Projection formulas for orthogonal polynomials

Włodzimierz Bryc*
Department of Mathematical Sciences
University of Cincinnati
Cincinnati, OH 45221-0025

Wojciech Matysiak†
Department of Mathematical Sciences
University of Cincinnati
Cincinnati, OH 45221-0025

Ryszard Szwarc‡
Instytut Matematyczny,
Uniwersytet Wrocławski,
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland.

Jacek Wesołowski
Faculty of Mathematics and Information Science
Warsaw University of Technology
pl. Politechniki 1
00-661 Warszawa, Poland

June 3, 2006; revised September 23, 2006

Abstract

We prove a new projection formula for the four-parameter family of orthogonal polynomials outside of the Askey-Wilson class. By carefully analyzing the recurrence relations we manage to overcome the lack of explicit expression for the orthogonality measure.

---

*Research partially supported by NSF grants #INT-03-32062, #DMS-05-04198, and by the C.P. Taft Memorial Fund.

†Permanent address: Faculty of Mathematics and Information Science, Warsaw University of Technology, pl. Politechniki 1, 00-661 Warszawa, Poland

‡Research partially supported by KBN (Poland) under grant 2 P03A 028 25

1 Introduction

Projection formulas of the type

$$q_n(x) = \int p_n(y)\nu_x(dy), \tag{1.1}$$

where $\{\nu_x\}$ is a family of probability measures, are of interest in the theory of orthogonal polynomials and in probability.

Explicit formulas for measure ν_x have been known since [2] when $q_n(x)$ and $p_n(y)$ are both Jacobi polynomials. These formulas were extended to pairs of Askey-Wilson polynomials in [11, 12] and to pairs of associated Askey-Wilson polynomials in [13]. The proofs rely on explicit evaluation of certain integrals, which is a topic of independent interest.

Projection formulas of the type (1.1) were used as a basis of construction of certain Markov processes in [7, 6, 3, 4]. The technique of proof in these papers is less constructive and relies on implicit definition of probability measure ν_x as the orthogonality measure of the auxiliary family of orthogonal polynomials. With the exception of [4], these projection formulas dealt with the pairs of polynomials within the Askey-Wilson class and in fact differ from [11, 12] only in the allowed ranges for the parameters. The purpose of this note is to provide a related projection formula outside of the Askey-Wilson class. Our method does not rely on the knowledge of explicit orthogonality measures and has more combinatorial character.

Our goal is to analyze in detail the family of orthogonal polynomials $\overline{p}_n(y;t) = \overline{p}_n^{(\eta,\theta,\tau,q)}(y;t)$ which appeared in the study of stochastic processes with linear regressions and quadratic conditional variances in [5, Theorem 4.5]. Let $\overline{p}_{-1} = 0$, $\overline{p}_0 = 1$. Fix $\eta, \theta \in \mathbb{R}$, $\tau \geq 0$, $-1 < q \leq 1$. For $t > 0$, $n \geq 0$ let

$$y\overline{p}_n(y;t) = \overline{p}_{n+1}(y;t) + b_n(t)\overline{p}_n(y;t) + a_{n-1}c_n(t)\overline{p}_{n-1}(y;t), \tag{1.2}$$

where for $\eta \neq 0$

$$\begin{aligned} a_n &= \eta^{-1} + \theta[n]_q + [n]_q^2\eta\tau, & (1.3)\\ b_n(t) &= (t\eta + \theta + ([n]_q + [n-1]_q)\eta\tau)\,[n]_q, & (1.4)\\ c_n(t) &= \eta(t + \tau[n-1]_q)[n]_q\ . & (1.5) \end{aligned}$$

For $\eta = 0$ we need to interpret $a_{n-1}c_n(t)$ as $(t + \tau[n-1]_q)[n]_q$. Our reason for the separation of these two factors is that for $\eta > 0$ we have

$$b_n(t) = a_n + c_n(t) - \frac{1}{\eta}, \tag{1.6}$$

a property which will be exploited later on. We use the notation

$$[n]_q = 1 + q + \cdots + q^{n-1},\ [n]_q! \ = \ [1]_q[2]_q\ldots[n]_q,\ \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!},$$

with the usual conventions $[0]_q = 0, [0]_q! = 1$.

Throughout this paper, by μ_t we denote the orthogonality measure of polynomials $\{\overline{p}_n(y;t)\}$. A sufficient condition for existence of such a probability measure is that $\eta\theta \geq 0$, $\tau \geq 0$, and $0 \leq q \leq 1$. It is possible that our results are valid for a more general range of the parameters (compare [3] and [4]), but an attempt to cover such a range is likely to lead to additional technical complications which should be avoided in a paper that already has a significant degree of computational complexity.

To compare the polynomials defined by (1.2) with the monic Askey-Wilson polynomials $\overline{w}_n$, recall that the latter are defined by the recurrence

$$x\overline{w}_n(x) = \overline{w}_{n+1}(x) + \frac{1}{2}(a + a^{-1} - (A_n + C_n))\overline{w}_n + \frac{1}{4}A_{n-1}C_n\overline{w}_{n-1}(x), \quad (1.7)$$

where

$$A_n = \frac{(1 - abcdq^{n-1})(1 - abq^n)(1 - acq^n)(1 - adq^n)}{a(1 - abcdq^{2n-1})(1 - abcdq^{2n})},$$

$$C_n = \frac{a(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})}.$$

A linear transformation $y = \alpha x + \beta$ with $\overline{p}_n(y) = \alpha^n\overline{w}_n(x)$ transforms this recurrence into

$$y\overline{p}_n(y) = \overline{p}_{n+1}(y) + \alpha(A_n + C_n)\overline{p}_n(y) + \alpha^2 C_n A_{n-1}\overline{p}_{n-1}(y).$$

On the other hand, recurrence (1.2) can be written as

$$y\overline{p}_n(y) = \overline{p}_{n+1}(y) + (\alpha_A A_n + \alpha_C C_n)\overline{p}_n(y) + \alpha_A\alpha_C A_{n-1}C_n\overline{p}_{n-1}(y),$$

with $d = 0$,

$$a = \frac{2\eta\tau}{2\eta\tau + (1-q)(\theta - \sqrt{\theta^2 - 4\tau})}, \quad b = \frac{2\eta\tau}{2\eta\tau + (1-q)(\theta + \sqrt{\theta^2 - 4\tau})},$$

$$c = \frac{\tau}{(1-q)t + \tau}, \quad \alpha_A = \frac{(1-q)^2 + \eta\theta(1-q) + \eta^2\tau}{\eta(1-q)^2}, \quad \alpha_C = \eta\frac{(1-q)t + \tau}{(1-q)^2}.$$

This is equivalent to the Askey-Wilson recurrence only when $\alpha_A = \alpha_C$, i.e. at a single value of $t = \frac{\theta}{\eta} + \frac{1-q}{\eta^2}$ only. (The latter plays a role in the proof of Lemma 4.2 below.)

Our main result is the following projection formula.

Theorem 1.1. *If $0 \leq s \leq t$, $0 \leq q \leq 1$, $\eta\theta \geq 0$, $\tau \geq 0$, then for all x in the support of μ_s there exists a unique probability measure $\nu_x = \nu_{x,t,s}$ such that*

$$\overline{p}_n(x;s) = \int \overline{p}_n(y;t)\nu_x(dy). \quad (1.8)$$

Of course, probability measure $\nu_x = \nu_{x,t,s}$ depends also on parameters $0 \leq s \leq t$ as well as on the remaining parameters η, θ, τ, q.

