Bounds on Turán determinants

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Abstract

Let μ denote a symmetric probability measure on $[-1,1]$ and let (p_n) be the corresponding orthogonal polynomials normalized such that $p_n(1)=1$. We prove that the normalized Turán determinant $\Delta_n(x)/(1-x^2)$, where $\Delta_n = p_n^2 - p_{n-1}p_{n+1}$, is a Turán determinant of order $n-1$ for orthogonal polynomials with respect to $(1-x^2)d\mu(x)$. We use this to prove lower and upper bounds for the normalized Turán determinant in the interval $-1<x<1$.

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1 Introduction

In the following we will deal with polynomial sequences (p_n) satisfying

$$\begin{aligned} &xp_n(x) = \gamma_n p_{n+1}(x) + \alpha_n p_{n-1}(x),\ n \ge 0,\\ &\alpha_n + \gamma_n = 1,\ \alpha_n > 0,\ \gamma_n > 0,\ n \ge 1,\\ &p_0(x) = 1,\ \alpha_0 = 0,\ 0 < \gamma_0 \le 1. \end{aligned} \tag{1}$$

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Note that (p_n) is uniquely determined by (1) from the recurrence coefficients α_n, γ_n and that p_n has a positive leading coefficient. It follows by Favard's theorem that the polynomials p_n are orthogonal with respect to a symmetric probability measure μ. From (1) we get for $x = 1$

$$\gamma_n(p_{n+1}(1) - p_n(1)) = (1 - \gamma_n)(p_n(1) - p_{n-1}(1)), \ n \geq 1, \tag{2}$$

hence

$$p_{n+1}(1) \geq p_n(1) \geq p_1(1) = \frac{1}{\gamma_0} \geq 1, \ n \geq 1, \tag{3}$$

so that $p_n(1) = 1$ for all n if $\gamma_0 = 1$, and $(p_n(1))$ is strictly increasing if $\gamma_0 < 1$.

We conclude that all zeros $x_{n,n} < \ldots < x_{1,n}$ of p_n belong to the interval $(-1, 1)$, hence $\mathrm{supp}(\mu) \subseteq [-1, 1]$. In fact, if there existed an integer n such that $x_{1,n} > 1$, then by assuming n smallest possible with this property, we get $x_{2,n} < 1 < x_{1,n}$ and hence $p_n(1) < 0$, a contradiction. By symmetry this implies that $-1 < x_{n,n}$.

Define the Turán determinant of order n by

$$\Delta_n(x) = p_n^2(x) - p_{n-1}(x)p_{n+1}(x), \ n \geq 1. \tag{4}$$

In [11] the second author proved non-negativity of the Turán determinant (4) under certain monotonicity conditions on the recurrence coefficients, thereby obtaining results for new classes of polynomials and unifying old results.

If $\gamma_0 = 1$ and hence $p_n(1) = 1$ for all n, the *normalized* Turán determinant $\Delta_n(x)/(1 - x^2)$ is a polynomial in x.

We shall prove estimates of the form

$$c\Delta_n(0) \leq \frac{\Delta_n(x)}{1 - x^2} \leq C\Delta_n(0), \quad -1 < x < 1, \tag{5}$$

under certain regularity conditions on the recurrence coefficients. We prove, e.g., an inequality of the left-hand type if (α_n) is increasing and concave, see Theorem 2.5. In Theorem 2.7 we give an inequality of the right-hand type if (α_n) is decreasing and satisfies a condition slightly stronger than convexity.

Our results depend on a simple relationship between the Turán determinants of order n and $n - 1$ (Proposition 2.1) and the following observation: the normalized Turán determinant is essentially a Turán determinant of order $n - 1$ for the polynomials (q_n) defined by (17) below, and if μ denotes the orthogonality measure of (p_n), then (q_n) are orthogonal with respect to the measure

$$(1-x^2)d\mu(x).$$

See Theorem 2.3 and Remark 2.4 for a precise statement.

In Proposition 2.11 we prove non-negativity of the Turán determinant for the normalized polynomials $q_n(x)/q_n(1)$ provided the sequence (α_n) is increasing and concave (or under the weaker condition (21)).

Our work is motivated by results about ultraspherical polynomials, which we describe next.

For $\alpha > -1$ let $R_n^{(\alpha,\alpha)}(x) = P_n^{(\alpha,\alpha)}(x)/P_n^{(\alpha,\alpha)}(1)$ denote the symmetric Jacobi polynomials normalized to be 1 for $x = 1$, i.e.,

$$R_n^{(\alpha,\alpha)}(x) = \frac{(-1)^n}{2^n(\alpha+1)_n}(1-x^2)^{-\alpha}\frac{d^n}{dx^n}(1-x^2)^{n+\alpha}, \tag{6}$$

cf. [10]. We have used the Pochhammer symbol

$$(a)_n = a(a+1)\dots(a+n-1).$$

The polynomials are orthogonal with respect to the symmetric weight function $c_\alpha(1-x^2)^\alpha$ on $(-1,1)$. Here $1/c_\alpha = B(\alpha+1, 1/2)$, so the weight is a probability density. We have $R_n^{(\alpha,\alpha)}(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$ with $\alpha = \lambda - \frac{1}{2}$, where $(P_n^{(\lambda)})$ are the ultraspherical polynomials in the notation of [10].

The corresponding Turán determinant of order n

$$\Delta_n^{(\alpha)}(x) = R_n^{(\alpha,\alpha)}(x)^2 - R_{n-1}^{(\alpha,\alpha)}(x)R_{n+1}^{(\alpha,\alpha)}(x), \tag{7}$$

is clearly a polynomial of degree n in x^2 and divisible by $1-x^2$ since it vanishes for $x = \pm 1$. The following Theorem was proved in [12, pp. 381-382] and in [14, sect. 6]:

Theorem 1.1. *The normalized Turán determinant*

$$f_n^{(\alpha)}(x) := \Delta_n^{(\alpha)}(x)/(1-x^2) \tag{8}$$

is

(i) strictly increasing for $0 \le x < \infty$ when $\alpha > -1/2$.

