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Letter to the Editor

# Kaczmarz algorithm in Hilbert space and tight frames ☆

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**Abstract**

We prove that any tight frame $\{g_n\}_{n=0}^{\infty}$, with $\|g_0\| = 1$, in a Hilbert space can be obtained by the Kaczmarz algorithm. The uniqueness of the correspondence is determined.


*Keywords:* Kaczmarz algorithm; Hilbert space; Bessel sequence; Tight frame



## 1. Introduction

Let $\{e_n\}_{n=0}^{\infty}$ be a linearly dense sequence of unit vectors in a Hilbert space $\mathcal{H}$. In 1937 Kaczmarz considered the problem of reconstructing vectors $x$ from the data $\langle x, e_n\rangle$. He proved that in the finite dimensional case we have $x_n \to x$ for any $x$, where elements $x_n$ are defined recursively by

$$x_0 = \langle x, e_0\rangle e_0,$$
$$x_n = x_{n-1} + \langle x - x_{n-1}, e_n\rangle e_n.$$

This formula is called the Kaczmarz algorithm [1].

It can be shown that if vectors $g_n$ are given by the recurrence relation

$$g_0 = e_0, \qquad g_n = e_n - \sum_{i=0}^{n-1} \langle e_n, e_i\rangle g_i, \tag{1}$$

then $g_0$ is orthogonal to $g_n$ for any $n \geqslant 1$ and

$$x_n = \sum_{i=0}^{n} \langle x, g_i\rangle e_i. \tag{2}$$

By (1) the vectors $\{g_n\}_{n=0}^{\infty}$ are linearly dense in $\mathcal{H}$. Also by definition of the algorithm the vectors $x - x_n$ and $e_n$ are orthogonal to each other. Hence

$$\|x\|^2 = \|x - x_0\|^2 + \left|\langle x, g_0\rangle\right|^2, \qquad \|x - x_{n-1}\|^2 = \|x - x_n\|^2 + \left|\langle x, g_n\rangle\right|^2, \quad n \geqslant 1. \tag{3}$$

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For $n \geqslant 1$ let $S_n$ denote a finite dimensional operator defined by the rule

$$S_n y = \sum_{j=0}^{n} \langle y, e_j \rangle g_j, \quad y \in \mathcal{H}. \tag{4}$$

Observe that the formulas (1) and (2) can be restated as

$$(I - S_{n-1})e_n = g_n, \tag{5}$$
$$(I - S_n^*)x = x - x_n. \tag{6}$$

Moreover by (3) it follows that

$$\|x - x_n\|^2 = \left\|(I - S_n^*)x\right\|^2 = \|x\|^2 - \sum_{j=0}^{n} \left|\langle x, g_j \rangle\right|^2. \tag{7}$$

In particular

$$\sum_{n=0}^{\infty} \left|\langle x, g_n \rangle\right|^2 \leqslant \|x\|^2, \quad x \in \mathcal{H}. \tag{8}$$

A sequence $\{e_n\}_{n=0}^{\infty}$ is called effective if $x_n \to x$ for any $x \in \mathcal{H}$. By virtue of (7) this is equivalent to $\|x\|^2 = \sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2$ for any $x \in \mathcal{H}$, which means $\{g_n\}_{n=0}^{\infty}$ is a tight frame. We refer to [2] for more information on the Kaczmarz algorithm and to [3] for the characterization of effective sequences through the Gram matrix of the sequence $\{e_n\}_{n=0}^{\infty}$.

## 2. Bessel sequences

**Definition 1.** A sequence of vectors $\{g_n\}_{n=0}^{\infty}$ in a Hilbert space $\mathcal{H}$ will be called a Bessel sequence if (8) holds. The sequence $\{g_n\}_{n=0}^{\infty}$ will be called a special Bessel sequence if in addition $\|g_0\| = 1$.

Observe that if $\{g_n\}_{n=0}^{\infty}$ is a special Bessel sequence then substituting $x = g_0$ into (8) implies $g_n \perp g_0$ for $n \geqslant 1$.

Let $P_n$ denote the orthogonal projection onto $e_n^{\perp}$, the orthogonal complement to the vector $e_n$. By [3, (1)] we have

$$I - S_n^* = P_n P_{n-1} \dots P_0, \tag{9}$$
$$I - S_n = P_0 \dots P_{n-1} P_n. \tag{10}$$

**Theorem 1.** *For any special Bessel sequence $\{g_n\}_{n=0}^{\infty}$ in a Hilbert space $\mathcal{H}$ there exists a sequence $\{e_n\}_{n=0}^{\infty}$ of unit vectors such that* (1) *holds. In other words, any special Bessel sequence can be obtained through the Kaczmarz algorithm.*