The proof of Theorem 1.1 appears in Section 5.4, after a number of preliminary results. The plan of the proof is as follows. In Section 5 we define a family of monic polynomials $\{\overline{Q}_n\}$ in variable y. We verify that the assumptions of Favard's theorem are satisfied for the relevant pairs (x, s), so that their orthogonality measure $\nu_{x,t,s}$ exists. We show that this measure is unique (a fact that is nontrivial only when $q = 1$). We then use the formula for the connection coefficients between polynomials $\{\overline{Q}_n\}$ and the monic version of polynomials $\{p_n\}$ to deduce (1.8).

When $\eta > 0$, we will find it convenient to consider the following non-monic polynomials

$$y p_n(y;t) = a_n p_{n+1}(y;t) + b_n(t) p_n(y;t) + c_n(t) p_{n-1}(y;t). \tag{1.9}$$

Clearly they have the same orthogonality measure μ_t as the monic polynomials.

2 Identities

We will need a number of auxiliary identities.

Lemma 2.1. *Fix a sequence $\{p_n : n \geq 0\}$ of real numbers. Let $\{\beta_{n,k} : 0 \leq k \leq n, n = 0, 1, \dots\}$ be defined by $\beta_{n,k} = 0$ for $n < 0$ or $k > n$ and for $0 \leq k \leq n$ by the recurrence*

$$[k+1]_q \beta_{n,k+1} = q^k [n-k]_q \beta_{n,k} + [n]_q \beta_{n-1,k}, \ 0 \leq k \leq n, \tag{2.1}$$

with the initial values $\beta_{n,0} = p_n$, $n = 0, 1, \dots$. Then

$$\beta_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{(k-j)(k-j-1)/2} p_{n-j}. \tag{2.2}$$

Proof. This follows by a routine induction argument with respect to k. Clearly (2.2) holds true for $k = 0$ and all $n \geq 0$. Suppose (2.2) holds true for some $k \geq 0$ and all $n \geq 0$. Then by (2.1) and the induction assumption, we have

$$\begin{aligned}
\beta_{n,k+1} = {} & \frac{[n-k]_q}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{k+(k-j)(k-j-1)/2} p_{n-j} + \\
& \frac{[n]_q}{[k+1]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{(k-j)(k-j-1)/2} p_{n-1-j} \\
= {} & \begin{bmatrix} n \\ k+1 \end{bmatrix} p_{n-(k+1)} + \begin{bmatrix} n \\ k+1 \end{bmatrix} q^{(k+1)k/2} p_n \\
& + \begin{bmatrix} n \\ k+1 \end{bmatrix}_q \sum_{j=1}^{k} \left(q^j \begin{bmatrix} k \\ j \end{bmatrix}_q + \begin{bmatrix} k \\ j-1 \end{bmatrix}_q \right) q^{(k+1-j)(k-j)/2} p_{n-j}.
\end{aligned}$$

The well known formula [9, (I.45)]

$$\left[\begin{array}{c} k+1 \\ j \end{array}\right]_q = q^j \left[\begin{array}{c} k \\ j \end{array}\right]_q + \left[\begin{array}{c} k \\ j-1 \end{array}\right]_q = \left[\begin{array}{c} k \\ j \end{array}\right]_q + q^{k-j+1} \left[\begin{array}{c} k \\ j-1 \end{array}\right]_q, \ 1 \leq j \leq k, \tag{2.3}$$

ends the proof. □

It turns out that expressions of the form (2.2) can sometimes be written as products.

Proposition 2.2. *If polynomials* $\{p_n(y;t)\}$ *satisfy recurrence* (1.9) *and*

$$A_n(x,s) \quad = \quad a_n + q^n x - s\eta q^n [n]_q \ , \tag{2.4}$$

then for all $k \geq 1$ *we have*

$$\sum_{j=0}^{k} \left[\begin{array}{c} k \\ j \end{array}\right]_q q^{(k-j)(k-j-1)/2} p_{k-j}(x;s) = \prod_{j=0}^{k-1} \frac{A_j(x,s)}{a_j}. \tag{2.5}$$

Proof. We proceed by induction with respect to k. Formula (2.5) holds true for $k=0$ by convention, and for $k=1$ by a calculation: $p_1(x;s)+p_0(x;s) = 1+\eta x$.

Let $\beta_{n,k}(x,s)$ be defined by (2.2) with $p_n = p_n(x;s)$, $n = 0,1,\dots$. The induction assumption says that

$$\beta_{k,k}(x,s) = \prod_{j=0}^{k-1} \frac{A_j(x,s)}{a_j} \tag{2.6}$$

for some $k \geq 1$. From (2.1) we see that

$$\beta_{k+1,k+1}(x,s) = \beta_{k,k}(x,s) + q^k \sum_{j=0}^{k} \left[\begin{array}{c} k \\ j \end{array}\right]_q q^{(k-j)(k-j-1)/2} p_{k+1-j}(x;s). \tag{2.7}$$

On the other hand, multiplying both sides of (2.6) by $\frac{A_k(x,s)}{a_k} = 1 + \frac{x - s\eta[k]_q}{a_k} q^k$ and using (1.9) we see that

$$\prod_{j=0}^{k} \frac{A_j(x,s)}{a_j} = \beta_{k,k}(x,s) - q^k \frac{s\eta[k]_q}{a_k} \sum_{j=0}^{k} \left[\begin{array}{c} k \\ j \end{array}\right]_q q^{(k-j)(k-j-1)/2} p_{k-j}(x;s)$$
$$+ \frac{q^k}{a_k} \sum_{j=0}^{k} \left[\begin{array}{c} k \\ j \end{array}\right]_q q^{(k-j)(k-j-1)/2} \big(a_{k-j} p_{k+1-j}(x;s) + b_{k-j}(s) p_{k-j}(x;s)$$
$$+ c_{k-j}(s) p_{k-1-j}(x;s)\big). \tag{2.8}$$

Writing the right hand side of (2.8) as

$$\beta_{k,k}(x,s) + \frac{q^k}{a_k} \sum_{j=0}^{k+1} \left[\begin{array}{c} k \\ j \end{array}\right]_q q^{(k-j)(k-j-1)/2} \gamma_{k,j} p_{k+1-j}(x;s), \tag{2.9}$$

from (1.3), (1.4) and (1.5) it is not difficult to see that $\gamma_{k,0} = a_k$ and $\gamma_{k,k+1} = -s\eta[k]_q + s\eta[k]_q = 0$. Similarly, for $1 \le j \le k$ we have

$$
\begin{aligned}
\gamma_{k,j} = a_{k-j} + \frac{q^{k-j}[j]_q}{[k+1-j]_q}\left(b_{k+1-j}(s) - s\eta[k]_q\right)& \\
+ \frac{q^{2k-2j+1}[j]_q[j-1]_q}{[k+2-j]_q[k+1-j]_q} c_{k+2-j}(s) = \eta^{-1} + \theta[k-j]_q + \eta\tau[k-j]_q^2 + [j]_q\theta q^{k-j}& \\
+ \eta\tau\Big([j]_q[k+1-j]q^{k-j} + [j]_q[k-j]_q q^{k-j} + [j]_q[j-1]_q q^{2k+1-2j}\Big)& \\
= \eta^{-1} + \theta\left([k-j]_q + q^{k-j}[j]_q\right)& \\
+ \eta\tau\left([k-j]_q\left([k-j]_q + q^{k-j}[j]_q\right) + [j]_q q^{k-j}\left([k+1-j] + [j-1]q^{k+1-j}\right)\right)& \\
= \eta^{-1} + \theta[k]_q + \eta\tau[k]_q^2.& \quad (2.10)
\end{aligned}
$$

(Here we used repeatedly the identity $[k-j]_q + q^{k-j}[j]_q = [k]_q$.) Thus $\gamma_{k,j} = a_k$, which shows that the right hand sides of equations (2.7) and (2.8) are equal. Therefore their left hand sides are equal, ending the proof. □

For $n \ge 0$, $\eta \ne 0$, and $q \ne 0$ let

$$x_n(s) = \frac{(s\eta q^n - \theta)[n]_q - \eta^{-1} - \eta\tau[n]_q^2}{q^n} \quad (2.11)$$

be the zero of $A_n(x, s)$, see (2.4). It turns out that (2.5) extends to higher order polynomials p_n when the polynomials are evaluated at x_k.