(ii) equal to 1 for $x \in \mathbb{R}$ when $\alpha = -1/2$.

(iii) strictly decreasing for $0 \le x < \infty$ *when* $-1 < \alpha < -1/2$.

It is easy to evaluate $f_n^{(\alpha)}$ at $x = 0, 1$ giving

$$f_n^{(\alpha)}(0) = \mu_{[n/2]}^{(\alpha)} \mu_{[(n+1)/2]}^{(\alpha)}, \quad f_n^{(\alpha)}(1) = \frac{1}{2\alpha + 2}, \tag{9}$$

where we have used the notation from [1]

$$\mu_n^{(\alpha)} = \frac{\mu_n}{\binom{n+\alpha}{n}}, \tag{10}$$

and μ_n is the normalized binomial mid-coefficient

$$\mu_n = 2^{-2n} \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot \ldots \cdot (2n)}. \tag{11}$$

Corollary 1.2. *For* $-1 < x < 1$ *we have*

$$f_n^{(\alpha)}(0)(1 - x^2) < \Delta_n^{(\alpha)}(x) < f_n^{(\alpha)}(1)(1 - x^2) \text{ for } \alpha > -1/2, \tag{12}$$

while the inequalities are reversed when $-1 < \alpha < -1/2$. *(For* $\alpha = -1/2$ *all three terms are equal to* $1 - x^2$*.)*

For $\alpha = 0$ the inequalities (12) reduce to $(-1 < x < 1)$

$$\mu_{[n/2]} \mu_{[(n+1)/2]} (1 - x^2) < P_n(x)^2 - P_{n-1}(x) P_{n+1}(x) < \frac{1}{2}(1 - x^2) \tag{13}$$

for Legendre polynomials (P_n). This result was recently published in [1] using a SumCracker Package by Manuel Kauers, and it was conjectured that the monotonicity result remains true for ultraspherical polynomials when $\alpha \ge -1/2$. Clearly the authors have not been aware of the early results above.[1] Turán [13] proved that $\Delta_n^{(0)}(x) > 0$ for $-1 < x < 1$. The proof in [12] of Theorem 1.1 is based on a formula relating the Turán determinant

$$\Delta_{n,\lambda}(x) = F_{n,\lambda}^2(x) - F_{n-1,\lambda}(x) F_{n+1,\lambda}(x)$$

of the normalized ultraspherical polynomials $F_{n,\lambda}(x) = P_n^{(\lambda)}(x)/P_n^{(\lambda)}(1)$ and the expression

$$D_{n,\lambda}(x) = [\frac{d}{dx} P_n^{(\lambda)}(x)]^2 - \frac{d}{dx} P_{n-1}^{(\lambda)}(x) \frac{d}{dx} P_{n+1}^{(\lambda)}(x),$$

[1] Motivated by this conjecture the present authors found a proof of Theorem 1.1 close to the old proofs. During the preparation of the paper we found the references [12], [14].

namely (see [12, (5.9)])

$$\frac{\Delta_{n,\lambda}(x)}{1-x^2} = \frac{D_{n,\lambda}(x)}{n(n+2\lambda)[P_n^{(\lambda)}(1)]^2}. \tag{14}$$

See also [3]. Using the well-known formula for differentiation of ultraspherical polynomials

$$\frac{d}{dx}P_n^{(\lambda)}(x) = 2\lambda P_{n-1}^{(\lambda+1)}(x),$$

we see that

$$D_{n,\lambda}(x) = (2\lambda)^2\left([P_{n-1}^{(\lambda+1)}(x)]^2 - P_{n-2}^{(\lambda+1)}(x)P_n^{(\lambda+1)}(x)\right). \tag{15}$$

Except for the factor $(2\lambda)^2$ this is the Turán determinant of order $n-1$ for the ultraspherical polynomials corresponding to the parameter $\lambda+1$.

We see that this result is generalized in Theorem 2.3.

Since the proof of the monotonicity in Theorem 1.1 depends on the fact that the ultraspherical polynomials satisfy a differential equation, there is little hope of extending the result to classes of orthogonal polynomials which do not satisfy a differential equation. We have instead attempted to find bounds for normalized Turán determinants without using monotonicity in the variable x.

This has also led us to consider the following lower boundedness property, which may or may not hold for a system of orthonormal polynomials (P_n):

$$\text{(LB)} \qquad \inf\{P_{n-1}^2(x) + P_n^2(x) \mid x \in \mathbb{R}, n \in \mathbb{N}\} > 0. \tag{16}$$

If property (LB) holds, then necessarily $\sum_{n=0}^{\infty} P_n^2(x) = \infty$ for all $x \in \mathbb{R}$. Therefore, the orthogonality measure of (P_n) is uniquely determined and has no mass points.

In Proposition 3.1 we prove that (LB) holds for symmetric orthonormal polynomials if the recurrence coefficients are increasing and bounded. It turns out that for the orthonormal symmetric Jacobi polynomials the condition (LB) holds if and only if $\alpha \geq 1/2$.

The theory is applied to continuous q-ultraspherical polynomials in Section 4.

Concerning the general theory of orthogonal polynomials we refer the reader to [10],[9],[6].

