server nodes having general service disciplines and exponentially distributed service times or having symmetric service disciplines with generally distributed service times. Moreover this is true also for networks with customers of different types entering the network and possibly changing their types during their passage through the network. Here one may allow additionally that the service time distributions at symmetric nodes are type dependent, see Daduna and Szekli [2000]. For generalizations to four step walk in Jackson networks see Daduna and Szekli [2003].

Sojourn times in closed networks

Intuitively, sojourn times in closed networks should be negatively correlated, but again negative association is a bit surprising as a property explaining this intuition. The next property for closed cycles of queues is taken from Daduna and Szekli [2004]. The expectations in this statement are taken with respect to the Palm measure defined with respect to the point process of transitions between two fixed consecutive stations.

Property 1.2.14 *Consider Gordon-Newell network* $(R/\boldsymbol{\mu}/J+N)$ *with constant* $\boldsymbol{\mu}$*, and cyclic structure of transitions, i.e.* $r_{i(i+1)} = 1$ *for* $i \leq J-1$ *and* $r_{J1} = 1$*. In equilibrium, for the successive sojourn times* $(\xi_1, \ldots, \xi_J)$ *of a customer at stations* $1, \ldots, J$*,*

$$E(f(\xi_i, i \in \mathrm{I})g(\xi_j, j \in \mathrm{I}^c)) \leq E(f(\xi_i, i \in \mathrm{I}))E(g(\xi_j, j \in \mathrm{I}^c)),$$

for all increasing f, g*, and all* $\mathrm{I} \subset \mathrm{J}$*, i.e.* $\boldsymbol{\xi}$ *is negatively associated.*

In a closed tandem system with fixed population size the conditional cycle time distribution of a customer increases in the strong stochastic ordering when the initial disposition of the other customers increases in the partial sum ordering. As a consequence of this property one obtains

Property 1.2.15 *Consider Gordon-Newell network* $(R/\boldsymbol{\mu}/J+N)$ *with constant* $\boldsymbol{\mu}$*, and cyclic structure of transitions, i.e.* $r_{i(i+1)} = 1$ *for* $i \leq J-1$ *and* $r_{J1} = 1$*. In equilibrium, the cycling time* $\xi_1 + \cdots + \xi_J)$ *of a customer going through stations* $1, \ldots, J$ *is stochastically increasing in* N*, the number of customers cycling.*

For negative association (NA) of sojourn times in the consecutive cycles made by a customer, see Daduna and Szekli [2004].

1.3 Properties of throughput in classical networks

Uniform conditional variability ordering, a relation between closed and open networks

The next property is taken from Whitt [1985]. Before formulating it we need some definitions.

Definition 1.3.1 *Suppose that μ, ν are probability measures which are not related by the stochastic ordering $\leq_{st}$, and are absolutely continuous with respect to Lebesque (counting) measure on $\mathbb{R}$ ($\mathbb{N}$) with densities (mass functions) f, g respectively, with supp$(\mu) \subset$ supp(ν). We say that*

1. *μ is uniformly conditionally less variable than ν, and write $\mu \prec_{uv} \nu$ if $f(t)/g(t)$ is unimodal on $t \in$ supp(ν), with the mode being the supremum.*
2. *μ is log-concave relative to ν, and write $\mu \prec_{lcv} \nu$ if supp$(\mu) \subset$ supp(ν) are intervals (connected sets of integers) and $f(t)/g(t)$ is log-concave on $t \in$ supp(μ).*
3. *$\mu \prec_{mlr} \nu$ if $f(t)/g(t)$ is nonincreasing on $t \in$ supp(μ).*

We use also $\prec_{lcv}$ and $\prec_{uv}$ to relate random variables using the above definition for their distributions.

If the number of sign changes $S(f-g) = 2$, and $\mu \prec_{lcv} \nu$ then $\mu \prec_{uv} \nu$. Moreover if $\mu(A), \nu(B) > 0$, $A \subset B$, $S(f-g) = 2$, and $\mu \prec_{lcv} \nu$ then

(i) if $E(\mu_A) \leq E(\nu_B)$ then $\mu_A \leq_{icx} \nu_B$

(ii) if $E(\mu_A) \geq E(\nu_B)$ then $\mu_A \leq_{dcx} \nu_B$

(iii) $E(\mu_A) = E(\nu_B)$ then $\mu_A \leq_{cx} \nu_B$,

where $E(\mu)$ denotes the expected value of μ, and μ_A denotes the conditional distribution of μ conditioned on A.

It is known (see Whitt [1985]) that for each Gordon-Newell network $(R/\boldsymbol{\mu}/J+N)$ there exist a Jackson network $(\lambda, \tilde{R}/\boldsymbol{\mu}/J)$, such that the stationary distribution of the network content in Gordon-Newell model is equal to the conditional stationary distribution in this Jackson model, conditioned on the fixed number of customers, that is $\pi^{(N,J)}(\boldsymbol{n}) = \tilde{\pi}^J(\boldsymbol{n} \mid \{\boldsymbol{n} : \sum_{i=1}^J n_i = N\})$. For each such pair of stationary network processes $\mathbf{X}, \tilde{\mathbf{X}}$ it is possible to compare variability of the corresponding one dimensional marginal distributions if for each i, $\mu_i(n)$ are nondecreasing functions of n.

Property 1.3.2 *In stationary conditions it holds that for all t*

$$X_i(t) \prec_{lcv} \tilde{X}_i(t),\ i = 1, \ldots, J.$$

From the above relation it follows that if $E(\sum_{i=1}^J \tilde{X}_i(t)) \leq N$ then for respective utilizations at each node i

$$E(\tilde{X}_i(t) \wedge s_i) \leq E(X_i(t) \wedge s_i),$$

*provided $\mu_i(n) = (n \wedge s_i)\mu$ for some $s_i \in \mathbb{N}$, and $\mu > 0$ or equivalently for **throughputs***

$$E(\mu_i(\tilde{X}_i(t))) \leq E(\mu_i(X_i(t))).$$

Effect of enlarging service rates in closed networks

Chen and Yao [2001] showed that if in a closed network, locally in some set of nodes the service rates will be increased then the number of customers in these nodes will decrease, but the number of customers elsewhere will increase (in $\prec_{mlr}$ sense). Moreover the overall throughput for the network will be larger.

Property 1.3.3 *Suppose that we consider two Gordon-Newell networks* $(R/\boldsymbol{\mu}/J+N)$ *and* $(R/\boldsymbol{\mu}'/J+N)$, *and the corresponding stationary queue length processes* $\mathbf{X}$, $\mathbf{X}'$, *such that for a subset* $A \subset \{1,\ldots,J\}$, $\mu_j \le \mu'_j$ *(pointwise) for* $j \in A$, *and* $\mu_j = \mu'_j$, *for* $j \in A^c$. *Then*

$$X'_j(t) \prec_{mlr} X_j(t)$$

for $j \in A$ *and*

$$X_j(t) \prec_{mlr} X'_j(t)$$

for $j \in A^c$. *Moreover if* $\mu_j(n)$ *are nondecreasing functions of* n *then*

$$TH(R/\boldsymbol{\mu}/J+N) \le TH(R/\boldsymbol{\mu}'/J+N).$$

From Shanthikumar and Yao [1988] we have

Remark 1.3.4 If we change the condition that $\mu_j \le \mu'_j$ (pointwise) for $j \in A$, by a stronger one: $\mu_j(m) \le \mu'_j(n)$ for $j \in A$, and all $m \le n$, $m,n \in \mathbb{N}$ then $TH(R/\boldsymbol{\mu}/J+N) \le TH(R/\boldsymbol{\mu}'/J+N)$ holds without assuming monotonicity of service rates. We have for example $TH(R/\boldsymbol{\mu}_{min}/J+N) \le TH(R/\boldsymbol{\mu}/J+N) \le TH(R/\boldsymbol{\mu}_{max}/J+N)$, whenever $\boldsymbol{\mu}_{min} = (\min_{n\ge1}\mu_1(n),\ldots,\min_{n\ge1}\mu_J(n))$ and $\boldsymbol{\mu}_{max} = (\max_{n\ge1}\mu_1(n),\ldots,\max_{n\ge1}\mu_J(n))$ are finite, positive.

Majorization, arrangement and proportional service rates

For two vectors $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ we define the relation $\mathbf{x} \prec_{\mathbf{m}} \mathbf{y}$ by

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]},\ k<n,\ \sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}\ ,$$

where $x_{[1]} \ge \ldots \ge x_{[n]}$ denotes non-increasing rearrangement of $\mathbf{x}$. This relation is the **majorization**.

For two vectors $\mathbf{x},\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x}$ is a permutation of $\mathbf{y}$ we define the relation $\mathbf{x} \prec_{\mathbf{a}} \mathbf{y}$ by requiring that $\mathbf{y}$ can be obtained from $\mathbf{x}$ by a sequence of transpositions such that after transposition the two transposed elements are in decreasing order.

For the next properties in this subsection see Shanthikumar [1987], and Chen and Yao [2001]. The first one exploits interplay between some special regularities of the service rates (fulfilled for example for linear service rates) and a perturbation of the routing in such a way that after perturbation more probable are visits to the stations with lower numbers, which leads to a larger throughput. The second one again assumes some special properties for the service rates (proportional to increasing concave function), and non-increasing routing vector (more probable visits to the stations with lower numbers), then a perturbation leading to more decreasingly arranged service rates (more service for the stations with lower numbers) implies larger throughput.

Property 1.3.5 *Consider two Gordon-Newell networks* $(R/\boldsymbol{\mu}/J+N)$, *and* $(R'/\boldsymbol{\mu}/J+N)$ *such that all* $\mu_j(n)$ *are nondecreasing and concave in* n, *and* $\mu_j(n)-\mu_{j+1}(n)$ *is nondecreasing in* n, *for* $j \leq J-1$. *If for the corresponding routing probabilities* $\boldsymbol{\eta} \prec_{\mathbf{a}} \boldsymbol{\eta}'$ *then*

$$TH(R/\boldsymbol{\mu}/J+N) \leq TH(R'/\boldsymbol{\mu}/J+N).$$

Property 1.3.6 *Consider two Gordon-Newell networks* $(R/\boldsymbol{\mu}/J+N)$, *and* $(R/\boldsymbol{\mu}'/J+N)$ *such that for all* j, $\mu_j(n)$ *and* $\mu'_j(n)$ *are proportional* $\mu_j(n)=\mu_j c(n)$, $\mu'_j(n)=\mu'_j c(n)$ *to some* $c(n)$ *which is nondecreasing and concave in* n, *and* $\boldsymbol{\eta}$ *is non-increasing. If for the corresponding service intensities* $\boldsymbol{\mu} \prec_{\mathbf{a}} \boldsymbol{\mu}'$ *then*

$$TH(R/\boldsymbol{\mu}/J+N) \leq TH(R/\boldsymbol{\mu}'/J+N).$$

Similar assumptions as above in the class of Jackson networks lead to the Schur-convex ordering of the state vectors, which here, intuitively speaking, describes a better performance of the network after the assumed perturbation (adjusting service capacities to the routing structure gives a better performance).

Property 1.3.7 *Consider two Jackson networks* $(\lambda, R/\boldsymbol{\mu}/J+N)$, *and* $(\lambda, R/\boldsymbol{\mu}'/J+N)$ *such that for all* j, $\mu_j(n)$ *and* $\mu'_j(n)$ *are proportional* $\mu_j(n)=\mu_j c(n)$, $\mu'_j(n)=\mu'_j c(n)$ *to some* $c(n)$ *which is nondecreasing and concave in* n, *and* $\tilde{\boldsymbol{\eta}}$ *is non-increasing. If for the corresponding service intensities* $\boldsymbol{\mu} \prec_{\mathbf{a}} \boldsymbol{\mu}'$ *then*

$$E(\psi(\tilde{X}(t)) \geq E(\psi(\tilde{X}'(t))$$

for all nondecreasing and Schur-convex functions ψ.

The next property shows that if the vector of ratios: probability of being in a station divided by its service intensity, has the property of being more equally distributed over the set of stations (in the sense of majorization) then it will lead to larger throughput provided the service function is increasing concave, and to smaller one if this function is increasing convex.

Property 1.3.8 *Consider two Gordon-Newell networks* $(R/\boldsymbol{\mu}/J+N)$, *and* $(R'/\boldsymbol{\mu}'/J+N)$ *such that for all* j, $\mu_j(n)$ *and* $\mu'_j(n)$ *are proportional* $\mu_j(n)=\mu_j c(n)$, $\mu'_j(n)=\mu'_j c(n)$ *to some* $c(n)$ *which is nondecreasing and concave (convex) in* n. *If*

$$(\eta_1/\mu_1,\ldots,\eta_J/\mu_J) \prec_{\mathbf{m}} (\eta'_1/\mu'_1,\ldots,\eta'_J/\mu'_J)$$

then

$$TH(R/\boldsymbol{\mu}/J+N) \geq (\leq) TH(R'/\boldsymbol{\mu}'/J+N).$$

An analog of the above property can be formulated for Jackson networks.

Property 1.3.9 *Consider two Jackson networks $(\lambda, R/\boldsymbol{\mu}/J)$, and $(\lambda', R'/\boldsymbol{\mu}'/J)$ such that for all j, $\mu_j(n)$ and $\mu'_j(n)$ are proportional $\mu_j(n) = \mu_j c(n)$, $\mu'_j(n) = \mu'_j c(n)$ to some $c(n)$ which is nondecreasing and concave in n. If*

$$(\tilde{\eta}_1/\mu_1, \ldots, \tilde{\eta}_J/\mu_J) \prec_{\mathbf{m}} (\tilde{\eta}'_1/\mu'_1, \ldots, \tilde{\eta}'_J/\mu'_J)$$

then in stationary conditions

$$E(\psi(\tilde{X}(t)) \leq E(\psi(\tilde{X}'(t))$$

for all nondecreasing and Schur-convex functions ψ.

A special case where for two networks the service rates are equal shows that the uniformly distributed routing gives the best throughput if the service function is increasing concave.