Lemma 2.3. *If $\{p_n(y;t)\}$ satisfies recurrence* (1.9) *and $\eta, q > 0$ then for $n \ge k$ we have*

$$\sum_{j=0}^{k} \begin{bmatrix} k \\ j \end{bmatrix}_q q^{(k-j)(k-j-1)/2} p_{n-j}(x_k(s); s) = \frac{(-1)^{n-k}}{q^{k(n-k)}} \prod_{j=0}^{k-1} \frac{A_j(x_k(s), s)}{a_j}. \quad (2.12)$$

(For $k = 0$ this should be interpreted as $p_n(x_0; s) = (-1)^n$, $n \ge 0$.)

Proof. Let $\beta_{n,k}(x, s)$ be defined by (2.2) with $p_n = p_n(x; s)$, $n = 0, 1, \ldots$. Then the left hand side of (2.12) is $\beta_{n,k}(x_k(s), s)/\begin{bmatrix} n \\ k \end{bmatrix}_q$. We first prove an auxiliary fact that for all $0 \le j < k \le n$ we have $\beta_{n,k}(x_j(s), s) = 0$. We prove this by induction with respect to $n - k$. Suppose there is $m \ge 0$ such that $\beta_{n,k}(x_j(s), s) = 0$ for all triplets (j, k, n) such that $0 \le j < k$ and $n - k = m$. By (2.5) this holds true for $m = 0$. Given $j < k$ and n such that $n - k = m + 1$ by (2.1) we have

$$([n]_q - [k]_q)\beta_{n,k}(x, s) = [k+1]_q\beta_{n,k+1}(x, s) - [n]\beta_{n-1,k}(x, s). \quad (2.13)$$

By induction assumption the right hand side of (2.13) evaluated at $(x_j(s), s)$ vanishes. As $q \ne 0$ and $n = k + m + 1 > k$, we have $[n]_q - [k]_q \ne 0$, so $\beta_{n,k}(x_j(s), s) = 0$.

We now prove (2.12). From (1.6) it is easy to see by induction that $p_n(x_0; s) = p_n(-\eta^{-1}; s) = (-1)^n$. For $k \geq 1$, we will prove (2.12) by induction with respect to n.

If $n = k$ then formula (2.12) holds by Proposition 2.2. Suppose (2.12) holds for some $n \geq k$. Then from (2.1) and the fact that $\beta_{n+1,k+1}(x_k(s), s) = 0$ we see that

$$\begin{aligned}\beta_{n+1,k}(x_k(s), s) &= -\frac{[n+1]_q}{q^k[n+1-k]_q}\beta_{n,k}(x_k(s), s) \\ &= -\frac{[n+1]_q}{q^k[n+1-k]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-1)^{n-k}}{q^{k(n-k)}}\prod_{j=0}^{k-1}\frac{A_j(x_k(s), s)}{a_j}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q q^{(k-j)(k-j-1)/2}p_{n+1-j}(x_k(s); s) &= \frac{\beta_{n+1,k}(x_k(s), s)}{\begin{bmatrix} n+1 \\ k \end{bmatrix}_q} \\ &= \frac{(-1)^{n+1-k}}{q^{k(n+1-k)}}\prod_{j=0}^{k-1}\frac{A_j(x_k(s), s)}{a_j}.\end{aligned}$$

□

We need to analyze equation (2.12) in more detail.

Lemma 2.4. *Fix $k \geq 1$, $q \neq 0$, $\Pi_k > 0$. Suppose that $(p_n)_{n\geq 0}$ is the general solution of the recurrence*

$$\sum_{j=0}^{k}\begin{bmatrix} k \\ j \end{bmatrix}_q q^{(k-j)(k-j-1)/2}p_{n-j} = \frac{(-1)^{n-k}}{q^{k(n-k)}}\Pi_k,\ n \geq k. \tag{2.14}$$

Then

$$p_n = (-1)^{n-k}q^{-nk}q^{k(k+1)/2}\left(\begin{bmatrix} n \\ k \end{bmatrix}_q + \sum_{r=1}^{k}C_r q^{nr}\begin{bmatrix} n \\ k-r \end{bmatrix}_q\right)\Pi_k,\ n \geq 0, \tag{2.15}$$

where $C_1, \ldots, C_k$ are arbitrary constants.

Proof. Substitute

$$y_n = \frac{(-1)^{n-k}q^{k(2n-k-1)/2}}{\Pi_k}p_n.$$

Then with $y = (y_n)_{n\geq 0}$ the equation takes the form of an initial value problem for a linear recurrence with constant coefficients:

$$(\Delta_{q,k}y)_n = 1,\ n \geq k, \tag{2.16}$$

where

$$(\Delta_{q,k}y)_n = \sum_{j=0}^{k}(-1)^j \begin{bmatrix} k \\ j \end{bmatrix}_q q^{j(j+1)/2} y_{n-j}.$$

We remark that when $q = 1$ we trivially have $\Delta_{1,k} = \Delta_{1,1}^k$. Since $\Delta_{1,1}$ is the usual difference operator, in this case the general solution of (2.16) is well known. The q-generalization of this formula follows from (2.3) by induction with respect to k. We have

$$\Delta_{q,k} = R_1 R_2 \ldots R_k, \tag{2.17}$$

where $R_j = \Delta_{q^j,1}$ are commuting difference operators, $(R_j y)_n = y_n - q^j y_{n-1}$ for $n \geq 1$.

The general theory of linear difference equations implies that (2.15) is a consequence of the following two observations.

Claim 2.5. *(i)* $(\Delta_{q,k}y)_n = 1$ *for* $n \geq k$ *when*

$$y_n = \begin{bmatrix} n \\ k \end{bmatrix}_q, \tag{2.18}$$

(ii) $(\Delta_{q,k}y)_n = 0$ *for* $n \geq k - r$ *when*

$$y_n = \begin{bmatrix} n \\ k-r \end{bmatrix}_q q^{rn},\ r = 1, 2, \ldots, k. \tag{2.19}$$

Proof of Claim 2.5. We note that (2.18) is just $r = 0$ case of (2.19).