2 Main results

Proposition 2.1. *In addition to* (1), *assume* $\alpha_n \neq \gamma_n$ *for* $n = 1, 2 \ldots$. *For* $n \geq 2$

$$\Delta_n = \frac{(\gamma_n - \alpha_n)\alpha_{n-1}}{(\gamma_{n-1} - \alpha_{n-1})\gamma_n}\Delta_{n-1} + \frac{\alpha_n - \alpha_{n-1}}{(\gamma_{n-1} - \alpha_{n-1})\gamma_n}(p_{n-1}^2 + p_n^2 - 2xp_{n-1}p_n).$$

Proof. By the recurrence relation we can remove either p_{n+1} or p_{n-1} from the formula defining Δ_n. This leads to two equations

$$\begin{aligned} \gamma_n\Delta_n &= \alpha_n p_{n-1}^2 + \gamma_n p_n^2 - xp_{n-1}p_n, \\ \alpha_n\Delta_n &= \alpha_n p_n^2 + \gamma_n p_{n+1}^2 - xp_n p_{n+1}. \end{aligned}$$

We replace n by $n-1$ in the second equation, multiply both sides by $\gamma_n - \alpha_n$ and subtract the resulting equation from the first one multiplied by $\gamma_{n-1} - \alpha_{n-1}$. In this way we obtain

$$\begin{aligned} &(\gamma_{n-1} - \alpha_{n-1})\gamma_n\Delta_n - (\gamma_n - \alpha_n)\alpha_{n-1}\Delta_{n-1} \\ &\quad = (\alpha_n\gamma_{n-1} - \alpha_{n-1}\gamma_n)(p_{n-1}^2 + p_n^2) - (\gamma_{n-1} - \gamma_n - \alpha_{n-1} + \alpha_n)\, xp_{n-1}p_n. \end{aligned}$$

Taking into account that $\alpha_k + \gamma_k = 1$ for $k \geq 1$ gives

$$(\gamma_{n-1} - \alpha_{n-1})\gamma_n\Delta_n - (\gamma_n - \alpha_n)\alpha_{n-1}\Delta_{n-1} = (\alpha_n - \alpha_{n-1})(p_{n-1}^2 + p_n^2 - 2xp_{n-1}p_n).$$

□

Proposition 2.1 implies

Corollary 2.2. [11, Thm. 1] *In addition to* (1), *assume that one of the following conditions holds:*

(i) (α_n) *is increasing and* $\alpha_n \leq \gamma_n$, $n \geq 1$.

(ii) (α_n) *is decreasing and* $\alpha_n \geq \gamma_n$, $n \geq 1$. *Furthermore, assume* $\gamma_0 = 1$ *or* $\gamma_0 \leq \gamma_1/(1 - \gamma_1)$.

Then $\Delta_n(x) > 0$ *for* $-1 < x < 1$.

Proof. Assume first the additional condition $\alpha_n \neq \gamma_n$ for all $n \geq 0$. Since for $-1 < x < 1$

$$p_{n-1}^2(x) + p_n^2(x) - 2xp_{n-1}(x)p_n(x) > 0,$$

it suffices in view of Proposition 2.1 to show $\Delta_1(x) > 0$ for $-1 < x < 1$. We have

$$\gamma_1\Delta_1(x) = \alpha_1 p_0^2 + \gamma_1 p_1^2 - xp_0p_1 = \frac{\alpha_1\gamma_0^2 + (\gamma_1 - \gamma_0)x^2}{\gamma_0^2},$$

hence $\Delta_1 > 0$ if $\gamma_1 \geq \gamma_0$. If $\gamma_1 < \gamma_0$ and $-1 < x < 1$, we have

$$\gamma_1\Delta_1(x) > \gamma_1\Delta_1(1) = \frac{\alpha_1(1-\gamma_0)(\gamma_1/\alpha_1 - \gamma_0)}{\gamma_0^2}.$$

The right-hand side is clearly non-negative in case (i) because $\gamma_1/\alpha_1 \geq 1$, but also non-negative in case (ii) because of the assumptions on γ_0.

Assume next in case (i) that there is an n such that $\alpha_n = \gamma_n$. Let $n_0 \geq 1$ be the smallest n with this property. Denoting $\alpha = \lim \alpha_n$, then clearly $\alpha_n \leq \alpha \leq 1 - \alpha \leq \gamma_n$ for all n and hence $\alpha_n = \gamma_n = 1/2$ for $n \geq n_0$. Therefore,

$$\Delta_n(x) = p_{n-1}^2(x) + p_n^2(x) - 2xp_{n-1}(x)p_n(x) > 0$$

for $n \geq n_0$, $-1 < x < 1$. The formula of Proposition 2.1 can be applied for $2 \leq n < n_0$ and the proof of the first case carries over. Equality in case (ii) is treated similarly. $\square$

From now on we will assume additionally that $\gamma_0 = 1$. In this case the polynomials p_n are normalized at $x = 1$ so that $p_n(1) = 1$. Since $p_n(-x) = (-1)^n p_n(x)$, we conclude that $p_n(-1) = (-1)^n$. Therefore, the polynomial $p_{n+2} - p_n$ is divisible by $x^2 - 1$ and

$$q_n(x) = \frac{p_{n+2}(x) - p_n(x)}{x^2 - 1}, \quad n \geq 0, \tag{17}$$

is a polynomial of degree n. Moreover, an easy calculation shows that the polynomials q_n are orthogonal with respect to the probability measure $d\nu(x) = \frac{1}{\gamma_1}(1 - x^2)d\mu(x)$. By the recurrence relation (1) with $\gamma_0 = 1$ we obtain that the polynomials q_n satisfy

$$xq_n(x) = \gamma_{n+2}q_{n+1}(x) + \alpha_n q_{n-1}(x),\ n \geq 0,\ q_0 = 1/\gamma_1. \tag{18}$$

The following theorem contains a fundamental formula relating the Turán determinants of the polynomials p_n and q_n.

