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Chain Sequences, Orthogonal Polynomials, and Jacobi Matrices

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Chain sequences are positive sequences $\{a_n\}$ of the form $a_n = g_n(1 - g_{n-1})$ for a nonnegative sequence $\{g_n\}$. This concept was introduced by Wall in connection with continued fractions. In his monograph on orthogonal polynomials, Chihara conjectured that if $a_n \geqslant \frac{1}{4}$ for each n then $\sum (a_n - \frac{1}{4}) \leqslant \frac{1}{4}$. We prove this conjecture and give other precise estimates for a_n. We also characterize the chain sequences $\{a_n\}$ whose terms are greater than $\frac{1}{4}$. We show connections to Jacobi matrices and orthogonal polynomials. In particular, we characterize the maximal chain sequences in terms of integrability properties of the spectral measure of the associated Jacobi matrix.

Key words: orthogonal polynomials; chain sequence; recurrence relation.

1. INTRODUCTION

The concept of chain sequences was introduced by Wall [7] in his monograph on continued fractions. Chain sequences are sequences $\{a_n\}_{n=1}^{\infty}$ for which there exists a sequence $\{g_n\}_{n=0}^{\infty}$ such that $0 \leqslant g_n \leqslant 1$ and

$$a_n = g_n(1 - g_{n-1}), \qquad \text{for} \quad n \geqslant 1.$$

The sequence $\{g_n\}$ is called a parameter sequence and need not be unique.

The connection to continued fractions is that a nonnegative sequence $\{a_n\}$ is a chain sequence if and only if the approximants of the continued fraction

$$\frac{1|}{|1} - \frac{a_1|}{|1} - \frac{a_2|}{|1} - \frac{a_3|}{|1} - \cdots$$

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are positive and converge to a limit. We refer to [2, 7] for basic facts about chain sequences.

The constant sequence $a_n = \frac{1}{4}$ is one of the simplest examples of a chain sequence. The constant $\frac{1}{4}$ cannot be enlarged and moreover if $a_n \geqslant \frac{1}{4}$ and $\{a_n\}$ is a chain sequence then $a_n \to \frac{1}{4}$. In [2, Theorem III.5.8] Chihara showed that if $\{a_n\}$ is a chain sequence such that $a_n \geqslant \frac{1}{4}$ then

$$\sum_{n=1}^{\infty} (a_n - \tfrac{1}{4}) \leqslant \tfrac{3}{8}.$$

In [2, Exercise III.5.6] he replaced $\frac{3}{8}$ with $(1+\sqrt{2})/8$ and conjectured that $\frac{1}{4}$ is sufficient. We show that this conjecture is correct. We also show that one has

$$\sum_{n=N}^{\infty} \left(a_n - \frac{1}{4}\right) \leqslant \frac{1}{4N},$$

and we determine when the equality holds.

Chain sequences have important applications to orthogonal polynomials (see [2]). Let p_n be symmetric orthogonal polynomials on the interval $[-1, 1]$ relative to a probability measure μ and satisfying the recurrence relation

$$xp_n(x) = \lambda_{n+1} p_{n+1}(x) + \lambda_n p_{n-1}(x), \qquad n \geqslant 1, \tag{1}$$

with initial conditions $p_0(x) = 1$ and $p_1(x) = \lambda_1/x$. It can be shown that the support of μ is contained in $[-1, 1]$ if and only if $\{\lambda_n^2\}$ is a chain sequence. The constant sequence $\lambda_n = \frac{1}{2}$ corresponds to the Chebyshev polynomials of the second kind. Their orthogonality measure $d\mu(x) = (2/\pi)(1-x^2)^{1/2}\,dx$ is supported in $[-1, 1]$. When $\lambda_n \geqslant \frac{1}{2}$ the orthogonality interval can be larger than $[-1, 1]$. The question arises: by how much can λ_n exceed $\frac{1}{2}$ so that the orthogonality measure is still supported in the interval $[-1, 1]$? This question is connected with estimating the norms or spectral radii of the Jacobi matrix associated with (1), because the orthogonality measure is supported in $[-1, 1]$ if and only if the spectral radius of the Jacobi matrix is less than or equal to 1. All this can readily be solved by means of chain sequences. We give necessary and sufficient conditions for sequences $\lambda_n \geqslant \frac{1}{2}$ such that $\operatorname{supp} \mu \subset [-1, 1]$. These conditions are useful in constructing such sequences.

We also discuss maximal sequences λ_n with the property that the Jacobi matrix associated with λ_n has a spectral radius equal to 1 and each Jacobi matrix associated with $\tilde{\lambda}_n \geqslant \lambda_n$ has a spectral radius equal to 1 if and only if $\tilde{\lambda}_n = \lambda_n$ for each n. We show that a sequence $\lambda_n > 0$ is maximal if and only if the series $\sum m_{2n}$ is divergent, where m_n are the moments of the orthogonality measure associated with J.