For fixed $r \geq 0$ and $n \geq k \geq r$ we have

$$\begin{aligned}
\left(R_k\left(\begin{bmatrix} n \\ k-r \end{bmatrix}_q q^{rn}\right)\right)_n &= \begin{bmatrix} n \\ k-r \end{bmatrix}_q q^{rn} - \begin{bmatrix} n-1 \\ k-r \end{bmatrix}_q q^{rn+k-r} \\
&= q^{rn}\frac{[n-k+r+1]_q \ldots [n-1]_q}{[k-r]_q!}\left([n]_q - q^{k-r}[n-k+r]_q\right) \\
&= q^{rn}\frac{[n-k+r+1]_q \ldots [n-1]_q}{[k-r]_q!}[k-r]_q \\
&= \begin{bmatrix} n-1 \\ k-1-r \end{bmatrix}_q q^{rn}.
\end{aligned}$$

Therefore

$$\left(R_{r+1}R_{r+2}\ldots R_k\left(\begin{bmatrix} n \\ k-r \end{bmatrix}_q q^{rn}\right)\right)_n = \begin{bmatrix} n-k+r \\ 0 \end{bmatrix}_q q^{rn} = q^{rn}.$$

If $r = 0$ this implies (2.18) by (2.17). If $r \geq 1$ then to prove (2.19) it remains to notice that since $n \geq k \geq r \geq 1$ we have $\left(R_r(q^{nr})\right)_n = q^{rn} - q^r q^{(n-1)r} = 0$. □

The constants $C_1, \dots, C_k$ are determined from the condition that formula (2.15) holds for $p_0, \dots, p_{k-1}$. $\square$

Proposition 2.6. *Suppose $\{p_n(y;t)\}$ satisfies recurrence* (1.9)*. Then there are constants $c_k(s)$ that do not depend on n such that:*

(i) if $0 < q < 1$ then $|p_n(x_k(s);s)| \leq c_k(s) q^{-kn}$;

(ii) if $q = 1$ then $|p_n(x_k(s);s)| \leq c_k(s) n^k$.

Proof. This follows from (2.15) and (2.12). $\square$

3 Uniqueness of the moment problem

Proposition 3.1. *Suppose $0 \leq q \leq 1$, $\eta > 0$, $\theta \geq 0$, $\tau \geq 0$. Let $\{p_n(y;t)\}$ be defined by* (1.9)*. Then the orthogonality measure μ_t of polynomials $\{p_n(y;t)\}$ is determined uniquely by moments.*

Proof. For $|q| < 1$, the coefficients of the recurrence are bounded, so the only case that requires proof is $q = 1$. Furthermore, the conclusion holds for $\tau = 0$, as in this case μ_t is a negative binomial law, see [8]. It therefore remains to consider the case $q = 1$, $\tau > 0$.

In this case, we use the fact that with $x_0 = -\eta^{-1}$ we have $p_n(x_0;t) = (-1)^n$, see Lemma 2.3. Let $q_n(y;t)$ be the associated polynomials which satisfy recurrence (1.9) for $n \geq 1$ with the initial terms $q_0 = 0$, $q_1 = 1/a_0$. Then

$$x_0 q_n(x_0) = a_n q_{n+1}(x_0) + (a_n + c_n + x_0) q_n(x_0) + c_n q_{n-1}(x_0).$$

Therefore with $f_n(t) := (-1)^{n-1} q_n(x_0;t)$ we have

$$\begin{aligned} f_{n+1} - f_n = \frac{c_n(t)}{a_n}(f_n - f_{n-1}) &= \frac{c_1(t)c_2(t)\dots c_n(t)}{a_1 a_2 \dots a_n}(f_1 - f_0) \\ &= \frac{c_1(t)c_2(t)\dots c_n(t)}{a_0 a_2 \dots a_{n-1}} \frac{1}{a_n}. \end{aligned}$$

Thus with a suitable convention for $n = 1$ we can write the solution as

$$f_{n+1}(t) = \sum_{k=0}^{n} \frac{c_1(t)c_2(t)\dots c_k(t)}{a_0 a_2 \dots a_{k-1}} \frac{1}{a_k}. \tag{3.1}$$

Let

$$\widetilde{p}_n(x;t) = \sqrt{\frac{a_0 a_2 \dots a_{n-1}}{c_1(t)c_2(t)\dots c_n(t)}}\, p_n(x;t) \tag{3.2}$$

and

$$\widetilde{q}_n(x;t) = \sqrt{\frac{a_0 a_2 \dots a_{n-1}}{c_1(t)c_2(t)\dots c_n(t)}}\, q_n(x;t)$$

be the corresponding orthonormal polynomials.

By [1, page 84], the moment problem is determined uniquely, if

$$\sum_n |\widetilde{p}_n(x_0)|^2 + \sum_n |\widetilde{q}_n(x_0)|^2 = \infty. \tag{3.3}$$

We have

$$|\widetilde{p}_n(x_0)|^2 = \frac{a_0 a_2 \dots a_{n-1}}{c_1(t)c_2(t)\dots c_n(t)},$$

and from (3.1) we get

$$|\widetilde{q}_n(x_0)|^2 = \frac{a_0 a_2 \dots a_{n-1}}{c_1(t)c_2(t)\dots c_n(t)} \left(\sum_{k=0}^{n} \frac{c_1(t)c_2(t)\dots c_k(t)}{a_0 a_2 \dots a_{k-1}} \, \frac{1}{a_k} \right)^2.$$

To verify (3.3) we use the fact that

$$a_n \approx \eta\tau n^2, \ \frac{c_{n+1}(t)}{a_n} = 1 + \frac{\alpha(t)}{n} + O(1/n^2), \ \frac{a_n}{c_{n+1}(t)} = 1 - \frac{\alpha(t)}{n} + O(1/n^2), \tag{3.4}$$

where

$$\alpha(t) = \frac{t\eta - \theta}{\eta\tau} + 1, \tag{3.5}$$

and $a_n \approx b_n$ means that $a_n/b_n \to 1$.

If $t \le \theta/\eta$ then $\alpha(t) \le 1$ and

$$\prod_{k=1}^{n} \left(1 - \frac{\alpha(t)}{k}\right) \approx \exp\left(-\alpha(t) \sum_{k=1}^{n} \frac{1}{k}\right) \approx n^{-\alpha(t)} \ge n^{-1},$$

so the first series in (3.3) diverges. On the other hand, if $t > \theta/\eta$ so that $\alpha(t) > 1$, then

$$|\widetilde{q}_n(x_0)|^2 \approx n^{-\alpha(t)} \left(\sum_{k=1}^{n-1} k^{\alpha(t)} \frac{1}{k^2} \right)^2 \approx n^{-\alpha(t)} \left(n^{\alpha(t)-1}\right)^2 = n^{\alpha(t)-2} > n^{-1},$$

so the second series in (3.3) diverges. □

4 Support of the orthogonality measure

Recall that μ_t denotes the orthogonality measure of polynomials $\{p_n(y;t)\}$. The following result will be used to define the orthogonality measure of auxiliary polynomials in Section 5.

Proposition 4.1. *Suppose* $0 \le q \le 1$, $\eta > 0$, $\theta \ge 0$, $\tau \ge 0$. *If* $x \in \operatorname{supp}(\mu_s)$, *then*

$$\prod_{j=0}^{n} A_j(x,s) \ge 0 \text{ for all } n \ge 0. \tag{4.1}$$

We prove Proposition 4.1 from rudimentary information about the support of μ_s.