**Proof.** We will construct the sequence $\{e_n\}_{n=0}^{\infty}$ recursively. Set $e_0 = g_0$. Assume the unit vectors $e_1, \dots, e_{N-1}$ have been constructed such that the formula (1) holds for $n = 0, \dots, N-1$. We want to find $y$ such that

$$(I - S_{N-1})y = g_N, \quad \|y\| = 1. \tag{11}$$

By (10) we have $(I - S_{N-1})e_{N-1} = 0$, i.e. the operator $I - S_{N-1}$ admits nontrivial kernel. Hence the solvability of (11) is equivalent to that of

$$(I - S_{N-1})y = g_N, \quad \|y\| \leqslant 1. \tag{12}$$

By the Fredholm alternative the equation $(I - S_{N-1})y = g_N$ is solvable if and only if $g_N$ is orthogonal to $\ker(I - S_{N-1}^*)$. We will check that this condition holds. Let $x \in \ker(I - S_{N-1}^*)$. Then by (7) and (8) we have

$$0 = \left\|(I - S_{N-1}^*)x\right\|^2 = \|x\|^2 - \sum_{j=0}^{N-1} \left|\langle x, g_j \rangle\right|^2 \geqslant \sum_{j=N}^{\infty} \left|\langle x, g_j \rangle\right|^2.$$

In particular $\langle x, g_N\rangle = 0$, i.e. $g_N \perp \ker(I - S^*_{N-1})$.

Let $y$ denote the unique solution to

$$(I - S_{N-1})y = g_N, \quad y \perp \ker(I - S_{N-1}).$$

The proof will be complete if we show that $\|y\| \leqslant 1$. Again by the Fredholm alternative we have $y \in \operatorname{Im}(I - S^*_{N-1})$. Let $y = (I - S^*_{N-1})x$ for some $x \in \mathcal{H}$. We may assume that $x \perp \ker(I - S^*_{N-1})$. In particular $\langle x, g_0\rangle = 0$, as (9) yields $g_0 \in \ker(I - S^*_{N-1})$. By (7) we have

$$\|y\|^2 = \left\|(I - S^*_{N-1})x\right\|^2 = \|x\|^2 - \sum_{j=1}^{N-1} \left|\langle x, g_j\rangle\right|^2.$$

On the other hand,

$$\|y\|^2 = \left\langle x, (I - S_{N-1})y\right\rangle = \langle x, g_N\rangle.$$

Therefore

$$\|y\|^2 - \|y\|^4 = \|x\|^2 - \sum_{j=1}^{N} \left|\langle x, g_j\rangle\right|^2 \geqslant 0,$$

which implies $\|y\| \leqslant 1$. $\square$

**Corollary 1.** *For any special tight frame $\{g_n\}_{n=0}^\infty$ in a Hilbert space $\mathcal{H}$ there exists an effective sequence $\{e_n\}_{n=0}^\infty$ of unit vectors such that* (1) *holds, i.e. any special tight frame can be obtained through the Kaczmarz algorithm.*

For a sequence $\{e_n\}_{n=0}^\infty$ of unit vectors the special Bessel sequence $\{g_n\}_{n=0}^\infty$ is determined uniquely by (1). However a given special Bessel sequence may correspond to many sequences of unit vectors due to two reasons. First of all for certain $N$ the dimension of the space $\ker(I - S_{N-1})$ may exceed 1. Secondly, if we fix a unit vector $u$ in $\ker(I - S_{N-1})$ the vector $e_N$ can be defined as $e_N = y + \lambda u$ for any complex $\lambda$ number such that $|\lambda|^2 + \|y\|^2 = 1$. In what follows we will indicate properties which guarantee a one to one correspondence between $\{e_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$.

**Definition 2.** A sequence of unit vectors $\{e_n\}_{n=0}^\infty$ will be called strongly redundant if the vectors $\{e_n\}_{n=N}^\infty$ are linearly dense for any $N$. A special Bessel sequence $\{g_n\}_{n=0}^\infty$ will be called strongly redundant if the vectors $\{g_0\} \cup \{g_n\}_{n=N}^\infty$ are linearly dense for any $N$.

**Proposition 1.** *Let sequences $\{e_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ satisfy* (1)*. The sequence $\{g_n\}_{n=0}^\infty$ is strongly redundant if and only if $\{e_n\}_{n=0}^\infty$ is strongly redundant and $\langle e_n, e_{n+1}\rangle \neq 0$ for any $n \geqslant 0$.*

**Proof.** Assume $\{g_n\}_{n=0}^\infty$ is strongly redundant. First we will show that the kernel of $I - S_{N-1}$ is one-dimensional and thus consists of the multiples of the vector $e_{N-1}$ (see (10)). Assume for a contradiction that $\dim\ker(I - S_{N-1}) \geqslant 2$. By the Fredholm alternative we get $\dim\ker(I - S^*_{N-1}) \geqslant 2$. Hence there exists a nonzero vector $x$ such that $x \perp g_0$ and $(I - S^*_{N-1})x = 0$. By (3) we obtain

$$\|x\|^2 = \sum_{n=1}^{N-1} \left|\langle x, g_n\rangle\right|^2.$$

This and the condition (8) imply that $x$ is orthogonal to all the vectors $g_0$ and $\{g_n\}_{n=N}^\infty$, which contradicts the strong redundancy assumption.

Assume $\langle e_{N-1}, e_N\rangle = 0$ for some $N \geqslant 1$. Then by (10) we have $e_{N-1}, e_N \in \ker(I - S_N)$ which is a contradiction as the kernel is one-dimensional.