Property 1.3.10 *Consider two Gordon-Newell networks $(R/\boldsymbol{\mu}/J+N)$, and $(R'/\boldsymbol{\mu}/J+N)$ such that for all j, $\mu_j(n)$ are equal and nondecreasing and concave in n. If for the corresponding routing probabilities $\boldsymbol{\eta} \prec_{\mathbf{m}} \boldsymbol{\eta}'$ then*

$$TH(R/\boldsymbol{\mu}/J+N) \geq TH(R'/\boldsymbol{\mu}/J+N).$$

Throughput and number of jobs

Van der Wal [1989] [1989] obtained the following intuitively clear property

Property 1.3.11 *Suppose that for a Gordon-Newell network $(R/\boldsymbol{\mu}/J+N)$ the service rates $\mu_i(n)$ are positive and nondecreasing functions of n, then in the stationary conditions $E(\mu_1(X_1(t)))$ is nondecreasing in N.*

From Chen and Yao [2001], Shanthikumar and Yao [1988] we have a more involved property.

Property 1.3.12 *Suppose that for a Gordon-Newell network $(R/\boldsymbol{\mu}/J+N)$ the service rates $\mu_i(n)$ are positive and nondecreasing concave (convex, starshaped, anti-starshaped, subadditive, superadditive) functions of n, then in the stationary conditions $TH(R/\boldsymbol{\mu}/J+N)$ has the same property as a function of N.*

The above property has an application to so called open - finite networks and blocking probabilities. Moreover Shanthikumar and Yao [1989] studied monotonicity of throughput in cyclic/finite buffer networks with respect to the convex ordering of service times, and of buffer capacities.

1.4 Routing and correlations

The general theory for comparison of Markov processes with respect to their internal dependence structure revealed that sometimes there is a complicated interplay of monotonicity properties with some generalized correlation structure of the processes. Such monotonicity requirement is not unexpected if we recall that the theory of association in time for Markovian processes is mainly developed for monotone Markov processes, for a review see Chapter II of Liggett [1985]. Association is a powerful tool in obtaining probability bounds e.g. in the realm of interacting processes of attractive particle systems. (A system is called attractive if it exhibits (strong) stochastic monotonicity.)

In the context of stochastic networks it turns out that similar connections between monotonicity and correlation are fundamental, but that due to the more complex structure of the processes we usually cannot hope to utilize the strong stochastic order, as required for association, or in the development in Hu and Pan [2000] and Daduna and Szekli [2006].

In this section we shall consider pairs of network processes related by some structural similarities, one can usually think of one network being obtained from the other by some structural perturbation. The perturbations we are mainly interested in are due to perturbation of the routing of individual customers. We will always give a precise meaning of what the perturbations are and of the resulting structural properties. Proofs of all results presented in this section can be found in Daduna and Szekli [2008].

We shall exhibit that the conditions that determine comparability of dependence, i.e., second order properties of processes having the same first order behavior (i.e. the same steady state), are closely connected with further properties of the asymptotic behavior of the processes: the asymptotic variance of certain functionals (performance measures and cost functions) of the network processes and the speed of convergence to stationarity via comparison of the spectral gap. A similar observation in a general setting was made also in Daduna and Szekli [2006].

Given a prescribed network in equilibrium, our conjecture is, that if we perturb the routing process (which governs the movements of the customers after being served at any node) in a way to make it more dependent in a specified way, than the joint queue length process after perturbation will be more dependent in some (possibly differently) specified way.

We concentrate especially on two ways in which the routing process is perturbed. The first way is by making routing more chaotic which is borrowed from statistical mechanics. There exists a well-established method to express more or less *chaotic behavior of a random walker*, if his itinerary is governed by doubly stochastic routing matrices, see Alberti and Uhlmann [1982]. We shall prove that if the routing is becoming more chaotic in this sense then the joint queue length process will show less internal dependency.

While the perturbation of the routing in this case is not connected with any order (numbering) of the nodes the second way of perturbing the routing is connected to some preassigned order of the nodes, which is expressed by a graph structure. Assuming that routing of customers is compatible with this graph structure we perturb it by shifting probability mass in the routing kernel along paths that are determined by the graph. We shall prove that if we shift the masses in a way that routing becomes more positive dependent then the internal dependence of the joint queue length process will increase.

We denote the Kronecker-Delta by $\delta_{i,j} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$, and for any real valued vector $\boldsymbol{\xi} = (\xi_i : 0 \leq i, j \leq J)$ we define the diagonal matrix with entries from $\boldsymbol{\xi}$ by

$$diag(\boldsymbol{\xi}) = (\delta_{i,j} \cdot \xi_i : 0 \leq i, j \leq J).$$

For $k = 1, \ldots, J$, the kth J-dimensional unit (row) vector is $\boldsymbol{e}_k := (\delta_{jk} : j = 1, \ldots, J)$.

For vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_J) \in \mathbb{R}^J$ the rank statistic $\mathcal{R}(\boldsymbol{\alpha}) = (\mathcal{R}_1(\boldsymbol{\alpha}), \ldots, \mathcal{R}_J(\boldsymbol{\alpha})) \in \mathbb{N}^J$ is defined by enumeration of the indices of the $\boldsymbol{\alpha}$ in the decreasing order of their associated $\alpha_{(\cdot)}-$values, i.e.

$$\alpha_{\mathcal{R}_i(\boldsymbol{\alpha})} \geq \alpha_{\mathcal{R}_{i+1}(\boldsymbol{\alpha})} \quad i = 1, \ldots, J-1,$$

and ties are resolved according to the natural order of the indices.
The vector $A\mathcal{R}(\boldsymbol{\alpha}) = (A\mathcal{R}_1(\alpha), \ldots, A\mathcal{R}_J(\boldsymbol{\alpha})) \in \mathbb{N}^J$ of antiranks of $\boldsymbol{\alpha}$ is defined by $A\mathcal{R}_j(\boldsymbol{\alpha}) = \mathcal{R}_{J+1-j}(\boldsymbol{\alpha})$ and so yields an enumeration of the indices of $\boldsymbol{\alpha}$ in increasing order of their associated $\alpha_{(\cdot)}-$values.

1.4.1 Correlation inequalities via generators

For a queue length network process $\tilde{\mathbf{X}}$ with generator $Q^{\tilde{\mathbf{X}}}$ and stationary distribution $\tilde{\pi}^J$ we are interested in *one step* correlation expressions

$$\langle f, Q^{\tilde{\mathbf{X}}} g \rangle_{\tilde{\pi}^J} \tag{1.4.1}$$

If $f = g$, then (1.4.1) is (the negative of) a quadratic form, because $-Q^{\tilde{\mathbf{X}}}$ is positive definite. (1.4.1) occurs in the definition of Cheeger's constant which is helpful to bound the second largest eigenvalue of $Q^{\tilde{\mathbf{X}}}$ (because division of (1.4.1) by $\langle f, f \rangle_{\tilde{\pi}^J}$ yields Rayleigh quotients), which essentially governs the speed of convergence of $\tilde{\mathbf{X}}$ to its equilibrium.

(1.4.1) can be utilized to determine the asymptotic variance of costs or performance measures associated with Markovian processes (network processes) and to compare the asymptotic variances of two such related processes.

In a natural way the correlations occur when comparing the dependence structure of $\tilde{\mathbf{X}}$ with that of a related process $\hat{\mathbf{X}}$ with the same stationary distribution $\tilde{\pi}^J$, where we evaluate

$$\langle f, Q^{\tilde{\mathbf{X}}} g \rangle_{\tilde{\pi}^J} - \langle f, Q^{\hat{\mathbf{X}}} g \rangle_{\tilde{\pi}^J}, \tag{1.4.2}$$

see e.g. (iv) and (v) in Theorem 1.4.15 below.

Because we are dealing with processes having bounded generators, properties connected with (1.4.1) can be turned into properties of

$$\langle f, I + \varepsilon Q^{\tilde{\mathbf{X}}} g \rangle_{\tilde{\pi}^J} = E_{\tilde{\pi}^J}(f(X_0)g(X_\tau)) \tag{1.4.3}$$

where I is the identity operator, and $\varepsilon > 0$ is sufficiently small such that $I + \varepsilon Q^{\tilde{\mathbf{X}}}$ is a stochastic matrix, and $\tau \sim \exp(\varepsilon)$ (exponentially distributed). This enables one to directly apply discrete time methods to characterize properties of continuous time processes in the range of problems sketched above.

We begin with an expressions which connects for continuous time processes the differences (1.4.2) of covariances for related network processes with some covariances for the corresponding routing matrices.

Property 1.4.1 *Suppose* $\tilde{\mathbf{X}}$ *is an ergodic Jackson network process with a routing matrix* $\tilde{R}$ *and* $\hat{\mathbf{X}}$ *is the Jackson network process having the same arrival and service intensities but with routing matrix* $\hat{R} = [\hat{r}_{ij}]$ *such that the* extended traffic solutions $\tilde{\boldsymbol{\eta}}$ *of the traffic equation for* $\tilde{R}$ *and for* $\hat{R}$ *coincide. Then for arbitrary real functions* f, g

$$\langle f, Q^{\tilde{\mathbf{X}}} g\rangle_{\tilde{\pi}^J} - \langle f, Q^{\hat{\mathbf{X}}} g\rangle_{\tilde{\pi}^J} = \lambda/\xi_0 E_{\tilde{\pi}^J}\left(tr\big(W^{g,f}(\tilde{X}_t)\cdot diag(\boldsymbol{\xi})\cdot(R-\hat{R})\big)\right),$$

where $\boldsymbol{\xi}$ *is the probability solution of the extended traffic equation* (1.1.3)*,* $\boldsymbol{e}_0 = (0, \ldots, 0)$*, and*

$$W^{g,f}(\mathbf{n}) = [g(\mathbf{n}+\boldsymbol{e}_i)f(\mathbf{n}+\boldsymbol{e}_j)]_{i,j=0,1,\ldots,J}.$$

Property 1.4.2 *Suppose* $\mathbf{X}$ *is an ergodic Gordon-Newell network process with routing matrix* R *and* $\hat{\mathbf{X}}$ *is the Gordon-Newell network process having the same service intensities but with routing matrix* $\hat{R} = [\hat{r}_{ij}]$ *such that the* stochastic traffic solutions $\boldsymbol{\eta}$ *of the traffic equation for* R *and for* $\hat{R}$ *coincide. Then for arbitrary real functions* f, g

$$\langle f, Q^{\mathbf{X}} g\rangle_{\pi(N,J)} - \langle f, Q^{\hat{\mathbf{X}}} g\rangle_{\pi(N,J)} = \frac{G(N-1,J)}{G(N,J)} E_{\pi(N-1,J)}\left(tr\big(W^{g,f}(X_t)\cdot diag(\boldsymbol{\eta})\cdot(R-\hat{R})\big)\right),$$

where $\boldsymbol{\eta}$ *is the probability solution of the traffic equation* (1.1.1)*,* $\boldsymbol{e}_0 = (0, \ldots, 0)$*, and*

$$W^{g,f}(\mathbf{n}) = [g(\mathbf{n}+\boldsymbol{e}_i)f(\mathbf{n}+\boldsymbol{e}_j)]_{i,j=1,\ldots,J}.$$

We can reformulate the results of these properties in a form which is of independent interest, because it immediately relates our results to methods dealt with in optimizing MCMC simulation. Introducing for convenience the notation $H^f(\mathbf{n}, i) := f(\mathbf{n}+\boldsymbol{e}_i)$ which in our framework occurs as $H^f(X_t, i) := f(X_t + \boldsymbol{e}_i)$ (and similarly for g), we obtain

Corollary 1.4.3 **(a)** *For Jackson network processes* $\tilde{\mathbf{X}}, \hat{\mathbf{X}}$ *as in Property 1.4.1, with* $\boldsymbol{\xi}$ *the probability solution of the extended traffic equation* (1.1.3)*, we have*

$$\langle f, Q^{\tilde{\mathbf{X}}} g\rangle_{\tilde{\pi}^J} - \langle f, Q^{\hat{\mathbf{X}}} g\rangle_{\tilde{\pi}^J} = \frac{\lambda}{\xi_0}\ E_{\tilde{\pi}^J}(\langle H^f(\tilde{X}_t, \cdot), \big(R-\hat{R}\big)H^g(\tilde{X}_t, \cdot)\rangle_{\boldsymbol{\xi}}) \tag{1.4.4}$$

(b) *For Gordon-Newell network processes* $\mathbf{X}, \hat{\mathbf{X}}$ *as in Proposition 1.4.2, with* $\boldsymbol{\eta}$ *the probability solution of the traffic equation, we have*

$$\langle f, Q^{\mathbf{X}} g\rangle_{\pi(N,J)} - \langle f, Q^{\hat{\mathbf{X}}} g\rangle_{\pi(N,J)} = \frac{G(N-1,J)}{G(N,J)} E_{\pi(N-1,J)}\ (\langle H^f(X_t, \cdot), \big(R-\tilde{R}\big)H^g(X_t, \cdot)\rangle_{\boldsymbol{\eta}}) \tag{1.4.5}$$

There are several appealing interpretations of the formulas (1.4.4) and (1.4.5) which will guide some of our forthcoming arguments. We discuss the closed network case (1.4.5).