Theorem 2.3. *For $n \geq 1$*

$$\frac{\Delta_n(x)}{1-x^2} = \alpha_n\gamma_n q_{n-1}^2(x) - \alpha_{n-1}\gamma_{n+1}q_{n-2}(x)q_n(x). \tag{19}$$

Proof. By (1)

$$\begin{aligned} p_{k+1} - xp_k &= \alpha_k(p_{k+1} - p_{k-1}) = \alpha_k(x^2-1)q_{k-1}, \\ xp_k - p_{k-1} &= \gamma_k(p_{k+1} - p_{k-1}) = \gamma_k(x^2-1)q_{k-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &(x^2-1)^2[\alpha_n\gamma_n q_{n-1}^2 - \alpha_{n-1}\gamma_{n+1}q_{n-2}q_n] \\ &\quad = (p_{n+1} - xp_n)(xp_n - p_{n-1}) - (p_n - xp_{n-1})(xp_{n+1} - p_n) \\ &\quad = (1-x^2)(p_n^2 - p_{n-1}p_{n+1}). \end{aligned}$$

□

Remark 2.4. Defining $\tilde{q}_0 = \gamma_1 q_0 = 1$ and

$$\tilde{q}_n = \frac{\gamma_1 \ldots \gamma_{n+1}}{\alpha_1 \ldots \alpha_n} q_n, \ n \geq 1,$$

we have

$$\frac{\Delta_n(x)}{1-x^2} = \frac{\gamma_n}{\alpha_n}\left(\frac{\alpha_1 \ldots \alpha_n}{\gamma_1 \ldots \gamma_n}\right)^2 \left[\tilde{q}_{n-1}^2(x) - \tilde{q}_{n-2}(x)\tilde{q}_n(x)\right], \tag{20}$$

showing that the normalized Turán determinant (19) is proportional to the Turán determinant of order $n-1$ of the renormalized polynomials $(\tilde{q}_n)$. They satisfy the recurrence relation $(\tilde{q}_{-1} := 0)$

$$x\tilde{q}_n = \alpha_{n+1}\tilde{q}_{n+1} + \gamma_{n+1}\tilde{q}_{n-1}, \ n \geq 0.$$

Theorem 2.5. *Assume that (p_n) satisfies (1) with $\gamma_0 = 1$. Let (α_n) be increasing, $\alpha_n \leq 1/2$ and*

$$\alpha_n - \alpha_{n-1} \geq \frac{\alpha_n}{1-\alpha_n}(\alpha_{n+1} - \alpha_n), \ n \geq 1. \tag{21}$$

Then $\Delta_n(x)$ defined by (4) satisfies

$$\frac{\Delta_n(x)}{1-x^2} \geq c\Delta_n(0), \qquad -1 < x < 1, \ n \geq 1,$$

where $c = 2\alpha_1\gamma_2/\gamma_1$. (Note that (21) holds if (α_n) is concave.)

Proof. Observe that (21) is equivalent to $(\alpha_n\gamma_{n+1})$ being increasing. Let

$$D_n(x) = \gamma_n q_{n-1}^2(x) - \gamma_{n+1}q_{n-2}(x)q_n(x).$$

Since $\alpha_n \geq \alpha_{n-1}$, Theorem 2.3 implies

$$\frac{\Delta_n(x)}{1-x^2} \geq \alpha_{n-1}D_n(x). \tag{22}$$

By (18) we can remove q_n or q_{n-2} from the expression defining D_n. In this way we obtain

$$\begin{aligned} D_n &= \alpha_{n-1}q_{n-2}^2 + \gamma_n q_{n-1}^2 - xq_{n-2}q_{n-1}, \qquad (23)\\ \frac{\alpha_{n-1}}{\gamma_{n+1}}D_n &= \frac{\alpha_{n-1}\gamma_n}{\gamma_{n+1}}q_{n-1}^2 + \gamma_{n+1}q_n^2 - xq_{n-1}q_n. \end{aligned}$$

Replacing n by $n-1$ in the second equation and subtracting it from the first, we find

$$D_n - \frac{\alpha_{n-2}}{\gamma_n}D_{n-1} = \frac{\alpha_{n-1}\gamma_n - \alpha_{n-2}\gamma_{n-1}}{\gamma_n}q_{n-2}^2 \geq 0.$$

By iterating the above inequality between D_n and D_{n-1}, we obtain

$$D_n \geq \frac{\alpha_1\ldots\alpha_{n-2}}{\gamma_3\cdots\gamma_n}D_2. \tag{24}$$

From (23) we get

$$D_2 = \alpha_1 q_0^2 + \gamma_2 q_1^2 - xq_0q_1 = \frac{\alpha_1}{\gamma_1^2} + \frac{\gamma_2 x^2}{\gamma_1^2\gamma_2^2} - \frac{x^2}{\gamma_1^2\gamma_2} = \frac{\alpha_1}{\gamma_1^2}, \tag{25}$$

so (22) implies

$$\frac{\Delta_n(x)}{1-x^2} \geq \frac{\alpha_1\ldots\alpha_n}{\gamma_1\ldots\gamma_n}\frac{\alpha_1\gamma_2}{\alpha_n\gamma_1} \geq \frac{\alpha_1\ldots\alpha_n}{\gamma_1\ldots\gamma_n}\frac{2\alpha_1\gamma_2}{\gamma_1},$$

and the conclusion follows from the next lemma. □

Lemma 2.6. *Under the assumptions of Theorem 2.5*

$$\Delta_n(0) \leq \frac{\alpha_1\ldots\alpha_n}{\gamma_1\ldots\gamma_n} \leq \frac{\gamma_1}{\alpha_1}\Delta_n(0), \quad n \geq 1.$$

Proof. Denote
$$h_n = \frac{\gamma_1 \cdots \gamma_n}{\alpha_1 \ldots \alpha_n}.$$
By (1) we have
$$p_{2n}(0) = (-1)^n \frac{\alpha_1\alpha_3 \ldots \alpha_{2n-1}}{\gamma_1\gamma_3 \cdots \gamma_{2n-1}},$$
hence
$$\Delta_{2n}(0)h_{2n} = p_{2n}^2(0)h_{2n} = \prod_{k=1}^{n} \frac{\alpha_{2k-1}}{\alpha_{2k}} \prod_{k=1}^{n} \frac{\gamma_{2k}}{\gamma_{2k-1}} \le 1.$$
On the other hand
$$\Delta_{2n+1}(0)h_{2n+1} = -p_{2n}(0)p_{2n+2}(0)h_{2n+1}$$
$$= \prod_{k=1}^{n} \frac{\alpha_{2k-1}}{\alpha_{2k}} \prod_{k=1}^{n} \frac{\gamma_{2k}}{\gamma_{2k-1}} = \Delta_{2n}(0)h_{2n} \le 1.$$
Moreover,
$$\Delta_{2n}(0)h_{2n} = \prod_{k=1}^{n} \frac{\alpha_{2k-1}}{\alpha_{2k}} \prod_{k=1}^{n} \frac{\gamma_{2k}}{\gamma_{2k-1}} \ge \prod_{k=2}^{2n} \frac{\alpha_{k-1}}{\alpha_k} \prod_{k=2}^{2n} \frac{\gamma_k}{\gamma_{k-1}} = \frac{\alpha_1\gamma_{2n}}{\gamma_1\alpha_{2n}} \ge \frac{\alpha_1}{\gamma_1}.$$
□