Lemma 4.2. *Let $x_j(t)$ be given by (2.11). Then the support of μ_t is a subset of the interval $[x_0(t), \infty)$. In addition, if $t > \frac{\theta}{\eta} + \frac{1-q}{\eta^2}$ then*

$$\mathrm{supp}(\mu_t) \subset \{x_0(t), x_1(t), \dots, x_{k_*}(t)\} \cup [y_*, \infty)$$

with $y_* = \max\{x_{k_*}(t), x_{k_*+1}(t)\}$ and

$$k_* = \begin{cases} \max\left\{k : t > \frac{\theta}{\eta} + 2\tau k\right\}, & q = 1; \\ \\ \max\left\{k : q^{2k} > \dfrac{(1-q)^2 + (1-q)\eta\theta + \eta^2\tau}{\eta^2(t(1-q)+\tau)}\right\}, & 0 < q < 1. \end{cases} \tag{4.2}$$

Remark 4.1. We note that $y_* = x_{m^*}(t)$ with

$$m^* = \begin{cases} \max\{j : t > \frac{\theta}{\eta} + 2\tau j - 1\}, & q = 1; \\ \max\{j : q^{2j-1}\eta^2(t(1-q)+\tau) > (1-q)^2 + (1-q)\eta\theta + \eta^2\tau\}, & q < 1. \end{cases}$$

We will use the following criterion to show that there are at most $k_* + 1$ atoms below y_*.

Theorem A. Suppose $p_n(x)$ are orthogonal polynomials with unique orthogonality measure μ. If the sequence $\{(-1)^n p_n(a) : n \geq 0\}$ changes sign k-times, then there is a finite set D with at most k points such that

$$\mathrm{supp}(\mu_t) \subset D \cup [a, \infty).$$

In particular, μ has at most k atoms in $(-\infty, a)$.

Proof. This follows from the interlacing property of zeros of orthogonal polynomials. The details are omitted. □

Proof of Lemma 4.2. We first observe that $\mathrm{supp}(\mu_s) \subset [-\eta^{-1}, \infty)$. This follows from the fact that by Proposition 3.1 measure μ_s is determined uniquely, so we can combine Lemma 2.3 applied to $k = 0$ with Theorem A applied to $a = x_0 = -\eta^{-1}$.

We now verify that if $t > \frac{\theta}{\eta} + \frac{1-q}{\eta^2}$ then there are $k_* + 1$ atoms at $\{x_0(t), x_1(t), \dots, x_{k_*}(t)\}$. Recall that $x_j(t)$ is an atom of μ_t if the orthonormal polynomials (3.2) are square-summable at $x = x_j(t)$, see [1, page 84]. We will consider separately the cases $q = 1$ and $0 < q < 1$.

Suppose $q = 1$. Then by Proposition 2.6 we have

$$\sum_n |\widetilde{p}_n(x_j; t)|^2 \leq c_j \sum_n n^{2j} \frac{a_0 a_2 \dots a_{n-1}}{c_1(t)c_2(t)\dots c_n(t)} \approx c_j n^{2j-\alpha(t)}.$$

(See (3.5).) Therefore from (3.4), the series converges if $t > \frac{\theta}{\eta} + 2j\tau$.

Suppose now that $0 < q < 1$. Then by Proposition 2.6 we have

$$\sum_n |\widetilde{p}_n(x_j;t)|^2 \le c_j \sum_n q^{-2nj} \frac{a_0 a_2 \dots a_{n-1}}{c_1(t)c_2(t)\dots c_n(t)}.$$

Since

$$\lim_{n\to\infty} \frac{a_n}{c_{n+1}(t)} = \frac{(1-q)^2 + (1-q)\eta\theta + \eta^2\tau}{\eta^2(t(1-q)+\tau)},$$

the series converges if $\frac{(1-q)^2+(1-q)\eta\theta+\eta^2\tau}{q^{2j}\eta^2(t(1-q)+\tau)} < 1$. This proves that $x_j(t)$, $0 \le j \le k_*$ is an atom under the condition (4.2).

To estimate that there are at most k_*+1 atoms below y_* we use Lemma 2.4 to verify that there are at most k atoms of μ_t below $x_k(t)$. Namely, Lemma 2.4 states that there exists a polynomial $r(x)$ of degree k such that

$$p_n(x_k(t);t) = \begin{cases} r(n), & q = 1; \\ r(q^{-n}), & 0 < q < 1. \end{cases}$$

Since $r(x) = 0$ has at most k real solutions, the sequence $\{p_n(x_k(t);t) : n \ge 0\}$ has at most k changes of sign. Proposition 3.1 implies that we can use Theorem A to end the proof. □

Proof of Proposition 4.1. If $q = 0$ then $A_n(x,s) = \eta^{-1}$ does not depend on x for $n \ge 1$. Since $A_0(x,s) = 1/\eta + \theta + \eta\tau + x$, (4.1) follows from $\mathrm{supp}(\mu_s) \subset [-\eta^{-1}, \infty)$.

In the remaining part of the proof, we assume $0 < q \le 1$. We use the trivial observation that $A_j(x,s)$ increases as a function of x and decreases as a function of s.

Suppose $0 \le s \le \theta/\eta + (1-q)/\eta^2$. From $x \in \mathrm{supp}(\mu_s) \subset [-\eta^{-1}, \infty)$ we get

$$A_n(x,s) \ge A_n\left(-\frac{1}{\eta}, \frac{\theta}{\eta} + \frac{1-q}{\eta^2}\right) = \frac{(1-q^n)^2}{\eta} + \theta[n]_q(1-q^n) + [n]_q^2\eta\tau \ge 0.$$

Thus (4.1) holds.

Suppose $s > \theta/\eta + (1-q)/\eta^2$ so that $k_* = k_*(s) \ge 0$ is well defined. We notice that

$$x_0(s) \le x_1(s) \le \dots \le x_{k_*}(s) \le y_* \text{ and } x_j(s) \le y_* \text{ for all } j > k_*. \tag{4.3}$$

Omitting the easier case of $q = 1$, write $x_n(s) = h(q^n)$, where

$$h(z) = -\frac{1}{\eta z} + \frac{(1-z)(sz\eta - \theta)}{z(1-q)} - \frac{(1-z)^2\eta\tau}{z(1-q)^2}.$$

A calculation shows that

$$h''(z) = -2\frac{(1-q)(1-q+\eta\theta) + \eta^2\tau}{(1-q)^2 z^3 \eta} < 0$$

on the interval $0 < z < 1$. Since h tends to $-\infty$ at the endpoints, therefore it has a unique maximum $z_* \in (0,1)$ given by

$$z_*^2 \eta^2 \left((1-q)\, s + \tau\right) = (1-q)\,(1-q+\eta\theta) + \eta^2 \tau.$$

In particular, $q^{k_*+1} \leq z_* < q^{k_*}$, so $h(z)$ increases on $(0, q^{k_*+1})$ and decreases on $(q^{k_*}, 1]$. Thus $h(q^{k_*+1}) \geq h(q^{k_*+2}) \geq \dots$, and $h(q^0) \leq h(q^1) \leq \dots \leq h(q^{k_*})$.

Inequality (4.3) ends the proof as follows. If $x = x_k(s)$ for some $k \leq k_*$ then $A_j(x,s) \geq A_j(x_j(s), s) = 0$ for $0 \leq j \leq k$, so (4.1) holds for $0 \leq n < k$. On the other hand, $A_k(x,s) = 0$, so (4.1) holds trivially for all $n \geq k$.

Suppose now that $x \geq y_*$. Then (4.3) implies $A_j(x,s) \geq A_j(y_*, s) \geq A_j(x_j(s), s) = 0$ for all $j = 0, 1, 2 \dots$. Thus (4.1) follows.

