Assume

$$a\alpha^2 + b\alpha + c = 0,$$

for some integers $a$, $b$ i $c$ so that $a, c \neq 0$. Consider the matrix

$$M = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$$

and the vector $v = (\alpha, 1)^t$. Then

$$(Mv, v) = a\alpha^2 + b\alpha + c = 0.$$

From

$$\alpha = \frac{r_n P_n + P_{n-1}}{r_n Q_n + Q_{n-1}}$$

we get

$$a(r_n P_n + P_{n-1})^2 + b(r_n P_n + P_{n-1})(x_n Q_n + Q_{n-1}) \\ + c(r_n Q_n + Q_{n-1})^2 = 0. \quad (1)$$

Let

$$U = \begin{bmatrix} P_n & P_{n-1} \\ Q_n & Q_{n-1} \end{bmatrix}$$

For $v = (r_n, 1)^t$ equation (1) takes the form

$$(MUv, Uv) = 0.$$

Hence

$$(U^t MUv, v) = 0.$$

Thus $r_n$ is the root of

$$A_n x_n^2 + B_n x_n + C_n = 0,$$

where

$$\begin{bmatrix} A_n & B_n/2 \\ B_n/2 & C_n \end{bmatrix} = U^t MU.$$

Moreover

$$A_n C_n - \frac{1}{4} B_n^2 = \det(U^t MU) = \det M = ac - \frac{1}{4} b^2.$$