Theorem 2.5 has the following counterpart and the proof is very similar:

Theorem 2.7. *Assume that (p_n) satisfies (1) with $\gamma_0 = 1$. Let $\alpha_n, n \ge 1$ be decreasing, $\alpha_n \ge \frac{1}{2}$ and*
$$\alpha_n - \alpha_{n-1} \le \frac{\alpha_n}{1-\alpha_n}(\alpha_{n+1} - \alpha_n), \;\; n \ge 2. \tag{26}$$
Then $\Delta_n(x)$ defined by (4) satisfies
$$\frac{\Delta_n(x)}{1-x^2} \le C\Delta_n(0), \qquad -1 < x < 1, \;\; n \ge 1,$$
where $C = 2\gamma_2$. (Note that (26) implies convexity of $\alpha_n, n \ge 1$.)

Remark 2.8. The normalized symmetric Jacobi polynomials $p_n(x) = R_n^{(\alpha,\alpha)}(x)$ given by (6) satisfy (1) with

$$\gamma_n = \frac{n+2\alpha+1}{2n+2\alpha+1}, \quad \alpha_n = \frac{n}{2n+2\alpha+1}. \tag{27}$$

(In the case of $\alpha = -1/2$, i.e., Chebyshev polynomials of the first kind, these formulas shall be interpreted for $n = 0$ as $\gamma_0 = 1, \alpha_0 = 0$.)

For $\alpha \geq -1/2$ the sequence (α_n) is increasing and concave. Furthermore, $c = 1$.

For $-1 < \alpha \leq -1/2$ the sequence (α_n) is decreasing, (26) holds and $C = 1$.

The statement about the constants c and C follows from Corollary 1.2. However, we cannot expect $c = 1$ in general, because it is easy to construct an example, where the normalized Turán determinant (19) is not monotone for $0 < x < 1$.

Consider the sequence $(\alpha_n) = (0, 1/2-3\varepsilon, 1/2-2\varepsilon, 1/2-\varepsilon, 1/2, 1/2, \ldots)$, which is increasing and concave for $0 < \varepsilon < 1/8$. In this case the Turán determinant $\tilde{q}_2^2 - \tilde{q}_1\tilde{q}_3$ is proportional to $f(x) = x^4 + A(\varepsilon)x^2 + B(\varepsilon)$, where

$$A(\varepsilon) = 4\varepsilon^2 + 3\varepsilon - 1/2,\ B(\varepsilon) = (1/2-3\varepsilon)^2(1/2-\varepsilon)(1/2+2\varepsilon)^2/\varepsilon.$$

Clearly, f is not monotone for $0 < x < 1$, when ε is small.

Corollary 2.9. *Under the assumptions of Theorem 2.5 and the additional hypothesis* $\lim \alpha_n = 1/2$*, the orthogonality measure* μ *is absolutely continuous on* $(-1,1)$ *with a strictly positive and continuous density* $g(x) = d\mu(x)/dx$ *satisfying*

$$g(x) \leq \frac{C}{\sqrt{1-x^2}}.$$

Proof. The corresponding orthonormal polynomials (P_n) satisfy

$$xP_n = \lambda_n P_{n+1} + \lambda_{n-1}P_{n-1}, \tag{28}$$

where $\lambda_n = \sqrt{\alpha_{n+1}\gamma_n}$. We also have $P_n = \delta_n p_n$, where

$$\delta_n = \sqrt{\frac{\gamma_0 \ldots \gamma_{n-1}}{\alpha_1 \ldots \alpha_n}},\ n \geq 1, \ \ \delta_0 = 1,$$

and $\lim \lambda_n = 1/2$. Since

$$\lambda_{n+1} - \lambda_n = \frac{\alpha_{n+2}(\gamma_{n+1} - \gamma_n) + \gamma_n(\alpha_{n+2} - \alpha_{n+1})}{\sqrt{\alpha_{n+2}\gamma_{n+1}} + \sqrt{\alpha_{n+1}\gamma_n}},$$

the monotonicity of $(\alpha_n), (\gamma_n)$ implies

$$\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty.$$

By the theorem in [8] we conclude that the orthogonality measure μ has a positive continuous density $g(x)$ for $-1 < x < 1$. Furthermore, it is known from this theorem that

$$\lim_{n\to\infty} [P_n^2(x) - P_{n-1}(x)P_{n+1}(x)] = \frac{2\sqrt{1-x^2}}{\pi g(x)},$$

uniformly on compact subsets of $(-1, 1)$. For another proof of this result see [5, p. 201], where it is also proved that $(P_n(x))$ is uniformly bounded on compact subsets of $(-1, 1)$ for $n \to \infty$. We have

$$\Delta_n(x) = \frac{1}{\delta_n^2}\left(P_n^2(x) - k_n P_{n-1}(x)P_{n+1}(x)\right),$$

where

$$k_n = \frac{\delta_n^2}{\delta_{n-1}\delta_{n+1}} = \sqrt{\frac{\alpha_{n+1}\gamma_{n-1}}{\alpha_n\gamma_n}},$$

and it follows that $\lim k_n = 1$. Using

$$\frac{\Delta_n(x)}{\Delta_n(0)} = \frac{P_n^2(x) - k_n P_{n-1}(x)P_{n+1}(x)}{P_n^2(0) - k_n P_{n-1}(0)P_{n+1}(0)},$$

we get the result. □

In analogy with the proof of Corollary 2.9 we get

Corollary 2.10. *Under the assumptions of Theorem 2.7 and the additional hypothesis* $\lim \alpha_n = 1/2$*, the orthogonality measure* μ *is absolutely continuous on* $(-1, 1)$ *with a strictly positive and continuous density* $g(x) = d\mu(x)/dx$ *satisfying*

$$g(x) \geq \frac{C}{\sqrt{1-x^2}}.$$

We now return to the polynomials (q_n) defined in (17) and prove that they have a non-negative Turán determinant after normalization to being 1 at 1. The polynomials q_n are orthogonal with respect to a measure supported by $[-1,1]$. Therefore, $q_n(1) > 0$.