□

4.1 Additional properties of orthogonality measure

Here we list without proof additional information about μ_t. Suppose $0 \leq q \leq 1$, $\eta > 0$, $\theta \geq 0$, $\tau > 0$.

(i) If $q = 1$, then there is at most countable set D such that $\mathrm{supp}(\mu_t) \subset D \cup [\alpha, \infty)$, where

$$\alpha = \frac{(\theta - \eta t)^2}{4\eta\tau} - \frac{1}{\eta}.$$

Moreover, α is an accumulation point of the support but $\mu_t(\{\alpha\}) = 0$. (This can be verified using [15].)

We conjecture that $D = \{x_0(t), x_1(t), \dots, x_{k_*}(t), y_*\}$.

(ii) If $0 < q < 1$, then μ_t has only absolutely continuous and discrete parts. The absolutely continuous part of μ_t has continuous density strictly positive on the interval $(\alpha - 2\sqrt{\beta}, \alpha - 2\sqrt{\beta})$, where

$$\alpha = \frac{\theta + t\eta}{1-q} + \frac{2\eta\tau}{(1-q)^2},\ \beta = \frac{((1-q)t + \tau)\left((1-q)^2 + \eta\theta(1-q) + \eta^2\tau\right)}{(1-q)^4}.$$

(This can be seen from [10], see also [14].)

(iii) If $q = 0$, then μ_t has only absolutely continuous and discrete parts. The absolutely continuous part of μ_s has continuous density strictly positive on the interval $(\alpha - 2\sqrt{\beta}, \alpha - 2\sqrt{\beta})$, where

$$\alpha = \theta + \eta t + 2\eta\tau,\ \beta = (t+\tau)(1 + \eta\theta + \eta^2\tau).$$

If $0 < t \leq \frac{\theta}{\eta} + \frac{1}{\eta^2}$, then there is no discrete part. If $t > \frac{\theta}{\eta} + \frac{1}{\eta^2}$, then the discrete part of μ_t is concentrated at $x_0 = -\eta^{-1}$. (This can be verified from the Stieltjes inversion formula, using the explicit formulas for the Cauchy transform for constant coefficient recursions.)

5 Auxiliary polynomials

For the proof of Theorem 1.1 we construct measure ν as a measure of orthogonality of auxiliary monic polynomials $\overline{Q}_n(y; x, t, s)$ in variable y. We begin with a non-monic version of these polynomials, defined by the three step recurrence

$$y\, Q_n(y; x, t, s) = A_n(x, s)Q_{n+1}(y; x, t, s) + B_n(x, t, s)Q_n(y; x, t, s) + C_n(t, s)Q_{n-1}(y; x, t, s), \quad (5.1)$$

where A_n is defined by (2.4) and

$$B_n(x, t, s) = b_n(t) + q^n x - (1 + q)q^{n-1}\eta s[n]_q\,, \quad (5.2)$$

$$C_n(t, s) = c_n(t) - q^{n-1}\eta s[n]_q\,. \quad (5.3)$$

with $Q_{-1} = 0$, $Q_0 = 1$. The Jacobi matrix of this recurrence arises as a solution of the q-commutation equation [5, (1)] with the appropriately modified initial condition; for more details see [4]. Polynomials $\{Q_n\}$ are well defined for all x, s, t as long as $x \notin \{x_0(s), x_1(s), \dots\}$.

5.1 Connection Coefficients

For $x \notin \{x_0(s), x_1(s), \dots\}$, the connection coefficients $\beta_{n,k}(x, t, s)$ are defined implicitly by

$$p_n(y; t) = \sum_{k=0}^{n} \beta_{n,k}(x, t, s)Q_k(y; x, t, s). \quad (5.4)$$

Our next goal is to find the connection coefficients $\beta_{n,k}(x, t, s)$ explicitly and to show that they do not depend on t.

Define two linear operators $K, L : \mathbb{R}^\infty \to \mathbb{R}^\infty$ acting on infinite matrices $\beta = [\beta_{n,k}]_{n,k\geq 0}$ by the rule

$$[K\beta]_{n,k} = a_n\beta_{n+1,k} + b_n(t)\beta_{n,k} + c_n(t)\beta_{n-1,k},$$

$$[L\beta]_{n,k} = A_{k-1}(x, s)\beta_{n,k-1} + B_k(x, t, s)\beta_{n,k} + C_{k+1}(t, s)\beta_{n,k+1}.$$

Let

$$H_t\beta = K\beta - L\beta. \quad (5.5)$$

The infinite triangular matrix $[\beta_{n,k}(x, t, s)]_{n\geq k\geq 0}$ is a unique solution of the discrete boundary value problem

$$H_t\beta = 0, \quad (5.6)$$

$$\beta_{n,n}(x, t, s) = \prod_{j=0}^{n-1} \frac{A_j(x, s)}{a_j},\ n \geq 0. \quad (5.7)$$

The boundary condition (5.7) arises from (5.4) by comparing the coefficients at y^n. Equation (5.6) follows directly from the recurrences; here we give an argument based on the fact that the orthogonality measure for polynomials $\{Q_n\}$ exists for an infinite set of x. For such x, we have

$$\beta_{n,k}(x,t,s) = \frac{\int p_n(y;t) Q_k(y;x,t,s)\nu_{x,t,s}(dy)}{\|Q_k\|_2^2}.$$

Since

$$\|Q_k\|_2^2 = \prod_{j=1}^{k} \frac{C_j(t,s)}{A_{j-1}(x,s)},$$

(5.6) follows from

$$\int [yp_n(y;t)] Q_k(y;x,t,s)\nu_{x,t,s}(dy) = \int p_n(y;t)[yQ_k(y;x,t,s)]\nu_{x,t,s}(dy).$$

by (1.9) and (5.1). Of course, once (5.6) holds for a large enough set of x, it holds for all x.

Lemma 5.1. *If $x \notin \{x_0(s), x_1(s), \dots\}$ then the coefficients $\beta_{n,k}(x,t,s)$ in* (5.4) *are determined uniquely, and do not depend on variable t. In fact, $\beta_{n,k}(x,t,s) = \beta_{n,k}(x,s)$ is defined by* (2.2) *with $p_n = p_n(x;s)$.*

Proof. Let $\beta_{n,k}(x,s)$ be defined by (2.1) with initial values $\beta_{n,0}(x,s) = p_n(x;s)$. Combining Lemma 2.1 with Proposition 2.2 we see that the initial condition (5.7) holds. Therefore, to conclude the proof we only need to verify the following.