The inner product $\langle H^f(X_t, \cdot), \big(R-\tilde{R}\big)H^g(X_t, \cdot)\rangle_{\boldsymbol{\eta}}$ can be evaluated path-wise for any ω, and whenever, e.g., the difference $R - \widetilde{R}$ is positive definite, the integral $E_{\pi^{I-1,J}}(\cdot)$ (across Ω) is over non negative functions. Recalling that $\boldsymbol{\eta}$ is invariant for R and $\hat{R}$ we obtain

$$\langle H^f(X_t, \cdot), \big(R-\tilde{R}\big)H^g(X_t, \cdot)\rangle_{\boldsymbol{\eta}} = E_{\boldsymbol{\eta}}\big(H^f(X_t, V_0)\cdot H^g(X_t, V_1)\big) - E_{\boldsymbol{\eta}}\big(H^f(X_t, \hat{V}_0)\cdot H^g(X_t, \hat{V}_1)\big),$$

where $V = (V_n : n \in \mathbb{N})$ and $\hat{V} = (\hat{V}_n : n \in \mathbb{N})$ are Markov (routing) chains with common steady state $\boldsymbol{\eta}$ and different transition matrices $R, \hat{R}$. If we consider formally a network process $\mathbf{X}$ and Markov chains V, resp. $\hat{V}$ that are independent of $\mathbf{X}$, we get

$$\begin{aligned}
&\langle f, Q^{\mathbf{X}} g\rangle_{\pi^{(N,J)}} - (f, Q^{\hat{\mathbf{X}}} g\rangle_{\pi^{(N,J)}} \\
&= \frac{G(N-1,J)}{G(N,J)} \\
&\quad \cdot \Big(E_{\pi^{(N-1,J)}}\, E_{\boldsymbol{\eta}}\big(H^f(X_t, V_0)\cdot H^g(X_t, V_1)\big) - E_{\pi^{(N-1,J)}}\, E_{\boldsymbol{\eta}}\big(H^f(X_t, \hat{V}_0)\cdot H^g(X_t, \hat{V}_1)\big)\Big) \\
&= \frac{G(N-1,J)}{G(N,J)} \\
&\quad \cdot \Big(E_{\boldsymbol{\eta}}\, E_{\pi^{(N-1,J)}}\big(H^f(X_t, V_0)\cdot H^g(X_t, V_1)\big) - E_{\boldsymbol{\eta}}\, E_{\pi^{(N-1,J)}}\big(H^f(X_t, \hat{V}_0)\cdot H^g(X_t, \hat{V}_1)\big)\Big),
\end{aligned}$$

the latter equality by the Fubini theorem. The last expression is a representation through comparison of stochastically ordered processes, whenever we can show that the difference of the covariances is nonnegative or non positive throughout.

Corollary 1.4.3 points out the relevance of the following orderings for transition matrices which are well known in the theory of optimal selection of transition kernels for MCMC simulation. In our investigations these orders will be utilized to compare routing processes via their transition matrices.

Definition 1.4.4 *Let $R = [r_{ij}]$ and $\hat{R} = [\hat{r}_{ij}]$ be transition matrices on a finite set $\mathbb{E}$ such that $\boldsymbol{\eta} R = \boldsymbol{\eta}\hat{R} = \boldsymbol{\eta}$.*

We say that $\hat{R}$ is smaller than R in the positive definite order , $\hat{R} \prec_{pd} R$, if their difference $R - \hat{R}$ is positive definite on $L_2(\mathbb{E}, \eta)$.

We say that $\hat{R}$ is smaller than R in the Peskun order, $\hat{R} \prec_P R$, if for all $j, i \in \mathbb{E}$ with $i \neq j$ holds $\hat{r}_{ji} \leq r_{ji}$, see Peskun [1973].

Peskun used the latter order to compare reversible transition matrices with the same stationary distribution and their asymptotic variance, and Tierney [1998] has shown that the main property used in the proof of Peskun, namely that $R \prec_P \hat{R}$ implies $\hat{R} \prec_{pd} R$, holds without reversibility assumptions.

Comparison of asymptotic variance Peskun and Tierney derived comparison theorems for the asymptotic variance of Markov chains for application to optimal selection of MCMC transition kernels in discrete time. These asymptotic variances occur as variance in the limiting distribution of central limit theorems (CLTs) for the MCMC estimators.

In the setting of queueing networks, performance measures of interest usually are steady state mean values of performance indices, $\pi(f) = E_{\tilde{\pi}^J}(f(\tilde{X}(t)))$, which can be estimated as time averages, justified by the ergodic theorem for Markov processes, i.e. in the discrete time we have for large n

$$E_{\tilde{\pi}^J}(f(\tilde{X}(t))) \sim \frac{1}{n}\sum_{k=1}^{n} f(X_k).$$

Under some regularity conditions on a homogeneous Markov chain with one step transition kernel K there is a CLT of the form (weak convergence $\equiv \xrightarrow{D}$)

$$\sqrt{n}(\frac{1}{n}\sum_{k=1}^{n} f(X_k) - E_{\tilde{\pi}^J}(f(\tilde{X}(t)))) \xrightarrow{D} N(0, v(f,K)),$$

where the asymptotic variance is

$$v(f,K) = \langle f,f\rangle_{\tilde{\pi}^J} - \pi(f) + 2\sum_{k=1}^{\infty}\langle f, K^k f\rangle_{\tilde{\pi}^J}. \tag{1.4.6}$$

To arrange a discrete time framework for our network processes $\mathbf{X}$ we consider the Markov chains with transition matrices

$$K = I + \varepsilon Q^{\tilde{\mathbf{X}}}$$

(with $\varepsilon > 0$ sufficiently small) that occur in the compound Poisson representation of the transition probabilities of the network processes.

The next properties show that perturbing routing in network can result in a larger asymptotic variance for the imbedded chain.

Property 1.4.5 (a) *Consider two ergodic Jackson networks with the same arrival and service intensities, and with queue length processes $\tilde{\mathbf{X}}$ and $\hat{\mathbf{X}}$. Assume that the* extended *routing matrices $\tilde{R}$ and $\hat{R}$ are reversible with respect to $\boldsymbol{\xi}$.*

If $\tilde{R}$ and $\hat{R}$ are ordered in the Peskun order, $\hat{R} \prec_P \tilde{R}$, then for any real function f we have

$$v(f, I + \varepsilon Q^{\hat{\mathbf{X}}}) \geq v(f, I + \varepsilon Q^{\tilde{\mathbf{X}}}). \tag{1.4.7}$$

(b) *Consider two ergodic Gordon-Newell networks with the same service intensities, and with queue length processes $\mathbf{X}$ and $\hat{\mathbf{X}}$. Assume that the routing matrices R and $\hat{R}$ are reversible with respect to $\boldsymbol{\eta}$.*

If $\hat{R} \prec_P R$, then for any real function f we have

$$v(f, I + \varepsilon Q^{\hat{\mathbf{X}}}) \geq v(f, I + \varepsilon Q^{\mathbf{X}}). \tag{1.4.8}$$

Comparison of spectral gaps Let $\mathbf{X}$ be a continuous time homogeneous ergodic Markov process with stationary probability π and generator $Q^{\mathbf{X}}$. The spectral gap of $\mathbf{X}$, resp. $Q^{\mathbf{X}}$ is

$$Gap(Q^{\mathbf{X}}) = \inf\{\langle f, -Q^{\mathbf{X}} f\rangle_\pi : f \in L_2(\mathbb{E},\pi), \pi(f) = 0,\ \langle f,f\rangle\pi = 1\}. \tag{1.4.9}$$

The spectral gap determines for $\mathbf{X}$ the speed of convergence to equilibrium π in $L_2(\mathbb{E},\pi)$-norm $\|\cdot\|_\pi$: $Gap(Q^{\mathbf{X}})$ is the largest number Δ such that for the transition semigroup $P = (P_t : t \geq 0)$ of $\mathbf{X}$ holds

$$\|P_t f - \pi(f)\|_\pi \leq e^{-\Delta t}\|f - \pi(f)\|_\pi \quad \forall f \in L_2(\mathbb{E},\pi).$$

For Gordon-Newell networks the spectral gap is always greater than zero, while for Jackson networks the situation is more delicate: zero gap and non zero gap can occur. Iscoe and McDonald [1994a], [1994b], and Lorek [2007] proved under some natural assumptions necessary and sufficient conditions for non zero spectral gap for Jackson networks. The case of positive gap is proved by using an attached vector of independent birth-death processes, to bound the gap away from zero.

It is interesting that for some classes of Jackson networks it is possible to strictly bound the gap of the queue length network process $\tilde{\mathbf{X}}$ from below by the gap of some multidimensional birth-death process, which will play in the next proposition the role of the network process $\hat{\mathbf{X}}$. Because we focus on the intuitive, but rather strong Peskun ordering of the routing matrices, we need additional assumptions on the routing. The assumption constitutes a detailed balance which determines an additional internal structure of a Markov chain and its global balance equation (= equilibrium equation). Such detailed balance equations are prevalent in many networks with (nearly) product form steady states, and often open the way to solve the global balance equation for the steady state. (1.4.10) equalizes the routing flow from any node into the (inner) network to the flow out of the (inner) network to that node.

Property 1.4.6 *Consider an ergodic Jackson network process $\tilde{\mathbf{X}}$ with positive external arrival rates $\lambda_i > 0$ at all nodes $i = 1, \ldots, J$. Assume that the extended routing matrix $\tilde{R} = [\tilde{r}_{ij}]_{i,j=0,1,\ldots,J}$ has strict positive departure probabilities $\tilde{r}_{i0} > 0$ from every node $i = 1, \ldots, J$. Assume further that the routing of $\tilde{\mathbf{X}}$ fulfills overall balance for all network nodes with respect to the solution $\tilde{\eta}_i, i = 1, \ldots, J,$ of the traffic equation* (1.1.3) *,i.e.,*

$$\tilde{\eta}_j \sum_{i=1}^{J} \tilde{r}_{j,i} = \sum_{i=1}^{J} \tilde{\eta}_i \tilde{r}_{i,j}, \quad \forall j = 1, \ldots, J. \tag{1.4.10}$$

Then there exists an ergodic Jackson network process $\hat{\mathbf{X}}$ of independent birth-death processes, the nodes of which have the same service intensities and external arrival rates $\hat{\lambda}_i = \lambda_i$ such that the $Gap(Q^{\tilde{\mathbf{X}}}) \geq Gap(Q^{\hat{\mathbf{X}}})$.

Extending this proposition to a more general setting we immediately obtain from (1.4.4) and (1.4.5) correlation inequalities which bound (1.4.2). So, we can immediately conclude for some networks that $Gap(Q^{\tilde{\mathbf{X}}}) \geq Gap(Q^{\hat{\mathbf{X}}})$ holds. A consequence which elaborates on the implication *Peskun yields positive definiteness* is, that if we perturb routing of customers in the networks by shifting mass from non diagonal entries to the diagonal (leaving the routing equilibrium fixed) and obtaining that way Peskun order of routing, then the speed of convergence of the perturbed process can only decrease. This is just what in optimization of MCMC was intended, and Peskun gave conditions for this. Similarly we see

Property 1.4.7 (a) *Consider ergodic Jackson networks with the same arrival and service intensities, and with the state processes $\tilde{\mathbf{X}}$ and $\hat{\mathbf{X}}$. Assume that for the* extended *routing matrices $\tilde{R}$ and $\hat{R}$ the stochastic solutions $\boldsymbol{\xi}$ of the traffic equation coincide. If $\tilde{R}$ and $\hat{R}$ are ordered in the positive definite order, $\tilde{R} \prec_{pd} \hat{R}$, then for any real function f we have*

$$\langle f, Q^{\hat{\mathbf{X}}} f \rangle_{\pi} \geq \langle f, Q^{\tilde{\mathbf{X}}} f \rangle_{\tilde{\pi}^J}, \quad and \quad Gap(Q^{\hat{\mathbf{X}})} \leq Gap(Q^{\tilde{\mathbf{X}}}). \tag{1.4.11}$$

(b) *Consider ergodic Gordon-Newell networks with the same service intensities, and with the state processes $\mathbf{X}$ and $\hat{\mathbf{X}}$. Assume that for the routing matrices R and $\hat{R}$ the stochastic solutions $\boldsymbol{\eta}$ of the traffic equation coincide. If $R \prec_{pd} \hat{R}$, then for any real function f we have*

$$\langle f, Q^{\hat{\mathbf{X}}} f \rangle_{\pi^{(N,J)}} \geq \langle f, Q^{\mathbf{X}} f \rangle_{\pi^{(N,J}}, \quad and \quad Gap(Q^{\hat{\mathbf{X}})} \leq Gap(Q^{\mathbf{X}}). \tag{1.4.12}$$

Comparison of dependencies The expression (1.4.1) for continuous time Markov processes is transformed via embedded uniformization chain (1.4.3) to a true covariance and via (1.4.2) to comparison of covariance functionals for two Markov processes and their at Poissonian times embedded chains, i.e., with $\tau \sim \exp(\eta)$ we obtain

$$E_{\tilde{\pi}^J}(f(\tilde{X}_0)g(\tilde{X}_\tau)) = \langle f, (I + \eta Q^{\tilde{\mathbf{X}}})g\rangle_{\tilde{\pi}^J} \leq \langle f, (I + \eta Q^{\hat{\mathbf{X}}})g\rangle_{\tilde{\pi}^J} = E_{\tilde{\pi}^J}(f(\hat{X}_0)g(\hat{X}_\tau)).$$

Transforming this property into analogous statements for the continuous time evolution (over many time points) will need in general additional monotonicity properties of the processes. It turns out that some form of stochastic monotonicity is in some cases a direct substitute for the strong reversibility assumption which is needed to prove Peskun's theorem.

1.4.2 Doubly stochastic routing

In this section the perturbation of a network process will be due to the fact that the routing of the customers will become more chaotic. In statistical physics there is a well-established method to express more or less *chaotic* behaviour of a random walker, if his itinerary is governed by doubly stochastic routing matrices. Alberti and Uhlmann provide an indepth study of *Stochasticity and Partial Order* that elaborates on these methods [1982]. Following their ideas in this section we consider (mainly) Gordon-Newell networks with doubly stochastic routing matrix.

Consider arbitrary row $r(i) := (r_{ij} : j = 1, 2, \dots, J)$ of the Gordon-Newell network's routing matrix R and a doubly stochastic matrix $T = [t_{ij}]_{i,j=1,\dots,J}$. Then the i-th row vector of the product $(R \cdot T)$ is smaller than $r(i)$ in the sense of the majorization ordering, see Marshall and Olkin [1979], p.18. This means that the probability mass is more equally distributed in each row after multiplication. The routing scheme is then more equally distributed too. Nevertheless, the solution of the traffic equation for $R \cdot T$ and therefore the steady state of the network under the $R \cdot T$ regime is the same as under R, namely, the normalized solution of the traffic equation (1.1.1) is in both cases the uniform distribution on $\{1, 2, \dots, J\}$.