Proposition 2.11. *Under the assumptions of Theorem 2.5*

$$\frac{q_n^2(x)}{q_n^2(1)} - \frac{q_{n-1}(x)}{q_{n-1}(1)}\frac{q_{n+1}(x)}{q_{n+1}(1)} \geq 0, \ -1 < x < 1, \ n \geq 1.$$

Proof. Indeed, let $Q_n(x) = q_n(x)/q_n(1)$. Then

$$xQ_n = c_nQ_{n+1} + (1-c_n)Q_{n-1},$$

where $c_n = \gamma_{n+2}(q_{n+1}(1)/q_n(1))$. We will show that (c_n) is decreasing and $c_n \geq 1/2$, and the conclusion then follows from Corollary 2.2. But $c_{n-1} \geq c_n$ is equivalent to

$$D_{n+1}(1) = \gamma_{n+1}q_n^2(1) - \gamma_{n+2}q_{n-1}(1)q_{n+1}(1) \geq 0,$$

which follows from (24) and (25). We will show that $c_n \geq 1/2$ by induction. We have

$$c_0 = \gamma_2\frac{q_1(1)}{q_0(1)} = 1.$$

Assume $c_{n-1} \geq 1/2$. By (21) the sequence $(\alpha_n\gamma_{n+1})$ is increasing. Putting $\alpha = \lim\alpha_n$, we then get

$$\alpha_n\gamma_{n+1} \leq \alpha(1-\alpha) \leq \frac{1}{4}.$$

Using this and (18) leads to

$$1 = c_n + \frac{\alpha_n\gamma_{n+1}}{c_{n-1}} \leq c_n + \frac{1}{4c_{n-1}} \leq c_n + \frac{1}{2},$$

hence $c_n \geq 1/2$. ☐

3 Lower bound estimates

It turns out that Turán determinants can be used to obtain lower bound estimates for orthonormal polynomials. Recall that if the polynomials (p_n)

satisfy the recurrence relation (1), then their orthonormal version (P_n) satisfies

$$xP_n = \lambda_n P_{n+1} + \lambda_{n-1} P_{n-1},$$

where $\lambda_n = \sqrt{\alpha_{n+1}\gamma_n}$.

Proposition 3.1. *Assume that the polynomials* (P_n) *satisfy*

$$xP_n = \lambda_n P_{n+1} + \lambda_{n-1} P_{n-1},\ n \geq 0, \tag{29}$$

with $P_{-1} = \lambda_{-1} = 0$, $\lambda_n > 0$, $n \geq 0$, *and* $P_0 = 1$. *Assume moreover that* (λ_n) *is increasing and* $\lim \lambda_n = L < \infty$. *Then the (LB) property* (16) *holds in the precise form*

$$P_n^2(x) + P_{n-1}^2(x) \geq \frac{\lambda_0^2}{2L^2},\ x \in \mathbb{R},\ n \geq 0.$$

Proof. This proof is inspired by [2, Thm. 3]. By replacing the polynomials $P_n(x)$ by $P_n(2Lx)$ we can assume that $\lim \lambda_n = 1/2$. This assumption implies that the corresponding Jacobi matrix acts as a contraction in ℓ^2, because it can be majorized by the Jacobi matrix with entries $\lambda_n = \frac{1}{2}$. Therefore, the orthogonality measure is supported by the interval $[-1, 1]$. In this way it suffices to consider x from $[-1, 1]$ because the functions $P_n^2(x)$ are increasing on $[1, +\infty[$ and $P_n^2(-x) = P_n^2(x)$. Let

$$\mathcal{D}_n(x) = \lambda_{n-1} P_n^2(x) - \lambda_n P_{n-1}(x) P_{n+1}(x), \quad n \geq 1.$$

By (29) we can remove P_{n+1} to get

$$\mathcal{D}_n = \lambda_{n-1} P_{n-1}^2 + \lambda_{n-1} P_n^2 - x P_{n-1} P_n. \tag{30}$$

Alternatively we can remove P_{n-1} and obtain

$$\frac{\lambda_{n-1}}{\lambda_n} \mathcal{D}_n = \lambda_n P_{n+1}^2 + \frac{\lambda_{n-1}^2}{\lambda_n} P_n^2 - x P_n P_{n+1}. \tag{31}$$

Replacing n by $n - 1$ in (31) and subtracting it from (30) gives

$$\mathcal{D}_n - \frac{\lambda_{n-2}}{\lambda_{n-1}} \mathcal{D}_{n-1} = \frac{\lambda_{n-1}^2 - \lambda_{n-2}^2}{\lambda_{n-1}} P_{n-1}^2 \geq 0. \tag{32}$$

Iterating the inequality $\mathcal{D}_n \geq (\lambda_{n-2}/\lambda_{n-1}) \mathcal{D}_{n-1}$ leads to

$$\mathcal{D}_n \geq \frac{\lambda_0}{\lambda_{n-1}} \mathcal{D}_1 = \frac{\lambda_0^2}{\lambda_{n-1}} \geq 2\lambda_0^2,$$

because $\mathcal{D}_1 = \lambda_0$ by (30), which for $|x| \le 1$ yields

$$\mathcal{D}_n \le \lambda_{n-1}P_{n-1}^2 + \lambda_{n-1}P_n^2 + \frac{1}{2}|x|(P_{n-1}^2 + P_n^2) \le P_{n-1}^2 + P_n^2. \qquad (33)$$

In the general case the lower bound is $2(\lambda_0/(2L))^2$. □

Corollary 3.2. *Under the assumptions of Proposition 3.1 with $L = 1/2$ the orthogonality measure μ is absolutely continuous with a continuous density $g = d\mu(x)/dx$ on $[-1, 1]$ satisfying*

$$g(x) \le \frac{1}{2\pi\lambda_0^2}\sqrt{1 - x^2}.$$

Furthermore, $g(x) > 0$ for $-1 < x < 1$.