Claim 5.2. *The matrix $\{\beta_{n,k}(x,s) : 0 \le k \le n\}$ as defined by* (2.1) *with $p_n = p_n(x;s)$ satisfies equation* (5.6)*.*

A straightforward computational proof goes as follows. Equation (5.6) is

$$a_n\beta_{n+1,k}(x,s) + b_n(t)\beta_{n,k}(x,s) + c_n(t)\beta_{n-1,k}(x,s) = A_{k-1}(x,s)\beta_{n,k-1}(x,s) + B_k(x,t,s)\beta_{n,k}(x,s) + C_{k+1}(t,s)\beta_{n,k+1}(x,s). \quad (5.8)$$

In view of (2.1), and using the explicit form (1.4), (1.5) we verify that the coefficients at variable t of this equation cancel out. Therefore, in (5.8) without loss of generality we may take $t = 0$. We now write $\beta_{n,k}(x,s)$ as $\sum_{j=0}^{k} \gamma_{n,k,j} p_{n-j}(x;s)$ where according to (2.2), we have

$$\gamma_{n,k,j} = q^{(k-j)(k-j-1)/2} \frac{[n]_q!}{[n-k]_q![j]_q![k-j]_q!}. \quad (5.9)$$

(We will also use the conventions that $\gamma_{n,k,j} = 0$ unless $0 \le k \le n$ and $0 \le j \le k$.) Then (5.8) is equivalent to a number of identities that arise from comparing the coefficients at $p_{n-j}(x;s)$. Here we use (1.9) to rewrite the terms $A_k(x,s)p_{n-j}(x;s)$ and $B_k(x,0,s)p_{n-j}(x;s)$ as the linear combinations of the polynomials $\{p_r(x;s)\}$. We get

$$
\begin{aligned}
a_n\gamma_{n+1,k,j+1} + b_n(0)\gamma_{n,k,j} + c_n(0)\gamma_{n-1,k,j-1} = a_{k-1}\gamma_{n,k-1,j}\\
- s\eta q^{k-1}[k-1]_q\gamma_{n,k-1,j} + q^{k-1}a_{n-j-1}\gamma_{n,k-1,j+1} + q^{k-1}b_{n-j}(s)\gamma_{n,k-1,j}\\
+ q^{k-1}c_{n+1-j}(s)\gamma_{n,k-1,j-1} + b_k(0)\gamma_{n,k,j} - (1+q)q^{k-1}s\eta[k]_q\gamma_{n,k,j}\\
+ q^k a_{n-j-1}(s)\gamma_{n,k,j+1} + q^k b_{n-j}(s)\gamma_{n,k,j}\\
+ q^k c_{n+1-j}(s)\gamma_{n,k,j-1} + c_{k+1}(0)\gamma_{n,k+1,j} - s\eta q^k[k+1]_q\gamma_{n,k+1,j}. \quad (5.10)
\end{aligned}
$$

Using the identities

$$
\begin{aligned}
\frac{\gamma_{n,k-1,j}}{\gamma_{n,k,j}} &= q^{j-k+1}\frac{[k-j]_q}{[n-k+1]_q}, & \frac{\gamma_{n,k-1,j-1}}{\gamma_{n,k,j}} &= \frac{[j]_q}{[n-k+1]_q},\\
\frac{\gamma_{n,k+1,j}}{\gamma_{n,k,j}} &= q^{k-j}\frac{[n-k]_q}{[k+1-j]_q}, & \frac{\gamma_{n+1,k,j+1}}{\gamma_{n,k,j}} &= q^{j-k+1}\frac{[n+1]_q[k-j]_q}{[n-k+1]_q[j+1]_q},\\
\frac{\gamma_{n-1,k,j-1}}{\gamma_{n,k,j}} &= q^{k-j}\frac{[n-k]_q[j]_q}{[n]_q[k+1-j]_q}, & \frac{\gamma_{n,k,j+1}}{\gamma_{n,k,j}} &= q^{j-k+1}\frac{[k-j]_q}{[j+1]_q},\\
\frac{\gamma_{n,k,j-1}}{\gamma_{n,k,j}} &= q^{k-j}\frac{[j]_q}{[k+1-j]_q}, & \frac{\gamma_{n,k-1,j+1}}{\gamma_{n,k,j}} &= q^{2j-2k+3}\frac{[k-j]_q[k-j-1]_q}{[n-k+1]_q[j+1]_q},
\end{aligned}
$$

equation (5.10) reduces to the following two identities between q-numbers. The first identity comes from comparing the coefficients at s,

$$
\begin{aligned}
0 = -q^j\frac{[k-1]_q[k-j]_q}{[n-k+1]_q} + q^j\frac{[n-j]_q[k-j]_q}{[n-k+1]_q} + q^{k-1}\frac{[n+1-j]_q[j]_q}{[n-k+1]_q}\\
-(1+q)q^{k-1}[k]_q + q^k[n-j]_q + q^{2k-j}\frac{[n+1-j]_q[j]_q}{[k+1-j]_q} - q^{2k-j}\frac{[k+1]_q[n-k]_q}{[k+1-j]_q}.
\end{aligned}
\quad (5.11)
$$

The second identity arises from comparing the coefficients free of s,

$$
\begin{aligned}
a_n q^{j-k+1}\frac{[n+1]_q[k-j]_q}{[n+1-k]_q[j+1]_q} + b_n(0) - b_k(0) + c_n(0)q^{k-j}\frac{[n-k]_q[j]_q}{[n]_q[k+1-j]_q}\\
= a_{k-1}q^{j-k+1}\frac{[k-j]_q}{[n+1-k]_q} + a_{n-j-1}q^{2j-k+2}\frac{[k-j]_q[k-j-1]_q}{[n+1-k]_q[j+1]_q}\\
+ b_{n-j}(0)q^j\frac{[k-j]_q}{[n+1-k]_q} + c_{n+1-j}(0)q^{k-1}\frac{[j]_q}{[n+1-k]_q} + a_{n-j-1}q^{j+1}\frac{[k-j]_q}{[j+1]_q}\\
+ b_{n-j}(0)q^k + c_{n+1-j}(0)q^{2k-j}\frac{[j]_q}{[k+1-j]_q} + c_{k+1}(0)q^{k-j}\frac{[n-k]_q}{[k+1-j]_q}. \quad (5.12)
\end{aligned}
$$

Identities (5.11) and (5.12) are in the form suitable for computer-assisted verification. We used Mathematica to confirm their validity. □

5.2 Monic polynomials

Let $\overline{Q}_n(y;x,t,s)$ denote the monic version of polynomials Q_n; these polynomials satisfy the recurrence

$$y\overline{Q}_n(y;x,t,s) = \overline{Q}_{n+1}(y;x,t,s) + B_n(x,t,s)\overline{Q}_n(y;x,t,s) \\ + A_{n-1}(x,s)C_n(t,s)\overline{Q}_{n-1}(y;x,t,s),\ n \geq 0, \tag{5.13}$$

with the usual initial conditions $\overline{Q}_{-1} = 0$, $\overline{Q}_0 = 1$. Here A_n, B_n, C_n are defined by (2.4), (5.2), and (5.3), respectively. Let $\overline{p}_n(y;t) = \overline{Q}_n(y;0,t,0)$ be the monic version of polynomials p_n. The monic polynomials $\{\overline{Q}_n\}$ are well defined for all x, s, leading to the following version of Lemma 5.1.

Corollary 5.3. *Suppose* $0 \leq q \leq 1$, $\eta > 0$, $\theta \geq 0$, $\tau \geq 0$. *For all* $s, t > 0$, $x, y \in \mathbb{R}$ *we have*

$$\overline{p}_n(y;t) = \sum_{k=0}^{n} \overline{\beta}_{n,k}(x,s)\overline{Q}_k(y;x,t,s),\ n \geq 0. \tag{5.14}$$

Proof. Suppose $x \notin \{x_0(s), x_1(s), \dots\}$. Then polynomials $Q_n(y;x,t,s)$ are well defined and from Lemma 5.1 we know that (5.4) holds with $\beta_{n,k}(x,t,s) = \beta_{n,k}(x,0,s)$ which do not depend on t.