More chaotic routing leads to less internal dependencies over time of the individual routing chains of the customers and will therefore lead to less internal dependence over time of the joint queue length process. Let

$$\mathcal{L} = \{f : \mathbb{E}_N \to \mathbb{R}_+ : f(n_1, \dots, n_J) = a + \sum_{i=1}^{J} \alpha_i \cdot n_i, \alpha_i \in \mathbb{R}, i = 1, \dots, J, a \in \mathbb{R}_+\},$$

be the convex cone of nonnegative affine-linear functions on $\mathbb{E}_N$.

Theorem 1.4.8 (Linear service rates) *Consider two ergodic Gordon-Newell network processes with common stationary distribution $\pi^{(N,J)}$: $\mathbf{X}$ with a doubly stochastic routing matrix R and $\hat{\mathbf{X}}$ with the routing matrix $\hat{R} = [\hat{r}_{ij}] = R \cdot T$, for a doubly stochastic matrix T. All other parameters of the networks are assumed to be the same.*

Consider pairs of nonnegative affine-linear functions

$$f : \mathbb{E}_N \to \mathbb{R}_+ : f(n_1, \ldots, n_J) = a + \sum_{i=1}^{J} \alpha_i \cdot n_i \in \mathcal{L}, \quad \textit{and}$$
$$g : \mathbb{E}_N \to \mathbb{R}_+ : g(n_1, \ldots, n_J) = b + \sum_{i=1}^{J} \beta_i \cdot n_i \in \mathcal{L}, \quad \textit{with}$$

$$\mathcal{R}(\alpha_1, \ldots, \alpha_J) = \mathcal{R}(\beta_1, \ldots, \beta_J).$$

Then for all such pairs of functions with $f, g \in \mathcal{I}_+^*(\mathbb{N}^J) \cap \mathcal{L}$, *and* $f, g \in \mathcal{D}_+^*(\mathbb{N}^J) \cap \mathcal{L}$, *holds*

$$\langle f, Q^{\hat{\mathbf{X}}} g \rangle_{\pi(N,J)} \le \langle f, Q^{\mathbf{X}} g \rangle_{\pi(N,J)}.$$

In Theorem 1.4.8, for $f = g$, the rank condition is trivially fulfilled. This yields

Corollary 1.4.9 *Under the assumptions of Theorem 1.4.8, for all* $f \in \mathcal{I}_+^*(\mathbb{N}^J) \cap \mathcal{L}$, *and* $f \in \mathcal{D}_+^*(\mathbb{N}^J) \cap \mathcal{L}$, *it holds*

$$\langle f, Q^{\hat{\mathbf{X}}} f \rangle_{\pi(N,J)} \le \langle f, Q^{\mathbf{X}} f \rangle_{\pi(N,J)}.$$

Note, that for $f = g$, $R(I - T)$ is nonnegative definite.

1.4.3 Robin-Hood transforms

If the node set is equipped with a partial order, which is relevant for the customers' migration, then it is tempting to consider perturbations of the routing processes that are in line with this order. To be more precise: We have an up-down relation between the nodes and the question is how the steady state performance reacts on routing more up, resp. down.

The construction of Corollary 2.1 and Example 3.1 in Daduna and Szekli [2006], which is sometimes called Robin-Hood transform because in a certain sense it equalizes the frequencies of the random walker to visit the different nodes, yields a change of routing such that it is more or less dependent in a well defined way. The construction is as follows:

Consider a homogeneous Markov chain (X_i) on a finite partially ordered state space $(\mathbb{E}, \prec)$ with transition matrix $[p(i,j)]_{i,j \in \mathbb{E}}$ and stationary distribution π.

Assume that for $a, b, c, d \in \mathbb{E}$ we have $a \prec c$ and $b \prec d$ such that $(a, d) \in \mathbb{E}^2$ and $(c, b) \in \mathbb{E}^2$ are not comparable with respect to the product order, and that $P^{(X_0,X_1)}(a, d) \ge \alpha$, $P^{(X_0,X_1)}(c, b) \ge \alpha$.

Construct the distribution $P^{(Y_0,Y_1)}$ of a random vector (Y_0, Y_1) from $P^{(X_0,X_1)}$ by

$$P^{(Y_0,Y_1)}(a,b) = P^{(X_0,X_1)}(a,b) + \alpha,\ P^{(Y_0,Y_1)}(c,d) = P^{(X_0,X_1)}(c,d) + \alpha, \text{ and}$$
$$P^{(Y_0,Y_1)}(a,d) = P^{(X_0,X_1)}(a,d) - \alpha,\ P^{(Y_0,Y_1)}(c,b) = P^{(X_0,X_1)}(c,b) - \alpha, \text{ and}$$
$$P^{(Y_0,Y_1)}(u,v) = P^{(X_0,X_1)}(u,v) \text{ for all other } (u,v) \in \mathbb{E}^2.$$

(This is the Robin-Hood transform.)

The one-dimensional marginals of both (X_0, X_1) and (Y_0, Y_1) are π and the conditional distribution $P(Y_1 = w \mid Y_0 = v) =: q(v, w)$ for $v, w \in \mathbb{E}$ is obtained from $[p(i, j)]$ as follows:

$$q(a,d) = p(a,d) - \frac{\alpha}{\pi(a)}, \quad q(c,b) = p(c,b) - \tfrac{\alpha}{\pi(c)}, \qquad (1.4.13)$$
$$q(a,b) = p(a,b) + \frac{\alpha}{\pi(a)}, \quad q(c,d) = p(c,d) + \tfrac{\alpha}{\pi(c)}, \quad q(u,v) = p(u,v) \quad \text{otherwise.}$$

Consider now a homogeneous Markov chain (Y_i) with the so defined transition matrix q, and consider (X_i) and (Y_i) as routing chains of a network process, where (Y_i) is obtained from (X_i) by a perturbation through the Robin-Hood transformation. Then according to Corollary 2.1 and Theorem 3.1 in Daduna and Szekli [2006] the routing governed by (Y_i) is more concordant than the routing governed by (X_i).

Definition 1.4.10 *Let $(\mathbb{E}, \prec)$ be a finite partially ordered set. The generalized partial sum order $\prec_*$ on $\mathbb{N}^{\mathbb{E}}$ is defined for $x = (x_i : i \in \mathbb{E}), y = (y_i : i \in \mathbb{E}) \in \mathbb{N}^{\mathbb{E}}$ by*

$$x \prec_* y :\Longleftrightarrow \left(\forall \text{ decreasing } K \subseteq \mathbb{E} \text{ holds } \sum_{k\in K} x_k \le \sum_{k \in K} y_k \right). \qquad (1.4.14)$$

Consider now a Jackson network $\tilde{\mathbf{X}}$ where the node set $\mathrm{J} = \{1, \dots, J\}$ is a partially ordered set $(\mathrm{J}, \prec)$ and the customers flow in line with the directions prescribed by this partial order, i.e. for the routing matrix $\tilde{R}$ holds (see Harris [1977]):

$$\tilde{r}_{ij} > 0 \Longrightarrow (i \prec j \vee j \prec i)\,. \qquad (1.4.15)$$

Then the Jackson network process $\tilde{\mathbf{X}}$ has the up-down property with respect to $\prec_*$, which means that for the generator $Q^{\tilde{\mathbf{X}}}$

$$q^{\tilde{\mathbf{X}}}(x, y) > 0 \Longrightarrow (x \prec_* y \vee y \prec_* x)\,. \qquad (1.4.16)$$

Lemma 1.4.11 *Consider an ergodic Jackson network with extended routing matrix $\tilde{R}$, and queue length process $\tilde{\mathbf{X}}$. We assume that the node set $\mathrm{J} = \{1, \dots, J\}$ is a partially ordered set. For some nodes $a, b, c, d \in \mathrm{J}$ (not necessarily distinct) let $a \prec c$ and $b \prec d$, and for some $\alpha > 0$ let*

$$\tilde{r}_{ad} \ge \alpha/\xi_a \qquad \text{and} \qquad \tilde{r}_{cb} \ge \alpha/\xi_c. \qquad (1.4.17)$$

Define a new network with queue length process $\hat{\mathbf{X}}$ as follows: The nodes, the nodes' structure, and the external arrival processes are the same as in the original network. The routing matrix $\hat{R}$ is computed by Robin-Hood transformation (1.4.13) *with the fixed a, b, c, d.*
Consider a pair of comonotone functions f, g (either both increasing or both decreasing) such that for all $n \in \mathbb{N}^J$ holds $(f(\mathbf{n} + \mathbf{e}_c) - f(\mathbf{n} + \mathbf{e}_a)) \cdot (g(\mathbf{n} + \mathbf{e}_d) - g(\mathbf{n} + \mathbf{e}_b)) \ge 0$.
Then

$$\langle f, Q^{\tilde{\mathbf{X}}} g\rangle_{\tilde{\pi}^J} \le \langle f, Q^{\hat{\mathbf{X}}} g\rangle_{\tilde{\pi}^J}. \qquad (1.4.18)$$

Immediately from this lemma we get

Theorem 1.4.12 *Consider an ergodic Jackson network with extended routing matrix $\tilde{R}$ according to* (1.1.3) *with queue length process $\tilde{\mathbf{X}}$. We assume that the node set is partially ordered* (J, $\prec$)*.*

Define a new network with queue length process $\hat{\mathbf{X}}$ as follows: The nodes, the nodes' structure, and the external arrival processes are the same as in the original network. The routing matrix $\hat{R}$ is computed by a sequence of $n \geq 1$ feasible Robin-Hood transformations according to (1.4.13) *for a sequence of nodes.*

Then for any pair of comonotone functions $f, g : \mathbb{N}^J \to \mathbb{R}_+$ with respect to the generalized partial sum order $\prec_*$ (either both increasing or both decreasing) it holds

$$\langle f, Q^{\tilde{\mathbf{X}}} g\rangle_{\tilde{\pi}^J} \leq \langle f, Q^{\hat{\mathbf{X}}} g\rangle_{\tilde{\pi}^J}.$$

1.4.4 Dependence orderings and monotonicity

We shall now generalize the concordance ordering.

Definition 1.4.13 **(a)** *Random elements $\mathbf{X}, \mathbf{Y}$ of $\mathbb{E}^n$ are called concordant stochastically ordered with respect to $\mathcal{F}$ (written as $\mathbf{X} \prec^n_{\mathcal{F}-cc} \mathbf{Y}$ or $\mathbf{Y} \succ^n_{\mathcal{F}-cc} \mathbf{X}$, often shortly: $\mathbf{X} \prec_{\mathcal{F}-cc} \mathbf{Y}$, resp., $\mathbf{Y} \succ_{\mathcal{F}-cc} \mathbf{X}$,) if*

$$E\left[\prod_{i=1}^{n} f_i(X_i)\right] \leq E\left[\prod_{i=1}^{n} f_i(Y_i)\right], \tag{1.4.19}$$

for all $f_i \in \mathcal{I}^*_+(\mathbb{E}) \cap \mathcal{F}$ and for all $f_i \in \mathcal{D}^*_+(\mathbb{E}) \cap \mathcal{F}, i = 1, \ldots, n$.

(b) *Let $T \subseteq \mathbb{R}$ be an index set for stochastic processes $\mathbf{X} = (X_t : t \in T)$ and $\mathbf{Y} = (Y_t : t \in T)$, $X_t, Y_t : (\Omega, \mathcal{F}, P) \to (\mathbb{E}, \mathcal{E}, \prec)$, $t \in T$. We say that $\mathbf{X}$ and $\mathbf{Y}$ are concordant stochastically ordered with respect to a class $\mathcal{F}$ of functions on $(\mathbb{E}, \mathcal{E}, \prec)$ (and write $\mathbf{X} \prec_{\mathcal{F}-cc} \mathbf{Y}$) if for all $n \geq 2$ and all $t_1 < t_2 < \ldots t_n$, we have on $\mathbb{E}^n$*

$$(X_{t_1}, \ldots, X_{t_n}) \prec_{\mathcal{F}-cc} (Y_{t_1}, \ldots, Y_{t_n}).$$

The setting of **(b)** will be applied to Markovian processes.

Taking in **(a)** for $\mathcal{F}$ the space of all measurable functions $\mathcal{M}$ on $\mathbb{E}$ we obtain the usual concordance ordering as in Daduna and Szekli [2006]. It is easy to see that the two-dimensional marginals of the Markov chains related by the Robin-Hood construction in (1.4.13) fulfill

$$(X_0, X_1) \leq_{\mathcal{M}-cc} (Y_0, Y_1).$$

For example, if $\mathcal{F}$ contains the indicator functions of point-generated increasing and decreasing sets, $\{i\}^{\uparrow} = \{j \in E : i \prec j\}$ and $\{i\}^{\downarrow} = \{j \in E : j \prec i\}$, for concordant stochastically ordered processes $\mathbf{X}$ and $\mathbf{Y}$ (with respect to $\mathcal{F}$) we can compare the probability of extreme events like

$$P(\inf (X_{t_1}, \ldots, X_{t_n}) \succ t) \leq P(\inf (Y_{t_1}, \ldots, Y_{t_n}) \succ t),$$

and

$$P(\sup(X_{t_1},\ldots,X_{t_n}) \prec s) \leq P(\sup(Y_{t_1},\ldots,Y_{t_n}) \prec s),$$

for fixed t and s. We mention, that in most cases $\mathcal{F}$ will be a convex cone of functions which is often additionally closed under point-wise convergence.

Discrete time. Let $\mathbf{X} = (X_t : t \in \mathbb{Z})$ and $\mathbf{Y} = (Y_t : t \in \mathbb{Z})$, $X_t, Y_t : (\Omega,\mathcal{F},P) \to (\mathbb{E},\mathcal{E},\prec)$, be discrete time, stationary, homogeneous Markov processes. Assume that π is an invariant (stationary) one–dimensional marginal distribution the same for both $\mathbf{X}$ and $\mathbf{Y}$, and denote the 1–step transition kernels for $\mathbf{X}$ and $\mathbf{Y}$, by $K^X : \mathbb{E} \times \mathcal{E} \to [0,1]$, and $K^Y : \mathbb{E} \times \mathcal{E} \to [0,1]$, respectively. Denote the respective transition kernels for the time reversed processes $\overleftarrow{\mathbf{X}}$, $\overleftarrow{\mathbf{Y}}$ by $\overleftarrow{K}^X$, $\overleftarrow{K}^Y$. We say that a stochastic kernel $K : \mathbb{E} \times \mathcal{E} \to [0,1]$ is $\mathcal{F}$-monotone if $\int f(x)K(s,dx) \in \mathcal{I}^*_+(\mathbb{E}) \cap \mathcal{F}$ for each $f \in \mathcal{I}^*_+(\mathbb{E}) \cap \mathcal{F}$.