Proof. By assumptions the orthogonality measure is supported by $[-1, 1]$. By the proof of Proposition 3.1 we have

$$\mathcal{D}_n(x) \ge 2\lambda_0^2.$$

On the other hand, by [8] and [5, p. 201] the orthogonality measure is absolutely continuous in the interval $(-1, 1)$ with a strictly positive and continuous density g such that

$$\lim_{n\to\infty} \frac{1}{\lambda_{n-1}}\mathcal{D}_n(x) = \frac{2\sqrt{1 - x^2}}{\pi g(x)},$$

uniformly on compact subsets of $(-1, 1)$, cf. the proof of Corollary 2.9. By property (LB) there are no masses at ± 1. □

Remark 3.3. Corollary 3.2 is also obtained in [4, p.758].

The Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ in the standard notation of Szegő, cf. [10], are discussed in the Introduction. The corresponding orthonormal polynomials are denoted $P_n(\alpha; x)$. We recall that

$$c_\alpha \int_{-1}^{1} [P_n^{(\alpha,\alpha)}(x)]^2(1 - x^2)^\alpha \, dx = \frac{2^{2\alpha+1}\Gamma(n + \alpha + 1)^2}{(2n + 2\alpha + 1)n!\Gamma(n + 2\alpha + 1)B(\alpha + 1, 1/2)}. \qquad (34)$$

Proposition 3.4. *Property (LB), defined in* (16), *holds for the orthonormal symmetric Jacobi polynomials* $(P_n(\alpha; x))$ *if and only if* $\alpha \geq 1/2$. *More precisely:*

(i) For $\alpha \geq 1/2$

$$\inf\{P_n^2(\alpha; x) + P_{n-1}^2(\alpha; x) \mid x \in \mathbb{R}, n \in \mathbb{N}\} \geq \frac{2}{2\alpha + 3}.$$

(ii) For $-1 < \alpha < 1/2$

$$\inf\{P_n^2(\alpha; x) + P_{n-1}^2(\alpha; x) \mid x \in \mathbb{R}, n \in \mathbb{N}\} = 0.$$

Proof. Assume $\alpha \geq 1/2$. In this case we get from (27)

$$\lambda_n^2 = \frac{1}{4}\left[1 - \frac{4\alpha^2 - 1}{4(n + \alpha + 1)^2 - 1}\right],$$

so (λ_n) is increasing with $\lim \lambda_n = 1/2$. By Proposition 3.1 we thus have

$$P_n^2 + P_{n-1}^2 \geq 2\lambda_0^2 = \frac{2}{2\alpha + 3},$$

which shows (i).

In order to show (ii) we use Theorem 31 on page 170 of [9] stating

$$w_n(x_{k,n}) p_{n-1}^2(w, x_{k,n}) \approx \sqrt{1 - x_{k,n}^2} \tag{35}$$

for a generalized Jacobi weight w. (For two positive sequences $(a_n), (b_n)$ we write $a_n \approx b_n$ if $0 < C_1 \leq a_n/b_n \leq C_2 < \infty$ for suitable constants C_j.) Applying this to the largest zero $x_{1,n}$ of the orthonormal symmetric Jacobi polynomials $(P_n(\alpha; x))$, we get

$$w_n(x_{1,n}) P_{n-1}^2(\alpha; x_{1,n}) \approx \sqrt{1 - x_{1,n}^2} \tag{36}$$

with

$$w_n(t) = (\sqrt{1-t} + \frac{1}{n})^{2\alpha}(\sqrt{1+t} + \frac{1}{n})^{2\alpha} > (1 - t^2)^\alpha.$$

This gives in particular

$$P_{n-1}^2(\alpha; x_{1,n}) \leq C(1 - x_{1,n}^2)^{1/2-\alpha},$$

hence $\lim P_{n-1}^2(\alpha; x_{1,n}) = 0$ for $\alpha < 1/2$. This shows (ii) because $P_n(\alpha; x_{1,n}) = 0$. □

Remark 3.5. For $-1 < \alpha < -1/2$ the observation of (ii) follows easily from the asymptotic result

$$P_n(\alpha;1) \sim d_\alpha n^{\alpha+1/2}, \quad n \to \infty,$$

where d_α is a suitable constant, but this simple asymptotic can not be used when $-1/2 \le \alpha < 1/2$.

The proof of (ii) presented above has kindly been communicated to us by Paul Nevai. Our original proof is based on Hilb's asymptotic formula [10, Thm 8.21.12]:

$$\theta^{-1/2}\left(\sin\frac{\theta}{2}\right)^{\alpha+1/2}\left(\cos\frac{\theta}{2}\right)^{\alpha+1/2} P_n^{(\alpha,\alpha)}(\cos\theta) = \frac{\Gamma(\alpha+n+1)}{n!\sqrt{2}N^\alpha} J_\alpha(N\theta) + O(n^{-3/2}), \quad (37)$$

where $\theta \in [c/n, \pi/2]$, $N = n + \alpha + \frac{1}{2}$ and $c > 0$ is fixed. Let j_α denote the smallest positive zero of the Bessel function J_α.