It is well known that the monic polynomials $\overline{Q}_n$ can be written as

$$\overline{Q}_n(y;x,t,s) = \frac{Q_n(y;x,t,s)}{\prod_{j=0}^{n-1} A_j(x,s)}. \tag{5.15}$$

Therefore for such $x \notin \{x_0(s), x_1(s), \dots\}$, we get (5.14) with

$$\overline{\beta}_{n,k}(x,s) = \frac{\beta_{n,k}(x,0,s)\prod_{j=0}^{k-1} A_j(x,s)}{\prod_{j=0}^{n-1} a_j}. \tag{5.16}$$

We now extend this relation to all x. From (2.1) and (5.16) we see that (5.14) is a relation between the polynomials in variable x and holds on an infinite set of x. Therefore, it extends to all $x \in \mathbb{R}$. $\square$

5.3 Uniqueness

It turns out that polynomials $\{Q_n\}$ can be interpreted as polynomials $\{p_n\}$ with modified parameters.

Lemma 5.4. *Suppose* $x \notin \{x_0(s), x_1(s), \dots\}$. *With*

$$\theta' = \frac{\theta - s\eta - (1-q)x}{1+\eta x},\ \tau' = \frac{\tau + (1-q)s}{1+\eta x}, \tag{5.17}$$

we have

$$Q_n^{(\theta,\tau)}(y;x,t,s) = p_n^{(\theta',\tau')}\left(\frac{y-x}{1+\eta x}; \frac{t-s}{1+\eta x}\right). \tag{5.18}$$

Proof. Write (5.1) as

$$\frac{y-x}{1+\eta x}Q_n(y) = \frac{A_n(x,s)}{1+\eta x}Q_{n+1}(y) + \frac{B_n(x,t,s)-x}{1+\eta x}Q_n(y) + \frac{C_n(t,s)}{1+\eta x}Q_{n-1}(y).$$

Consider polynomials $r_n(y')$ such that $r_{-1}=0$, $r_0=1$, and

$$y'r_n(y') = \frac{A_n(x,s)}{1+\eta x}r_{n+1}(y') + \frac{B_n(x,t,s)-x}{1+\eta x}r_n(y') + \frac{C_n(t,s)}{1+\eta x}r_{n-1}(y').$$

Since $r_{-1}=Q_{-1}$ and $r_0=Q_0$, setting $y'=\frac{y-x}{1+\eta x}$ we have

$$r_n(y') = Q_n(y),\ n\geq 1.$$

Using (5.17) we get

$$\begin{aligned} y'r_n(y') = {} & \left(\eta^{-1}+\theta'[n]_q+\tau'\eta[n]_q^2\right) r_{n+1}(y') \\ & + \left(\eta t'+\theta'+([n]_q+[n-1]_q)\tau'\eta\right)[n]_q r_n(y') \\ & + \eta\left(t'+\tau'[n-1]_q\right)[n]_q r_{n-1}(y'). \end{aligned}$$

This means that polynomials $r_n(y')$ satisfy the same recurrence as polynomials $p_n(y';t')$ with parameters θ' and τ'. Thus (5.18) follows. ☐

Polynomials $\{Q_n\}$ are just a reparametrized version of polynomials p_n, see Proposition 2.2, so their orthogonality measure $\nu_{x,t,s}$ is also determined by moments. Since the orthogonality measure of polynomials $\overline{Q}_n$ may differ only for $x\in\{x_0(s),x_1(s),\dots\}$ in which case it has finite support, we get the following.

Corollary 5.5. *For all x such that (4.1) holds, the orthogonality measure $\nu_{x,t,s}$ of polynomials $\overline{Q}(\cdot;x,t,s)$ is unique.*

5.4 Proof of Theorem 1.1

Proof of Theorem 1.1. Replacing x by $-x$ in (1.2), changes η,θ to $-\eta,-\theta$. So without loss of generality we may assume $\eta\geq 0$. Furthermore, the case $\eta=0$ is known from [7], so we only consider $\eta>0$.

Let $\nu_{x,t,s}(dy)$ be the orthogonality measure of polynomials $\overline{Q}_n(y;x,t,s)$, see (5.13). By Proposition 4.1, measure $\nu_{x,t,s}(dy)$ is well defined for all $0<s<t$, $x\in\operatorname{supp}(\mu_s)$.

Corollary 5.3 implies that

$$\overline{\beta}_{n,k}(x,s) = \frac{\int \overline{p}_n(y;t)\overline{Q}_k(y;x,t,s)\nu_{x,t,s}(dy)}{\|\overline{Q}_k\|_2^2}.$$

Since $\overline{p}_n(x;s)=\overline{\beta}_{n,0}(x,s)$, using the above with $k=0$ we see that projection formula (1.8) holds for all $x\in\operatorname{supp}(\mu_s)$.

Projection formula (1.8) determines the moments of ν_x. By Corollary 5.5, this determines ν_x uniquely. ☐

Acknowledgement

The first-named author (WB) thanks Mourad Ismail for several helpful conversations.

References

[1] N. I. Akhiezer, *The classical moment problem and some related questions in analysis*, Translated by N. Kemmer, Hafner Publishing Co., New York, 1965.

[2] R. Askey and J. Fitch, *Integral representations for Jacobi polynomials and some applications.*, J. Math. Anal. Appl., 26 (1969), pp. 411–437.

[3] W. Bryc, W. Matysiak, and J. Wesołowski, *The bi-Poisson process: a quadratic harness.* arxiv.org/abs/math.PR/0510208, 2005.

[4] ——, *Free quadratic harness (tentative title).* In preparation, 2005.

[5] ——, *Quadratic harnesses, q-commutations, and orthogonal martingale polynomials*, Trans. Amer. Math. Soc., (2005). (to appear), arxiv.org/abs/math.PR/0504194.

[6] W. Bryc and J. Wesołowski, *Bi-Poisson process.* Submitted. arxiv.org/abs/math.PR/0404241, 2004.

[7] ——, *Conditional moments of q-Meixner processes*, Probab. Theory Related Fields, 131 (2005), pp. 415–441. arxiv.org/abs/math.PR/0403016.

[8] ——, *Classical bi-Poisson process: an invertible quadratic harness*, Statist. Probab. Lett., 76 (2006), pp. 1664–1674. arxiv.org/abs/math.PR/0508383.

[9] G. Gasper and M. Rahman, *Basic hypergeometric series*, vol. 35 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1990. With a foreword by Richard Askey.

[10] A. Máté and P. Nevai, *Orthogonal polynomials and absolutely continuous measures*, in Approximation theory, IV (College Station, Tex., 1983), Academic Press, New York, 1983, pp. 611–617.

[11] B. Nassrallah and M. Rahman, *Projection formulas, a reproducing kernel and a generating function for q-Wilson polynomials*, SIAM J. Math. Anal., 16 (1985), pp. 186–197.

[12] M. Rahman, *A projection formula for the Askey-Wilson polynomials and an application*, Proc. Amer. Math. Soc., 103 (1988), pp. 1099–1107.

[13] M. Rahman and Q. M. Tariq, *A projection formula and a reproducing kernel for the associated Askey-Wilson polynomials*, Int. J. Math. Stat. Sci., 6 (1997), pp. 141–160.

[14] R. Szwarc, *Uniform subexponential growth of orthogonal polynomials*, J. Approx. Theory, 81 (1995), pp. 296–302.

[15] ———, *Sharp estimates for Jacobi matrices and chain sequences*, J. Approx. Theory, 118 (2002), pp. 94–105.