The following property recently proved to be useful in comparing second order properties of Markov processes, see Hu and Pan [2000], Daduna and Szekli [1995], Baeuerle and Rolski [1998], Daduna and Szekli [2006]. It will be convenient to impose this condition here as well. A pair $\mathbf{X}$ and $\mathbf{Y}$ of discrete time Markov processes having the same invariant probability measure fulfils

$\mathcal{F}$-symmetric monotonicity if : *Either K^Y and $\overleftarrow{K}^X$ are $\mathcal{F}$-monotone, or K^X and $\overleftarrow{K}^Y$ are $\mathcal{F}$- monotone.*

The following theorem is an analog of Theorem 3.1 in Daduna and Szekli [2006].

Theorem 1.4.14 (concordance ordering under $\mathcal{F}$- symmetric monotonicity) *For the stationary Markov processes $\mathbf{X}, \mathbf{Y}$ defined above with a common invariant distribution π, fulfilling $\mathcal{F}$-symmetric monotonicity, the following relations are equivalent*

(i) $\mathbf{X} \prec_{\mathcal{F}-cc} \mathbf{Y}$

(ii) $(X_0, X_1) \prec^2_{\mathcal{F}-cc} (Y_0, Y_1)$

(iii) $\langle f, K^X g\rangle_\pi \leq \langle f, K^Y g\rangle_\pi$ *for all* $f, g \in \mathcal{I}^*_+(\mathbb{E}) \cap \mathcal{F}$, *and for all* $f, g \in \mathcal{D}^*_+(\mathbb{E}) \cap \mathcal{F}$

(iv) $\langle f, \overleftarrow{K}^X g\rangle_\pi \leq \langle f, \overleftarrow{K}^Y g\rangle_\pi$ *for all* $f, g \in \mathcal{I}^*_+(\mathbb{E}) \cap \mathcal{F}$, *and for all* $f, g \in \mathcal{D}^*_+(\mathbb{E}) \cap \mathcal{F}$

Continuous time. Let $\mathbf{X} = (X_t : t \in \mathbb{R})$ and $\mathbf{Y} = (Y_t : t \in \mathbb{R})$, $X_t, Y_t : (\Omega,\mathcal{F},P) \to (\mathbb{E},\mathcal{E},\prec)$, be stationary homogeneous Markov processes with countable state spaces. Denote the corresponding families of transition kernels of $\mathbf{X}$, and $\mathbf{Y}$, by $\mathbb{K}^X = (K^X_t : \mathbb{E} \times \mathcal{E} \to [0,1] : t \geq 0)$, and $\mathbb{K}^Y = (K^Y_t : \mathbb{E} \times \mathcal{E} \to [0,1] : t \geq 0)$, respectively, and the respective transition kernels for the stationary time reversed processes $\overleftarrow{\mathbf{X}}$, $\overleftarrow{\mathbf{Y}}$ by $\overleftarrow{\mathbb{K}}^X = (\overleftarrow{K}^X_t : \mathbb{E} \times \mathcal{E} \to [0,1] : t \geq 0)$, and $\overleftarrow{\mathbb{K}}^Y = (\overleftarrow{K}^Y_t : \mathbb{E} \times \mathcal{E} \to [0,1] : t \geq 0)$, respectively. Assume that π is an invariant distribution common for both $\mathbb{K}^X$ and $\mathbb{K}^Y$, that is $\int K^X_t(x,dy)\pi(dx) = \int K^Y_t(x,dy)\pi(dx) = \pi(dy)$, for all $t > 0$.

For the time reversed processes we use the corresponding notation $\overleftarrow{Q}^X$ and $\overleftarrow{Q}^Y$. We say that $I\!K^X = (K_t^X : \mathbb{E}\times\mathcal{E} \to [0,1] : t \geq 0)$ is $\mathcal{F}$-time monotone if for each $t \geq 0$, K_t^X is $\mathcal{F}$- monotone.

Analogously to the discrete case we define: A pair **X** and **Y** of continuous time Markov processes having the same invariant probability measure fulfills

$\mathcal{F}$-time symmetric monotonicity if : *Either $I\!K^Y$ and $\overleftarrow{I\!K}^X$ are $\mathcal{F}$-time monotone, or $I\!K^X$ and $\overleftarrow{I\!K}^Y$ are $\mathcal{F}$- time monotone.*

Using similar arguments as in Theorem 3.3 in Daduna and Szekli [2006] we have

Theorem 1.4.15 *Suppose that $(\mathbb{E}, \mathcal{E}, \prec)$ is countable and the above defined stationary chains* **X** *and* **Y** *have bounded intensity matrices Q^X and Q^Y, respectively. Then under $\mathcal{F}$-time symmetric monotonicity the following properties are equivalent*

(i) $\mathbf{X} \prec_{\mathcal{F}-cc} \mathbf{Y}$

(ii) $(X_0, X_t) \prec^2_{\mathcal{F}-cc} (Y_0, Y_t) \quad \forall t > 0,$

(iii) $\langle f, T_t^X g\rangle_\pi \leq \langle f, T_t^Y g\rangle_\pi$ *for all* $f, g \in \mathcal{I}_+^*(\mathbb{E}) \cap \mathcal{F}$, *and for all* $f, g \in \mathcal{D}_+^*(\mathbb{E}) \cap \mathcal{F}, \forall t > 0$

(iv) $\langle f, Q^X g\rangle_\pi \leq \langle f, Q^Y g\rangle_\pi$ *for all* $f, g \in \mathcal{I}_+^*(\mathbb{E}) \cap \mathcal{F}$, *and for all* $f, g \in \mathcal{D}_+^*(\mathbb{E}) \cap \mathcal{F}$

(v) $\langle f, \overleftarrow{Q}^X g\rangle_\pi \leq \langle f, \overleftarrow{Q}^Y g\rangle_\pi$ *for all* $f, g \in \mathcal{I}_+^*(\mathbb{E}) \cap \mathcal{F}$, *and for all* $f, g \in \mathcal{D}_+^*(\mathbb{E}) \cap \mathcal{F}$

Reducing the class of functions from $\mathcal{M}$ to some smaller class $\mathcal{F}$ makes this theorem much more versatile for applications as we shall demonstrate below.

From Theorem 1.4.15 we conclude that problem of comparing correlations for stochastic network processes in continuous time is an interplay of two tasks:
• proving monotonicity, the form of which we identified as $\mathcal{F}$- time symmetric monotonicity, and
• additionally proving generator inequalities.

Generator inequalities have been presented in the previous paragraphs. We shall continue with presenting the concept of time symmetric monotonicity for network processes.

From the recent literature on dependence structure of Markovian processes with one dimensional (linearly ordered) discrete state spaces we conclude that *$\mathcal{F}$-time symmetric monotonicity* (in continuous time) and *$\mathcal{F}$ symmetric monotonicity* (in discrete time) plays a central role, see e.g., Hu and Pan [2000]. This property occurred independently in the literature several times, see e.g., Baeuerle and Rolski [1998], Daduna and Szekli [1995][Lemma 3.2].

So in general we cannot hope to dispense from these assumptions when proving dependence properties in the more complex network setting. Nevertheless, the necessity of these assumptions is still an unsolved problem, some counterexamples, where dependence structures of Markovian processes over a finite time horizon are proved without *$\mathcal{F}$ symmetric monotonicity* are provided in Daduna and Szekli [2006][Section 3.3].

On the other hand the need for some monotonicity is emphasized further by the related theory of association in time for Markov processes, which strongly relies on monotonicity of the processes, see for a review Liggett [1985][chapter II], and Daduna and Szekli [1995].

For stochastic networks, which are in general not reversible, the property of *time symmetric monotonicity* seems to ba a natural property: Every Jackson network process $\mathbf{X}$ with service rates that are at all nodes nondecreasing functions of the local queue length [Daduna and Szekli [1995],Corollary 4.1] is stochastically monotone with respect to strong stochastic ordering on the set of all probability measures on $(\mathbb{N}^J, \leq)$. Because the time reversed process of a Jackson network process is the state process of a suitably defined Jackson network with the same properties for the service rates, any pair of Jackson network processes with the same steady state distribution fulfills *$\mathcal{F}$-time symmetric monotonicity*, where $\mathcal{F} = \mathcal{I}^*(\mathbb{N}^J, \leq)$.

We only mention that by a similar observation *$\mathcal{F}$-time symmetric monotonicity* holds for Gordon-Newell networks with respect to strong stochastic ordering.

In the investigations found in the literature $\mathcal{F}$ is always the class of all (bounded) increasing functions with respect to the natural linear order. The weaker concept of *$\mathcal{F}$-(time) symmetric monotonicity* for smaller sets of functions is suggested by the concept of integral orders with respect to subclasses of the class of increasing functions, see Mueller and Stoyan [2002] or Li and Shaked [1994]. The problems arising with this concept: We need the *closure property*, that $\mathcal{F}$-functions are transformed into $\mathcal{F}$-functions, or at least into the maximal generator of the respective order, Mueller and Stoyan [2002][Definition 2.3.3] or Li and Shaked [1994] (Definition 3.2).

The balance between having a small class of $\mathcal{F}$-functions and the necessity of obtaining the closure property is demonstrated next. The first example is in the spirit of the classical Gordon-Newell networks but with a smaller set $\mathcal{F}$. Recall that $\mathcal{L}$ is the set of nonnegative affine-linear functions on $\mathbb{E}_N$.

Property 1.4.16 (Linear service rates) *Consider two Gordon-Newell network processes $\mathbf{X}, \hat{\mathbf{X}}$ on state space $\mathbb{E}_N \subseteq \mathbb{N}^J$ equipped with the coordinate-wise order $\leq$, both with stationary distribution $\pi^{N,J}$. Assume that the service rates in both networks at all nodes are linear functions of the local queue length, $\mu_j(n_j) = \mu_j \cdot n_j, n_j \geq 0$ for all $j = 1, \ldots, J$.*

Then the pair $\mathbf{X}, \hat{\mathbf{X}}$ of Gordon-Newell network processes is $\mathcal{L}$-time symmetric monotone.

Property 1.4.17 (Generalized tandem network) *Consider an open tandem network process $\tilde{\mathbf{X}}$ on the state space $\mathbb{N}^J$ equipped with the partial sum order $\leq_*$ with stationary distribution $\tilde{\pi}^J$. The routing for $\tilde{\mathbf{X}}$ is* linear *as follows:*

- *customers enter the network only through node 1: $\lambda_1 > 0, \lambda_j = 0, j = 2, \ldots, J$,*
- *customers depart from the network only from node J: $\tilde{r}_{J0} > 0, r_{j0} = 0, j = 1, \ldots, J-1$,*
- *customers move only stepwise: $\tilde{r}_{j(j+1)} > 0, j = 1, \ldots, J-1$, and $\tilde{r}_{j(j-1)} \geq 0, j = 2, \ldots, J$, and $\tilde{r}_{jj} \geq 0, j = 1, \ldots, J$, and $\tilde{r}_{ji} = 0$ in any other case.*

Let $\hat{\mathbf{X}}$ be another generalized tandem network process with stationary distribution $\tilde{\pi}^J$, and with routing subject to the same restriction as described for $\tilde{\mathbf{X}}$.

Assume that the arrival rates and the (nondecreasing) service rates in both networks are the same and bounded.

Then the pair $\tilde{\mathbf{X}}, \hat{\mathbf{X}}$ is $\mathcal{I}^*(\mathbb{R}^J, \leq_*) \cup \mathcal{D}^*(\mathbb{R}^J, \leq_*)$-time symmetric monotone.

Property 1.4.18 (Functions of the total population size) *Consider two Jackson networks with linear service rates, i.e., $\mu_j(n_j) = \mu_j \cdot n_j$ for all $h \in \mathbb{N}$ and for all $j = 1, \dots, J$, which have the same stationary distribution. Assume further that inside both networks the effective departure rates from all nodes are the same, i.e., $\mu_j \cdot r_{j0}$ is invariant for all $j = 1, \dots, J$, (and therefore > 0). Let*

$$\mathcal{F} = \{f : \mathbb{N}^J \to \mathbb{R}_+ : f(n_1, \dots, n_J) = \hat{f}(n_1 + \cdots + n_J) \text{ for some } \hat{f} : \mathbb{R} \to \mathbb{R}_+\}$$

be the set of real valued functions on $\mathbb{N}^J$, which depend on the sum of the arguments only. Then the state processes in these networks constitute a $\mathcal{F}-$time symmetric monotone pair.

Let $\boldsymbol{\rho} = (\rho_1, \dots, \rho_J)$ be an ordered sequence of the numbers $\{1, 2, \dots, J\}$ (without repetition) which will serve as rank vector for the linear factors of functions in

$$\begin{aligned}\mathcal{L}(\boldsymbol{\rho}) &= \{f : S(I,J) \to \mathbb{R}_+ : f(n_1, \dots, n_J) \\ &\qquad = a + \sum_{i=1}^{J} \alpha_i \cdot n_i, \alpha_i \in \mathbb{R}, i = 1, \dots, J, a \in \mathbb{R}_+, \mathcal{R}(\alpha_1, \dots, \alpha_J) = \boldsymbol{\rho}\} \subseteq \mathcal{L}.\end{aligned} \tag{1.4.20}$$

Theorem 1.4.19 *Consider two ergodic Gordon-Newell network processes with common stationary distribution $\pi^{(N,J)}$: $\mathbf{X}$ with a doubly stochastic routing matrix $R = [r_{ij}]$ and $\hat{\mathbf{X}}$ with the routing matrix $\hat{R} = R \cdot T$, for a doubly stochastic matrix $T = [t_{ij} : i, j = 1, \dots, J]$. The service rates $\mu_j(n_j) = \mu_j \cdot n_j$ are in both networks the same.*

Let $A\mathcal{R}(\boldsymbol{\mu}) = \rho = (\rho_1, \dots, \rho_J)$ denote the antirank vector of the unit service intensity vector $\boldsymbol{\mu} = (\mu_1, \dots, \mu_J)$. Then

$$\mathbf{X} \geq_{\mathcal{L}(\boldsymbol{\rho})-cc} \hat{\mathbf{X}}. \tag{1.4.21}$$

Example 1.4.20 *In many applications the functions in $\mathcal{F}$ serve as cost or reward functions connected with the network's performance. A typical cost function is as follows:*

Per customer at node j and per time unit a cost of amount α_j occurs, so $f_j(X_j(t)) = \alpha_j \cdot X_j(t)$ is the cost at node j. Incorporating a fixed constant cost a then in state $(n_1, \dots, n_J)$ the total cost per time unit is $f(n_1, \dots, n_J) = a + \sum_{i=1}^{J} \alpha_i \cdot n_i$. When we put the natural assumption that the costs increase when the service speed decreases, this situation is covered by the preceeding theorem.