Defining $\theta_n = j_\alpha/N$, we get

$$\begin{aligned} n^{-\alpha} P_n^{(\alpha,\alpha)}(\theta_n) &= O(n^{-3/2}), \\ n^{-\alpha} P_{n-1}^{(\alpha,\alpha)}(\theta_n) &= (1/\sqrt{2} + o(1)) J_\alpha\Big(j_\alpha \frac{n+\alpha-1/2}{n+\alpha+1/2}\Big) + O(n^{-3/2}) = O(n^{-1}). \end{aligned}$$

By (34) and Stirling's formula

$$c_\alpha \int_{-1}^{1} [P_n^{(\alpha,\alpha)}(x)]^2 (1-x^2)^\alpha \, dx \sim \frac{2^{2\alpha}}{B(\alpha+1, 1/2)} n^{-1},$$

and hence

$$P_n^2(\alpha; \cos\theta_n) = O(n^{2\alpha-2}), \quad P_{n-1}^2(\alpha; \cos\theta_n) = O(n^{2\alpha-1}).$$

This shows that

$$P_n^2(\alpha;\cos\theta_n) + P_{n-1}^2(\alpha;\cos\theta_n) \to 0 \text{ when } \alpha < 1/2.$$

Remark 3.6. The example of symmetric Jacobi polynomials suggests that if (λ_n) is decreasing, then property (LB) does not hold. This is not true, however, because for $\frac{1}{2} < \lambda_0 < \frac{1}{\sqrt{2}}$ and $\lambda_n = \frac{1}{2}$ for $n \geq 1$ we have a decreasing sequence. The corresponding Jacobi matrix has norm 1 because this is so for the cases $\lambda_0 = \frac{1}{2}$ and $\lambda_0 = 1/\sqrt{2}$, which correspond to the Chebyshev polynomials of the second and first kind respectively. Furthermore, for $n \geq 2$ we have by (30) and (32)

$$\mathcal{D}_n = \lambda_{n-1}P_n^2 - \lambda_n P_{n-1}P_{n+1} = \mathcal{D}_2 = \frac{2}{\lambda_0^2}[\lambda_0^4 - (\lambda_0^2 - \frac{1}{4})x^2]$$

and for $-1 < x < 1$

$$\mathcal{D}_2(x) > \mathcal{D}_2(1) = \frac{2}{\lambda_0^2}(\lambda_0^2 - \frac{1}{2})^2 > 0.$$

On the other hand, (33) applies for $n \geq 2$, and we see that the orthonormal polynomials satisfy

$$\inf\{P_n^2(x) + P_{n-1}^2(x) \mid x \in \mathbb{R}, n \in \mathbb{N}\} \geq \frac{2}{\lambda_0^2}(\lambda_0^2 - \frac{1}{2})^2.$$

4 Continuous q-ultraspherical polynomials

The continuous q-ultraspherical polynomials $C_n(x;\beta|q)$ depend on two real parameters q, β, and for $|q|, |\beta| < 1$ they are orthogonal with respect to a continuous weight function on $(-1, 1)$, cf. [6],[7]. The three term recurrence relation is

$$xC_n(x;\beta|q) = \frac{1-q^{n+1}}{2(1-\beta q^n)}C_{n+1}(x;\beta|q) + \frac{1-\beta^2 q^{n-1}}{2(1-\beta q^n)}C_{n-1}(x;\beta|q), \quad n \geq 0 \tag{38}$$

with $C_{-1} = 0, C_0 = 1$. The orthonormal version $\mathcal{C}_n(x;\beta|q)$ satisfies equation (29) with

$$\lambda_n = \frac{1}{2}\sqrt{\frac{(1-q^{n+1})(1-\beta^2 q^n)}{(1-\beta q^n)(1-\beta q^{n+1})}}. \tag{39}$$

The value $C_n(1;\beta|q)$ is not explicitly known, and therefore we can only obtain the recurrence coefficients α_n, γ_n from (1) for $p_n(x) = C_n(x;\beta|q)/C_n(1;\beta|q)$ as given by the recursion formulas

$$\alpha_{n+1} = \frac{\lambda_n^2}{1-\alpha_n}, \quad \alpha_0 = 0,\ \gamma_n = 1 - \alpha_n, \tag{40}$$

which we get from the relation $\lambda_n = \sqrt{\alpha_{n+1}\gamma_n}$.

Theorem 4.1. *(i) Assume* $0 \leq \beta \leq q < 1$. *Then the recurrence coefficients* (λ_n) *form an increasing sequence with limit* $1/2$, *and therefore* $(\mathcal{C}_n(x;\beta|q))$ *satisfies (LB).*

(ii) Assume $0 \leq q \leq \beta < 1$. *Then the recurrence coefficients* (λ_n) *form a decreasing sequence with limit* $1/2$, *and the sequence* (α_n) *is increasing and concave with limit* $1/2$. *In particular, we have*

$$\frac{\Delta_n(x)}{1-x^2} \geq c\Delta_n(0), \quad -1 < x < 1, \ n \geq 1,$$

with $c = 2\alpha_1(1-\alpha_2)/(1-\alpha_1)$.

Proof. The function

$$\psi(x) = \frac{(1-qx)(1-\beta^2 x)}{(1-\beta x)(1-\beta qx)} = 1 + (1-\beta)(\beta-q)\frac{x}{(1-\beta x)(1-\beta qx)}$$

is decreasing for $0 \leq \beta \leq q < 1$ and increasing for $0 \leq q \leq \beta < 1$. This shows that $\lambda_n = (1/2)\sqrt{\psi(q^n)}$ is increasing in case (i) and decreasing in case (ii). In both cases the limit is 1/2.

In case (ii) we therefore have $\lambda_n^2 \geq 1/4$ and hence

$$\alpha_{n+1} \geq \frac{1}{4(1-\alpha_n)} \geq \alpha_n,$$

because $4x(1-x) \leq 1$ for $0 \leq x \leq 1$. This shows that (α_n) is increasing and hence with limit 1/2. We further have

$$\alpha_{n+1} - \alpha_n = 2(\lambda_n^2 - \frac{1}{4}) + 2(\frac{1}{2} - \alpha_n)(\frac{1}{2} - \alpha_{n+1}),$$

showing that $\alpha_{n+1} - \alpha_n$ is decreasing, i.e., (α_n) is concave. We can now apply Theorem 2.5. ☐

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