Concerning strong redundancy of $\{e_n\}_{n=0}^\infty$, assume a vector $y$ is orthogonal to all the vectors $\{e_n\}_{n=N}^\infty$. In particular $y$ is orthogonal to $e_N$. Since $\ker(I - S_N) = \mathbb{C}e_N$, by the Fredholm alternative $y$ belongs to $\operatorname{Im}(I - S^*_N)$. Let $y = (I - S^*_N)x$ for some $x \in \mathcal{H}$. We may assume that $x \perp g_0$ as $g_0 \in \ker(I - S^*_N)$. By (9), since $y$ is orthogonal to $e_n$ for

$n \geqslant N$, we get $y = (I - S_n^*)x = (I - S_N^*)x$ for $n \geqslant N$. On the other hand, by (2) and (6) we obtain that $\langle x, g_n \rangle = 0$ for $n > N + 1$. Since $x \perp g_0$, by strong redundancy assumptions we obtain $x = 0$ and thus $y = 0$.

For the converse implication assume $\{e_n\}_{n=0}^\infty$ is strongly redundant and $\langle e_n, e_{n+1} \rangle \neq 0$. By the inequality (see [2])

$$\|x - x_n\| \geqslant \left|\langle e_{n-1}, e_n \rangle\right| \|x - x_{n-1}\|$$

we get that $x - x_n \neq 0$ for any $x \perp e_0$. Since $x - x_n = (I - S_n^*)x$, the kernel of $I - S_n^*$ consists of the multiples of $e_0 = g_0$, only.

Let $x$ be orthogonal to $\{g_0\} \cup \{g_n\}_{n \geqslant N+1}$ for some $N \geqslant 1$. By (2) we obtain that $x_n = x_N$ for $n \geqslant N$. By the definition of the Kaczmarz algorithm we get $x - x_N \perp e_n$ for $n \geqslant N + 1$. Now strong redundancy of $\{e_n\}_{n=0}^\infty$ implies $x - x_N = 0$. By (6) we obtain $(I - S_N^*)x = 0$. This yields $x = 0$ since the kernel is one-dimensional and consists of the multiples of $g_0$. □

For sequences $\{e_n\}_{n=0}^\infty$ and $\{\sigma_n e_n\}_{n=0}^\infty$, where $\sigma_n$ are complex numbers of absolute value 1, the Kaczmarz algorithm coincides. Therefore we will restrict our attention to *admissible* sequences of unit vectors $\{e_n\}_{n=0}^\infty$ such that $\langle e_n, e_{n+1} \rangle \geqslant 0$.

**Theorem 2.** *Let $\{g_n\}_{n=0}^\infty$ be a strongly redundant special Bessel sequence. Then there exists a unique admissible sequence $\{e_n\}_{n=0}^\infty$ of unit vectors such that* (1) *holds. Moreover, the sequence $\{e_n\}_{n=0}^\infty$ is strongly redundant.*

**Proof.** The proof will go by induction. The vector $e_0$ is determined by $e_0 = g_0$. Assume the vectors $e_0, \dots, e_{N-1}$ were determined uniquely. We have to show that the problem

$$(I - S_{N-1})y = g_N, \quad \|y\| = 1, \ \langle y, e_{N-1} \rangle \geqslant 0$$

has a unique solution $y$.

By the proof of Proposition 1 the kernel of $I - S_{N-1}$ is one-dimensional and thus consists of the multiples of the vector $e_{N-1}$. By the proof of Theorem 1 there exists a unique solution $y_N$ to the problem

$$(I - S_{N-1})y = g_N, \quad y \perp \ker(I - S_{N-1})$$

and $\|y_N\| \leqslant 1$. Moreover by this proof $\|y_N\| = 1$ if and only if

$$\|x\|^2 - \sum_{j=1}^{N} \left|\langle x, g_j \rangle\right|^2 = 0,$$

where $y_N = (I - S_{N-1}^*)x$ and $x \perp \ker(I - S_{N-1}^*)$. This leads to a contradiction because by inequality (8) we get that $x$ is orthogonal to all the vectors $g_0$ and $\{g_n\}_{n=N}^\infty$. Hence $\|y_N\| < 1$.

At this stage we know that any solution to the equation

$$(I - S_{N-1})y = 0$$

is of the form

$$y = y_N + \lambda e_{N-1}, \quad \lambda \in \mathbb{C},$$

because $\ker(I - S_{N-1}) = \mathbb{C}e_{N-1}$. Since $\|y_N\| < 1$ and $y_N \perp e_{N-1}$ there exists a unique solution $y$ satisfying $\|y\| = 1$ and $\langle y, e_{N-1} \rangle \geqslant 0$ namely the one corresponding to $\lambda = \sqrt{1 - \|y_N\|^2}$. □

**Corollary 2.** *Let $\{g_n\}_{n=0}^\infty$ be a strongly redundant special tight frame. Then there exists a unique admissible effective sequence $\{e_n\}_{n=0}^\infty$ of unit vectors such that* (1) *holds. Moreover, the sequence $\{e_n\}_{n=0}^\infty$ is strongly redundant.*

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