Our next theorem is in the realm of generalized tandem networks as described in Proposition 1.4.17. Robin-Hood transforms under this graph structure are of the following form: Shift (probability) mass $\alpha > 0$ from arcs $r_{j,j+1}$ and $r_{j+1,j}$ to arcs $r_{j,j}$ and $r_{j+1,j+1}$. This has the following consequences.

Theorem 1.4.21 (General tandem) *Consider Jackson network processes $\tilde{\mathbf{X}}, \hat{\mathbf{X}}$ on state space $\mathbb{N}^J$ equipped with partial sum order $\leq_*$, which have the same stationary distribution $\tilde{\pi}^J$.*

Assume further that for some fixed $j \in \{1, \dots, J-1\}$ and $\alpha > 0$ it holds $\tilde{r}_{j(j+1)} > \alpha$ and $\tilde{r}_{(j+1)j} \geq \alpha$, and that the routing for $\hat{\mathbf{X}}$ is obtained by Robin-Hood transformation according to (1.4.13), where $a = b = j$ and $c = d = j+1$.

Then with $\mathcal{PS} := \mathcal{I}^*(\mathbb{R}^J, \leq_*) \cup \mathcal{D}^*(\mathbb{R}^J, \leq_*)$ we have

$$\tilde{\mathbf{X}} \leq_{\mathcal{PS}-cc} \hat{\mathbf{X}}. \tag{1.4.22}$$

It is worth mentioning that the Robin-Hood transformation applied to the tandem routing yields Peskun ordering between the routing matrices (see Definition 1.4.4) but we do not need reversibility in the above theorem which is substituted by the time symmetric monotonicity.

1.5 Jackson networks with breakdowns

The class of Jackson networks can be reasonably extended. Assume the servers at the nodes in the Jackson network to be unreliable, i.e., the nodes may break down. The breakdown event may occur in different ways. Nodes may break down as an isolated event or in groups simultaneously, and the repair of the nodes may end for each node individually or in groups as well. It is not required that those nodes which stopped service simultaneously return to service at the same time instant. To describe the system's evolution we have to enlarge the state space for the network process as will be described below. For a detailed description see Sauer and Daduna [2003].

Control of breakdowns and repairs is as follows:

Let $\mathrm{I} \subset \mathrm{J}$ be the set of nodes in down status and $\mathrm{H} \subset \mathrm{J}\backslash\mathrm{I}, \mathrm{H} \neq \emptyset$, be some subset of nodes in up status. Then the nodes of H break down with intensity $\alpha(\mathrm{I}, \mathrm{I} \cup \mathrm{H})$.

Nodes in down status neither accept new customers nor continue serving the old customers which will wait for the server's return. (At nodes i under repair the service intensities $\mu_i(n_i)$ are set to 0). Therefore, the routing matrix has to be changed so that customers attending to join a node in down status are rerouted to nodes in up status or to the outside. We describe three possible rerouting schemes below.

Assume the nodes in I are under repair, $\mathrm{I} \subset \mathrm{J}, \mathrm{I} \neq \emptyset$. Then if $\mathrm{H} \subset \mathrm{I}, \mathrm{H} \neq \emptyset$, the nodes of H return from repair as a batch group with intensity $\beta(\mathrm{I}, \mathrm{I} \setminus \mathrm{H})$ and immediately resume their services. Routing then has to be updated again as will be described below.

The intensities for occurrence of breakdowns and repairs have to be set under constraints. A rather general versatile class is defined as follows.

Definition 1.5.1 *Let* I *be the set of nodes in down status. The intensities for breakdowns, resp. repairs for* $\mathrm{H} \neq \emptyset$ *are defined by*

$$\alpha(\mathrm{I}, \mathrm{I} \cup \mathrm{H}) := \frac{a(\mathrm{I} \cup \mathrm{H})}{a(\mathrm{I})}, \quad \textit{resp.} \quad \beta(\mathrm{I}, \mathrm{I} \setminus \mathrm{H}) := \frac{b(\mathrm{I})}{b(\mathrm{I}\backslash\mathrm{H})}, \tag{1.5.1}$$

where a *and* b *are any functions,* $a, b : \mathcal{P}(\mathrm{J}) \to [0, \infty)$ *whereas* $\frac{0}{0} := 0$.
The above intensities are assumed henceforth to be finite.

The rerouting matrices of interest are as follows.

Definition 1.5.2 (BLOCKING) *Assume that the routing matrix of the original process is reversible. Assume the nodes in* I *are the nodes of the Jackson network presently under repair. Then the routing probabilities are redefined on* $\mathrm{J}_0\backslash\mathrm{I}$ *according to*

$$\tilde{r}_{ij}^{\mathrm{I}} = \begin{cases} \tilde{r}_{ij}, & i, j \in \mathrm{J}_0\backslash\mathrm{I},\ i \neq j, \\ \tilde{r}_{ii} + \sum_{k\in\mathrm{I}} \tilde{r}_{ik}, & i \in \mathrm{J}_0\backslash\mathrm{I},\ i = j. \end{cases} \tag{1.5.2}$$

Note that even in case of $\tilde{r}_{00} = 0$*, external arrivals may be now rejected with positive probability to an immediate departure, because arrivals to nodes under repair are rerouted:*

$$\tilde{r}^{\mathrm{I}}_{00} = \tilde{r}_{00} + \sum_{k \in \mathrm{I}} \tilde{r}_{0k} \geq 0.$$

Definition 1.5.3 (STALLING) *If there is any breakdown of either a single node or a group of nodes, then all arrival streams to the network and all service processes at the nodes in up status are completely interrupted and resumed only when all nodes are repaired again.*

Definition 1.5.4 (SKIPPING) *Assume that the nodes in* I *are the nodes presently under repair. Then the routing matrix is redefined on* $\mathrm{J}_0 \setminus \mathrm{I}$ *according to:*

$$\tilde{r}^{\mathrm{I}}_{jk} = \tilde{r}_{jk} + \textstyle\sum_{i \in \mathrm{I}} \tilde{r}_{ji} \tilde{r}^{\mathrm{I}}_{ik}, \quad k, j \in \mathrm{J}_0 \backslash \mathrm{I},$$

$$\tilde{r}^{\mathrm{I}}_{ik} = \tilde{r}_{ik} + \textstyle\sum_{l \in \mathrm{I}} \tilde{r}_{il} \tilde{r}^{\mathrm{I}}_{lk}, \quad i \in \mathrm{I}, k \in \mathrm{J}_0 \backslash \mathrm{I}.$$

For describing the breakdown of nodes in Jackson networks we have to attach to the state spaces $\mathbb{E} = \mathbb{N}^J$ of the corresponding network processes an additional component which carries information of the reliability behavior of the system described by a process $\mathbf{Y}$. We introduce states of the form

$$(\mathrm{I};\ n_1, n_2, \ldots, n_J) \in \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J.$$

The meaning of such a prototype state is:

I is the set of nodes under repair. For $j \in \mathrm{J} \backslash \mathrm{I}$, the numbers $n_j \in \mathbb{N}$ indicate that at nodes j which work in a normal up status, there are n_j customers present; for $i \in \mathrm{I}$ the numbers $n_i \in \mathbb{N}$ indicate that at each node i which is in down status there are n_i customers that wait at node i for the return of the repaired server. Collecting these states we define for the networks new Markov processes $\tilde{\mathbf{Z}} = (\mathbf{Y}, \hat{\mathbf{X}})$ on

$$\widetilde{\mathbb{E}} = \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J. \tag{1.5.3}$$

For such general models with breakdowns and repairs and with the above rerouting principles it was shown in Sauer and Daduna [2003] that on the state space $\widetilde{\mathbb{E}}$ the steady state distribution for $\tilde{\mathbf{Z}}$ is of product form. Note that the breakdown/repair process $\mathbf{Y}$ is Markovian on the state space $\mathcal{P}(\mathrm{J})$ of all subsets of J, but that the network process component $\hat{\mathbf{X}}$ is in this setting **not a Markov** process.

Theorem 1.5.5 *The process* $\tilde{\mathbf{Z}}$ *with breakdown and repair intensities given by Eq. (1.5.1) and rerouting according to either* BLOCKING *or* STALLING*, or* SKIPPING *has a stationary distribution of product form given by:*

$$\tilde{\pi}^{Y,J}(\mathrm{I};\ n_1, n_2, \ldots, n_J) = \pi^Y(\mathrm{I})\, \tilde{\pi}^J(n_1, n_2, \ldots, n_J)$$

with

$$\pi^Y(\mathrm{I}) = \left(1 + \sum_{\substack{\mathrm{K} \subset \mathrm{J} \\ \mathrm{K} \neq \emptyset}} \frac{a(\mathrm{K})}{b(\mathrm{K})}\right)^{-1} \frac{a(\mathrm{I})}{b(\mathrm{I})} \quad \text{for } \mathrm{I} \subset \mathrm{J}$$

and $\tilde{\pi}^J(n_1, n_2, \ldots, n_J)$ *the equilibrium distribution in the standard Jackson network.*

Note that time evolution of the queueing process $\hat{\mathbf{X}}$ is different in all cases (standard Jackson, BLOCKING, STALLING, SKIPPING). At the same time, it is possible to change the breakdown/repair intensities in such a way that the stationary distribution for the joint process $\tilde{\mathbf{Z}}$ remains unchanged.

1.5.1 Bounds via dependence ordering for networks with breakdowns

Dependence ordering of Jackson networks with breakdowns

Consider Markov processes $\tilde{\mathbf{Z}} = (\mathbf{Y}, \hat{\mathbf{X}})$ on $\widetilde{\mathbb{E}} = \mathcal{P}(\mathrm{J}) \times \mathbb{N}^J$ describing the state of the Jackson network with breakdowns. For a given set K denote by $\{\mathrm{K}\}^{\uparrow}$, $\{\mathrm{K}\}^{\downarrow}$, $\{\mathrm{K}\}^{\prec}$ the sets of its ancestors, descendants and relatives, respectively, i.e.

$$
\begin{aligned}
\{\mathrm{K}\}^{\uparrow} &:= \{\mathrm{I} \subseteq \mathrm{J} : \mathrm{K} \subset \mathrm{I}, \mathrm{K} \neq \mathrm{I}\}, \\
\{\mathrm{K}\}^{\downarrow} &:= \{\mathrm{I} \subseteq \mathrm{J} : \mathrm{I} \subset \mathrm{K}, \mathrm{K} \neq \mathrm{I}\}, \\
\{\mathrm{K}\}^{\prec} &:= \{\mathrm{K}\}^{\downarrow} \cup \{\mathrm{K}\}^{\uparrow}.
\end{aligned}
$$

Recall that $\mathbf{Y} = (Y(t), t \geq 0)$ is a cadlag Markov process on the state space $\mathcal{P}(\mathrm{J})$ which describes availability of the network's components over time, i.e. $Y(t) = \mathrm{K}$, $\mathrm{K} \in \mathcal{P}(\mathrm{J})$, means that at time t the set K consists of the nodes which are under repair. We have

$$
q^Y(\mathrm{K}, \mathrm{H}) = \begin{cases} \alpha(\mathrm{K}, \mathrm{H}) = \frac{a(\mathrm{H})}{a(\mathrm{K})}, & \text{if } \mathrm{H} \in \{\mathrm{K}\}^{\uparrow}_{+}, \\ \beta(\mathrm{K}, \mathrm{H}) = \frac{b(\mathrm{K})}{b(\mathrm{H})}, & \text{if } \mathrm{H} \in \{\mathrm{K}\}^{\downarrow}, \\ -\sum_{\mathrm{I} \in \{\mathrm{K}\}^{\uparrow}} \frac{a(\mathrm{I})}{a(\mathrm{K})} - \sum_{\mathrm{I} \in \{\mathrm{K}\}^{\downarrow}} \frac{b(\mathrm{K})}{b(\mathrm{I})}, & \text{if } \mathrm{H} = \mathrm{K}, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5.4}
$$

We define for fixed $\mathrm{I}_1 \subset \mathrm{I}_2$, $\mathrm{J}_1 \subset \mathrm{J}_2$ new intensities by

$$
q^{Y^{\varepsilon}}(\mathrm{K}, \mathrm{H}) = \begin{cases} q^Y(\mathrm{K}, \mathrm{H}) + \frac{\varepsilon}{\pi^Y(\mathrm{K})}, & \text{if } (\mathrm{K} = \mathrm{I}_1, \mathrm{H} = \mathrm{J}_1) \text{ or } (\mathrm{K} = \mathrm{I}_2, \mathrm{H} = \mathrm{J}_2), \\ q^Y(\mathrm{K}, \mathrm{H}) - \frac{\varepsilon}{\pi^Y(\mathrm{K})}, & \text{if } (\mathrm{K} = \mathrm{I}_1, \mathrm{H} = \mathrm{J}_2) \text{ or } (\mathrm{K} = \mathrm{I}_2, \mathrm{H} = \mathrm{J}_1), \\ q^Y(\mathrm{K}, \mathrm{H}), & \text{otherwise}. \end{cases} \tag{1.5.5}
$$

Consider the processes $\mathbf{Y}$, $\mathbf{Y}^{\varepsilon}$ on state space $(\mathcal{P}(\mathrm{J}), \subseteq)$ and two processes $\tilde{\mathbf{Z}} = (\mathbf{Y}, \hat{\mathbf{X}})$, $\tilde{\mathbf{Z}}^{\varepsilon} = (\mathbf{Y}^{\varepsilon}, \hat{\mathbf{X}}^{\varepsilon})$ which have the same routing matrices and service intensities but different breakdown/repair processes $\mathbf{Y}$ and $\mathbf{Y}^{\varepsilon}$.

The following property is taken from Daduna et al [2006]. Note that both processes, before and after modification, have the same product form invariant distribution, but they are different in their time evolution. The modification results in a higher rate to change sets under repair to "similar" ones. Of course such a transformation can be iterated, which leads to eliminate transitions between not ordered sets. Note that processes under comparison are not Markovian (the "big" process $\mathbf{Z}$ is Markovian, but $\hat{\mathbf{X}}$ usually not).

Property 1.5.6 (Enlarging dependence in time evolution by structure of breakdowns)
Assume that two Jackson networks have the same arrival intensities, the same rerouting matrices according to either BLOCKING *or* STALLING *or* SKIPPING *and breakdown/repair intensity*

matrices are given by (1.5.4) *and* (1.5.5)*. Assume also that breakdown/repair intensity matrices and its time-reversal counterparts are stochastically monotone. Then in equilibrium, for all* $n \geq 2$ *and* $t_1 \leq \cdots \leq t_n$,

$$E\left[f\left(\hat{X}(t_1), \ldots, \hat{X}(t_n)\right)\right] \leq E\left[f\left(\hat{X}^{\varepsilon}(t_1), \ldots, \hat{X}^{\varepsilon}(t_n)\right)\right],$$

for all functions f *with isotone differences on* $(\widetilde{\mathbb{E}}^n, (\leq^J)^n)$.

1.6 General networks

Consider an open network of J, k_j-server, FCFS nodes, $j \in \mathrm{J} = \{1, \ldots, J\}$. We set $\boldsymbol{k} = (k_1, \ldots, k_J)$. Denote by $\boldsymbol{N}^0 = (N^1, \ldots, N^J)$ the vector of counting processes of arrivals from outside to the nodes, by $\boldsymbol{S} = (S^1, \ldots, S^J)$ the vector of service time sequences $S^j = (S_1^j, \ldots)$, where S_n^j denotes the service time received by the n-th initiated job at station j. Denote by $\boldsymbol{V} = (V^1, \ldots, V^J)$ the vector of destination sequences $V^j = (V_1^j, \ldots)$, where V_n^j denotes the number of the node visited by the job that is the n-th departing from the node j or $V_n^j = 0$ if the job leaves the network. Let $\hat{\mathbf{X}} = (\hat{X}(t) : t \geq 0)$ denote the vector process recording the joint queue lengths in the network for time t. For $t \in \mathbb{R}_+$, $\hat{X}(t) = (\hat{X}_1(t), \ldots, \hat{X}_J(t))$ means that at time t there are $\hat{X}_j(t)$ customers present at node j, either in service or waiting. Given an initial content $\hat{X}(0) = (\hat{X}_1(0), \ldots, \hat{X}_J(0))$, such a general network is determined by the arrival, service and routing variables and will be denoted therefore by $(\boldsymbol{N}^0, \boldsymbol{V})/\boldsymbol{S}, \boldsymbol{k}/J$. The corresponding closed network, which starts with N customers and does not admit arrivals from outside will be denoted by $\boldsymbol{V}/\boldsymbol{S}, \boldsymbol{k}/J + N$. Denote by $\boldsymbol{N}^d = (N^{1,\cdot}, \ldots, N^{J,\cdot})$ the vector of point processes of departures from the nodes, and by $\boldsymbol{N}^a = (N^{\cdot,1}, \ldots, N^{\cdot,J})$ the vector of all arrivals to the nodes. The limits (if they exist) $\lim_{t\to\infty} N^{j,\cdot}(t)/t$, which are the throughputs of the consecutive nodes will be denoted by $TH_j(\boldsymbol{V}/\boldsymbol{S}, \boldsymbol{k}/J + N)$, $j \in \mathrm{J}$.

For an open network of J, k_j-server, FCFS nodes, with finite waiting rooms of sizes $B_1, \ldots, B_J$ we introduce additional parameter $\boldsymbol{B} = (B_1, \ldots, B_j)$ and use notation $(\boldsymbol{N}^0, \boldsymbol{V})/\boldsymbol{S}, \boldsymbol{k}, \boldsymbol{B}/J$ for open networks, and $\boldsymbol{V}/\boldsymbol{S}, \boldsymbol{k}, \boldsymbol{B}/J + N$ for closed networks. An arriving job from outside that finds the selected node full is lost. A job that completes service in node j proceeds to the next node according to V^j unless the latter is full. In this case we consider *manufacturing blocking*: the job has to wait until there is an empty space in the selected node, i.e. the server at node j is idle (blocked); or we consider *communication blocking*: if a job completes service at j and finds the next node full, it has to repeat service at j.

An alternative description of a J-variate arrival process is the one given by a sequence $\boldsymbol{\Phi} \equiv \{(T_n^1, \ldots, T_n^J)\}_{n=-\infty}^{\infty}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, such that $T_0^i \leq 0 < T_1^i$, $T_n^i < T_{n+1}^i$, $i = 1, \ldots, J$, $n \in \mathbb{Z}$ and $\lim_{n\to\pm\infty} T_n^i = \pm\infty$ ($\boldsymbol{\Phi}$ is nonexplosive). Denote by $\{X_n^i\}_{n=-\infty}^{\infty}$ a sequence of interpoint distances, i.e. $X_n^i = T_n^i - T_{n-1}^i$ (the interval X_1^i contains 0). Then a J-variate point process $\boldsymbol{\Phi}$ can be seen as a random element assuming its values in $(\mathbb{R}_+^{\infty})^J$.

Let $\mathcal{N}$ be a set of locally finite integer valued measures on $\mathbb{R}$. Equivalently, we view $\boldsymbol{\Phi}$ as a random measure $\boldsymbol{\Phi} : \Omega \to \mathcal{N}^k$ with the coordinate functions $\boldsymbol{\Phi} = (\Phi^1, \ldots, \Phi^k)$, $\Phi^i : \Omega \to \mathcal{N}$. Then for all Borel sets B, $N_{\boldsymbol{\Phi}}^i(B) := \Phi^i(B)$ is the corresponding counting variable. However, if it is clear which point process do we mean we shall write shortly N^i instead of $N^i_{\mathbf{\Phi}}$. The corresponding counting processes $(N^i(t), t \geq 0)$, $i = 1, \ldots, J$ are given by $N^i(t) := N^i((0,t])$.

It will be convenient to have notation for another point process $\mathbf{\Psi}$ with the corresponding points $\{(\mathcal{T}_n^1, \ldots, \mathcal{T}_n^k)\}_{n\geq 1}$, $k \leq \infty$ and interpoint distances $U_n^i = \mathcal{T}_n^i - \mathcal{T}_{n-1}^i$, $i = 1, \ldots, k$.

In the case $k = 1$ we shall write T_n (X_n, N, λ) and $\mathcal{T}_n$ (U_n) instead of writing these quantities with the superscript 1.

We denote by $\mathcal{L}_{\text{st}}$ ($\mathcal{L}_{\text{cx}}$, $\mathcal{L}_{\text{icx}}$) the class of increasing (convex, increasing and convex) functions $f : \mathbb{R} \to \mathbb{R}$.

Define for $1 \leq l \leq m$, $\epsilon > 0$ and arbitrary function $\varphi : \mathbb{R}^m \to \mathbb{R}$ the difference operator Δ_l^ϵ by

$$\Delta_l^\epsilon \varphi(u_1, \ldots, u_m) = \varphi(u_1, \ldots, u_{l-1}, u_l + \epsilon, u_{l+1}, \ldots, u_m) - \varphi(u_1, \ldots, u_m)$$

for given $u_1, \ldots, u_m$.

We denote arbitrary m-dimensional intervals by $\mathcal{J} \subseteq \mathbb{R}^m$, i.e. $\mathcal{J} = I^1 \times \cdots \times I^m$, where I^j is a (possibly infinite ended) interval on $\mathbb{R}$ for $j = 1, \ldots, m$. A function $\varphi : \mathbb{R}^m \to \mathbb{R}$ is *supermodular* on $\mathcal{J}$ if for all $1 \leq l < j \leq m$, $\epsilon_l, \epsilon_j > 0$ and $\mathbf{u} = (u_1, \ldots, u_m) \in \mathcal{J}$ such that $(u_1, \ldots, u_{l-1}, u_l + \epsilon_l, u_{l+1}, \ldots, u_m) \in \mathcal{J}$ we have

$$\Delta_l^{\epsilon_l} \Delta_j^{\epsilon_j} \varphi(\mathbf{u}) \geq 0\,.$$

A function $\varphi : \mathbb{R}^m \to \mathbb{R}$ is *directionally convex* on $\mathcal{J}$ if it is supermodular on $\mathcal{J}$ and convex w.r.t. each coordinate on I^j, $j = 1, \ldots, m$ or, equivalently

$$\Delta_l^{\epsilon_l} \Delta_j^{\epsilon_j} \varphi(\mathbf{u}) \geq 0$$

for all $1 \leq l \leq j \leq m$. We denote by $\mathcal{L}_{\text{sm}}(\mathcal{J})$ ($\mathcal{L}_{\text{dcx}}(\mathcal{J})$) the class of all supermodular (directionally convex) functions on $\mathcal{J}$. Moreover, we denote the class of increasing directionally convex functions on $\mathcal{J}$ by $\mathcal{L}_{\text{idcx}}(\mathcal{J})$ and symmetric supermodular functions on $\mathcal{J}$ by $\mathcal{L}_{\text{ssm}}(\mathcal{J})$. We skip $\mathcal{J}$ in this notation if $\mathcal{J} = \mathbb{R}^m$.

For arbitrary random vectors $(Y_1, \ldots, Y_n)$, $(\tilde{Y}_1, \ldots, \tilde{Y}_n)$ defined on probability spaces $(\Omega, \mathcal{F}, P)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ respectively, we write

$$(Y_1, \ldots, Y_n) <_{\text{a}} (\tilde{Y}_1, \ldots, \tilde{Y}_n)$$

if

$$E[\varphi(Y_1, \ldots, Y_n)] \leq E[\varphi(\tilde{Y}_1, \ldots, \tilde{Y}_n)]$$

for all $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that $\varphi \in \mathcal{L}_{\text{a}}$, where $\mathcal{L}_{\text{a}}$ denotes one of the classes $\mathcal{L}_{\text{sm}}$, $\mathcal{L}_{\text{dcx}}$, $\mathcal{L}_{\text{idcx}}$. Similarly, for random sequences $\{Y_n\}_{n\geq 1}$ and $\{\tilde{Y}_n\}_{n\geq 1}$ we write $\{Y_n\} <_{\text{a}} \{\tilde{Y}_n\}$ if for all $n \geq 1$, $(Y_1, \ldots, Y_n) <_{\text{a}} (\tilde{Y}_1, \ldots, \tilde{Y}_n)$.

Let $\mathbf{\Psi}$ ($\tilde{\mathbf{\Psi}}$) be a J-variate stationary point process with the corresponding interpoint distances $\{U_n^i\}$ ($\{\tilde{U}_n^i\}$), $i = 1, \ldots, k$. We write

- $\mathbf{\Psi} <_{\text{m-a-}\infty} \tilde{\mathbf{\Psi}}$ if $(\{U_n^1\}, \ldots, \{U_n^J\}) <_{\text{a}} (\{\tilde{U}_n^1\}, \ldots, \{\tilde{U}_n^J\})$, i.e. if for all $n \geq 1$, ,

$$((U_1^1, \ldots, U_n^1), \ldots, (U_1^J, \ldots, U_n^J)) <_{\text{a}} \left(\left(\tilde{U}_1^1, \ldots, \tilde{U}_n^1\right), \ldots, \left(\tilde{U}_1^J, \ldots, \tilde{U}_n^J\right)\right)\,.$$

Let $\mathbf{\Phi}$ ($\mathbf{\Phi}$) be a J-variate point process with the corresponding counting measures N^i ($\tilde{N}^i$), $i = 1, \ldots, J$. We write

- $\mathbf{\Phi} <_{\mathrm{m-a-D}} \tilde{\mathbf{\Phi}}$ if for all $0 \le t_1 < t_2 < \cdots < t_r$, $r \ge 1$,

$$(N^i(t_1), \ldots, N^i(t_r), i = 1, \ldots, J) <_{\mathrm{a}} (\tilde{N}^i(t_1), \ldots, \tilde{N}^i(t_r), i = 1, \ldots, J)\,.$$

Let $\mathcal{I} = \{I_n\}_{n \ge 1}$ be a partition of $\mathbb{R}_+$ such that I_r, $r \ge 1$ have the same length. We write

- $\mathbf{\Phi} <_{\mathrm{m-a-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ if for all $(I_1, \ldots, I_r)$, $r \ge 1$,

$$(N^i(I_1), \ldots, N^i(I_r), i = 1, \ldots, J) <_{\mathrm{a}} (\tilde{N}^i(I_1), \ldots, \tilde{N}^i(I_r), i = 1, \ldots, J)\,.$$

Here $<_{\cdot\cdot\infty}$ ($<_{\cdot\cdot\mathcal{N}}$, $<_{\cdot\cdot\mathrm{D}}$) stands for the comparison of point processes considered as random elements of $(\mathbb{R}_+^\infty)^J$, $(\mathcal{N}^k$, $(\mathrm{D}([0,\infty)))^k J)$, where $\mathrm{D}([0,\infty))$ is the space of right-hand-side continuous functions with left-hand-side limits.
For 1-variate point processes ($J = 1$) we shall omit subscript 1, and write $<_{\mathrm{a-D}}$, $<_{\mathrm{a-}\mathcal{N}}$, $<_{\mathrm{a-}\infty}$ coincides with orderings defined in Kwieciński and Szekli [1991].

1.6.1 Dependence and variability in input

The next property proved by Meester and Shanthikumar [1993] is a general result connected with so called Ross's conjecture, which still receives some attention in the context of single queues.

Property 1.6.1 *Consider two open networks with finite waiting rooms* $(\boldsymbol{N}^0, \boldsymbol{V})/\boldsymbol{S}, \mathbf{1}, \boldsymbol{B}/J$, *and* $(\boldsymbol{N}'^0, \boldsymbol{V})/\boldsymbol{S}, \mathbf{1}, \boldsymbol{B}/J$ *which operate according to the manufacturing blocking (***1** *denotes the vector with 1 on each coordinate). Assume that in* $\boldsymbol{N}^0$, *and* $\boldsymbol{N}'^0$ *only the first coordinates are non-trivial, and* $V^j = (j+1, j+1, \ldots)$, *i.e. these networks are open tandems. If* $\boldsymbol{S}$ *is a vector of independent sequences of independent exponential random variables with rates* $\mu_j(k)$ *when there are k jobs at station j which are increasing and concave functions in k then* $N^{0,1} <_{idcx-\mathcal{N}} N'^{0,1}$ *implies that* $(N^l(t), N^l(t) + \tilde{X}_1(t), \ldots, N^l(t) + \tilde{X}_1(t) + \cdots + X_J(t)) <_{idcx}$ $(N'^l(t), N'^l(t) + \tilde{X}'_1(t), \ldots, N'^l(t) + \tilde{X}'_1(t) + \cdots + \tilde{X}'_J(t)$, *where* N^l *denotes the point process of lost jobs.*

Chang et al. [1991] considered a special case where the authors assumed infinite buffers, and doubly stochastic Poisson input point process N^1, obtaining this result only for the number of jobs. Moreover for finite buffers they obtained the result for the number of lost jobs. For a more recent research of this type, where the arrival stream consists of multiple on-off sources, see Koole and Liu [1998].

1.6.2 Comparison of workloads

Assume that for the routing vector $\boldsymbol{V} = (V^1, \ldots, V^J)$, we have $V^j_n = 0$ for all $j \in \mathrm{J}$, and $n \in \boldsymbol{N}$. That is the arrivals are routed to one of the J queues with infinite waiting room and after receiving service depart from the system. Arrivals are characterized by $\boldsymbol{N}^0 = (N^1, \ldots, N^J)$

which can be seen as a marked point process (τ_n, Z_n), where τ_n denotes the epoch of the nth arrival and Z_n denotes the number of the station this arrival is routed to. Consider a parallel system with *resequencing synchronization*, which means that the nth customer departs from the system provided that all the customers that arrived earlier have been served. Denote by $\boldsymbol{W}(t) = (W^1(t), \dots, W^J(t))$ the amount of work in the queues at time t. The next property comes from Chang Cheng-Shang [1992].

Property 1.6.2 *Suppose that in a parallel system described above, (Z_n) is a stationary Markov chain independent of (τ_n) and $\boldsymbol{S}$, with the transition probabilities $P(Z_{n+1} = j \mid Z_n = i) = (1-\sigma)/J$, $i \neq j$, and $P(Z_{n+1} = i \mid Z_n = i) = \sigma + (1-\sigma)/J$, for some parameter $\sigma \in [0,1]$. Then for each t, $E(f(\boldsymbol{W}(t)))$ is increasing as a function of σ, provided f is coordinate-wise increasing, symmetric, submodular, and convex in each variable.*

Workload in parallel queues

Consider a queueing system of J parallel $G/G/1$ FIFO queues. The input is generated by kJ-variate point processes $\mathbf{\Phi}$ (interarrival times) and $\mathbf{\Psi}$ (service times), independent of $\mathbf{\Phi}$. For $t \geq 0$ and $I = (a,b]$ define

$$M^i(t) = \sum_{n=1}^{N^i(t)} U_n^i, \quad i = 1, \dots, J$$

and

$$M^i(I) = \sum_{n=N^i(a)+1}^{N^i(b)} U_n^i, \quad i = 1, \dots, J\,.$$

Call M^i, $i = 1, \dots, k$ cumulative processes. Denote by

$$\mathbf{W}(t) \equiv (W^1(t), \dots, W^J(t))$$

the vector of transient workloads, which is known to fulfill

$$W^i(t) = \max_{0 \leq u \leq t} (0, M^i(t) - M^i(u) - (t-u))$$

(Borovkov [1976, p. 23]). Similarly, for J-variate point processes $\tilde{\mathbf{\Phi}}$, $\tilde{\mathbf{\Psi}}$ define

$$\tilde{M}^i(t) = \sum_{n=1}^{\tilde{N}_i(t)} \tilde{U}_n^i, \quad i = 1, \dots, J$$

and as above $\tilde{M}^i(I)$ and $\tilde{\mathbf{W}}(t)$. The following property is taken from Kulik and Szekli [2005].

Property 1.6.3 (i) *Assume that $\mathbf{\Phi} <_{\mathrm{m-idcx-}\mathcal{N}} \tilde{\mathbf{\Phi}}$, $\mathbf{\Psi} = \tilde{\mathbf{\Psi}}$ and $\mathbf{\Psi}$ consists of mutually independent iid sequences. Then for all $0 < t_1 < \cdots < t_r$,*

$$(\mathbf{W}(t_1), \dots, \mathbf{W}(t_r)) <_{\mathrm{idcx}} (\tilde{\mathbf{W}}(t_1), \dots, \tilde{\mathbf{W}}(t_r))\,.$$

(ii) *Assume that $\mathbf{\Psi} <_{\mathrm{m-idcx-}\infty} \tilde{\mathbf{\Psi}}$, $\mathbf{\Phi} = \tilde{\mathbf{\Phi}}$. Then for all $0 < t_1 < \cdots < t_r$,*

$$(\mathbf{W}(t_1), \dots, \mathbf{W}(t_r)) <_{\mathrm{idcx}} (\tilde{\mathbf{W}}(t_1), \dots, \tilde{\mathbf{W}}(t_r))\,.$$

Workload in batch queues

Consider a queueing system of J parallel $G/GI/1$ FIFO queues. The input is generated by J-variate point processes $\mathbf{\Phi}$ (arrival times) and $\mathbf{\Psi}$ (batch sizes), independent of $\mathbf{\Phi}$. For $t \geq 0$ and $I = (a, b]$ define

$$K^i(t) = \sum_{n=1}^{N^i(t)} U_n^i, \quad i = 1, \ldots, J\,,$$

and

$$K^i(I) = \sum_{n=N^i(a)+1}^{N^i(b)} U_n^i, \quad i = 1, \ldots, kJ\,.$$

Here, $K^i(t)$ represents the number of jobs brought to a queue i up to time t. For $\{S_n^i\}_{n\geq 1}$, $i = 1, \ldots, J$, iid mutually independent service times, independent of $\mathbf{\Phi}$ and $\mathbf{\Psi}$ define cumulative processes

$$M^i(t) = \sum_{n=1}^{K^i(t)} S_n^i, \quad i = 1, \ldots, J\,,$$

and

$$M^i(I) = \sum_{n=K^i(a)+1}^{K^i(b)} S_n^i, \quad i = 1, \ldots, J\,.$$

Then the transient workload is given by

$$W^i(t) = \max_{0\leq u\leq t}(0, M^i(t) - M^i(u) - (t-u))\,.$$

Denote by

$$\mathbf{W}(t) \equiv (W^1(t), \ldots, W^J(t))$$

the vector of transient workload. Similarly, having arrival process $\tilde{\mathbf{\Phi}} = \mathbf{\Phi}$, batch size process $\tilde{\mathbf{\Psi}}$ and the same service times, we define $\tilde{K}^i(t)$, $\tilde{K}^i(I)$, $\tilde{M}^i(t)$, $\tilde{M}^i(I)$, $\tilde{W}^i(t)$ and $\tilde{\mathbf{W}}(t)$.

From Kulik and Szekli [2005] we have

Property 1.6.4 *Assume that* $\{(U_n^1, \ldots, U_n^J)\}_{n\geq 1}$, $\{(\tilde{U}_n^1, \ldots, \tilde{U}_n^J)\}_{n\geq 1}$ *are sequences of independent random variables such that for all* $n \geq 1$, $(U_n^1, \ldots, U_n^J) <_{\mathrm{sm}} (\tilde{U}_n^1, \ldots, \tilde{U}_n^J)$. *Then for all* $0 < t_1 < \cdots < t_r$,

$$(\mathbf{W}(t_1), \ldots, \mathbf{W}(t_r)) <_{\mathrm{idcx}} (\tilde{\mathbf{W}}(t_1), \ldots, \tilde{\mathbf{W}}(t_r))\,.$$

The assumptions in the above properties can be described in a more detailed way. Let $\mathbf{\Phi}$, $\tilde{\mathbf{\Phi}}$ be J-variate arrival processes with interarrival times X_n^i, $\tilde{X}_n^i$, $i = 1, \ldots, J$. If $\{X_n^1, \ldots, X_n^J\}_{n\geq 1}$ and $\{\tilde{X}_n^1, \ldots, \tilde{X}_n^J\}_{n\geq 1}$ are sequences of independent random vectors and for all $n \geq 1$,

$$(X_n^1, \ldots, X_n^J) <_{\mathrm{sm}} (\tilde{X}_n^1, \ldots, \tilde{X}_n^J),$$

then $\mathbf{\Phi} <_{\mathrm{m-sm-}\mathcal{N}} \tilde{\mathbf{\Phi}}$ (Li and Xu [2000]). Assume that $X_n =^d X_n^i =^d X_n^j$, $i, j = 1, \ldots, J$, $n \geq 1$. From Lorentz inequality one obtains that $(X_n^1, \ldots, X_n^J) <_{\mathrm{sm}} (X_n, \ldots, X_n)$. Therefore, synchronization give the upper bound (in $<_{\mathrm{sm}}$ and hence in $<_{\mathrm{idcx}}$-order) for arrival processes and hence, using previous results, for workload in parallel queues.

1.6.3 Throughput in general networks

For general networks results about throughput were obtained by Shanthikumar and Yao [1989a], and by Tsoucas and Walrand [1989]. Since the formulations of the following properties are self-explaining we shall skip comments on them.

Property 1.6.5 *Consider two general closed networks $\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}/J+N$ with an independent initial content $\hat{X}(0)$ and $\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}/J+N'$ with an independent initial content $\hat{X}'(0)$ such that $\hat{X}(0) \le_{st} \hat{X}'(0)$. Then $\boldsymbol{N}^a <_{st\text{-}\mathcal{D}} \boldsymbol{N}'^a$, $\boldsymbol{N}^d <_{st\text{-}\mathcal{D}} \boldsymbol{N}'^d$, and*

$$TH_j(\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}/J+N) \le TH_j(\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}/J+N'),\ j \in \mathrm{J}.$$

Property 1.6.6 *Consider two general closed networks $\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}/J+N$ with an initial content $\hat{X}(0)$ and $\boldsymbol{V}/\boldsymbol{S}',\boldsymbol{k}/J+N$ with equal initial content such that service time sequences are independent of the initial content and of $\boldsymbol{V}$, and $\boldsymbol{S} \ge_{st} \boldsymbol{S}'$. Then $\boldsymbol{N}^a <_{st\text{-}\mathcal{D}} \boldsymbol{N}'^a$, $\boldsymbol{N}^d <_{st\text{-}\mathcal{D}} \boldsymbol{N}'^d$, and*

$$TH_j(\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}/J+N) \le TH_j(\boldsymbol{V}/\boldsymbol{S}',\boldsymbol{k}/J+N),\ j \in \mathrm{J}.$$

Property 1.6.7 *Consider two general closed networks $\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}/J+N$ with an initial content $\hat{X}(0)$ and $\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}'/J+N$ with equal initial content such that $\boldsymbol{k} \ge \boldsymbol{k}'$. Then $\boldsymbol{N}^a <_{st\text{-}\mathcal{D}} \boldsymbol{N}'^a$, $\boldsymbol{N}^d <_{st\text{-}\mathcal{D}} \boldsymbol{N}'^d$, and*

$$TH_j(\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}/J+N) \le TH_j(\boldsymbol{V}/\boldsymbol{S},\boldsymbol{k}'/J+N),\ j \in \mathrm{J}.$$

From Tsoucas and Walrand [1989] we have

Property 1.6.8 *Consider two open networks with finite waiting rooms $(\boldsymbol{N}^0,\boldsymbol{V})/\boldsymbol{S},\boldsymbol{k},\boldsymbol{B}/J$, and $(\boldsymbol{N}^0,\boldsymbol{V})/\boldsymbol{S},\boldsymbol{k}',\boldsymbol{B}'/J$ which operate according to the manufacturing blocking. Assume that in $\boldsymbol{N}^0$ only the first coordinate is non-trivial, and $V^j = (j+1, j+1, \ldots)$, i.e. these networks are open tandems. If $\boldsymbol{N}^0$ and $\boldsymbol{S}$ are independent and $\boldsymbol{k} \le \boldsymbol{k}'$ and $\boldsymbol{B} \le \boldsymbol{B}'$ then*

$$N^{acc} <_{st\text{-}\mathcal{D}} N'^{acc},$$

where N^{acc} denotes the point process of accepted jobs to the tandem.

From Meester and Shanthikumar [1990], also Anantharam, Tsoucas [1990] we have

Property 1.6.9 *Consider open network with finite waiting rooms $(\boldsymbol{N}^0,\boldsymbol{V})/\boldsymbol{S},\mathbf{1},\boldsymbol{B}/J$, which operates according to the manufacturing blocking (**1** denotes the vector with 1 on each coordinate). Assume that in $\boldsymbol{N}^0$ only the first coordinate is non-trivial, and $V^j = (j+1, j+1, \ldots)$, i.e. these network is an open tandem. If $\boldsymbol{S}$ is a vector of independent sequences of iid exponential random variables, and $B_1 = \infty$ then the throughput of this tandem is increasing and concave as a function of $\boldsymbol{B}$.*

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