The origin of our interest in polynomials p_n orthogonal on the interval $[-1,1]$ and such that $\lambda_n \geqslant \frac{1}{2}$ comes from the nonnegative linearization problem. If we express the product $p_n(x)\,p_m(x)$ in terms of $p_k(x)$ we get the linearization formula

$$p_n(x)\,p_m(x) = \sum_{k=|n-m|}^{n+m} c(n,m,k)\,p_k(x).$$

By [6, Prop. 1] we get that $c(n,m,k)$ are nonnegative for all $n, m, k \geqslant 0$ provided that $\lambda_n \geqslant \frac{1}{2}$ and $\operatorname{supp}\mu \subset [-1,1]$ (see also [5, Theorem 3]).

2. JACOBI MATRICES AND CHAIN SEQUENCES

A given sequence of real numbers λ_n determines a Jacobi matrix J as follows:

$$J = \begin{pmatrix} 0 & \lambda_1 & 0 & 0 & \cdots \\ \lambda_1 & 0 & \lambda_2 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \lambda_3 & \ddots \\ 0 & 0 & \lambda_3 & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2}$$

The connection between Jacobi matrices and chain sequences is exhibited in the next proposition.

PROPOSITION 1. *The Jacobi matrix J corresponds to a bounded linear operator on square summable real valued sequences, with operator norm less than or equal to* 1 *if and only if λ_n^2 is a chain sequence.*

Proof. Since J is a symmetric matrix we have

$$\|J\|_{\ell^2 \to \ell^2} = \sup\{x^{\mathrm{T}} J x \mid x^{\mathrm{T}} x \leqslant 1\}.$$

On the other hand,

$$x^{\mathrm{T}} J x = 2 \sum_{n=1}^{\infty} \lambda_n x_n x_{n+1}.$$

Now the conclusion follows from [7, Theorem 20.1] (see also [2, Exercise III.5 13]). ∎

3. ESTIMATES FOR CHAIN SEQUENCES

The next two lemmas are known. We prove them in order to remain self-contained. Note that the proof of Lemma 2 is entirely different from the one in [2, p. 99].

LEMMA 1 (Wall [7]). *Let a_n be a chain sequence with a parameter sequence g_n. If $a_n \geqslant \frac{1}{4}$ the sequence g_n is increasing and it tends to $\frac{1}{2}$. In particular, a_n tends to $\frac{1}{4}$.*

Proof. We have

$$g_n = \frac{a_n}{1 - g_{n-1}} \geqslant \frac{1}{4(1 - g_{n-1})} \geqslant g_{n-1}.$$

Thus $g_n \nearrow g$ and $a_n \to g(1-g) \geqslant \frac{1}{4}$. Hence $g = \frac{1}{2}$.

LEMMA 2 ([2], p. 99). *Let $a_n \geqslant \frac{1}{4}$ be a chain sequence with a parameter sequence g_n. Then*

$$0 \leqslant \frac{1}{2} - g_n \leqslant \frac{1}{2(n+1)}.$$

Proof. Let

$$\delta_n = 1 - 2g_n.$$

In view of the preceding lemma we have $0 \leqslant \delta_n \leqslant 1$ and

$$\delta_n = \frac{\delta_{n-1} - (4a_n - 1)}{1 + \delta_{n-1}} \leqslant \frac{\delta_{n-1}}{1 + \delta_{n-1}}.$$

Hence

$$\delta_n \leqslant f(\delta_{n-1}), \qquad \text{where} \quad f(x) = \frac{x}{x+1}.$$

Therefore

$$\delta_n \leqslant \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}(\delta_0) = \frac{\delta_0}{n\delta_0 + 1} \leqslant \frac{1}{n+1}.$$

This gives the conclusion. ∎

THEOREM 1. *Let $a_n \geqslant 1/4$ be a chain sequence. Then*

$$\sum_{m=n}^{\infty}\left(a_m - \frac{1}{4}\right) \leqslant \frac{1}{4n}, \qquad n \geqslant 1. \tag{3}$$

If for some $n \geqslant 1$ equality holds in (3) then $a_m = 1/4$ for $m > n$ and $a_n = 1/4 + 1/4n$.

In particular

$$\sum_{n=1}^{\infty}\left(a_n - \frac{1}{4}\right) \leqslant \frac{1}{4} \tag{4}$$

and the equality holds if and only if $a_1 = 1/2$ and $a_n = 1/4$ for $n \geqslant 2$.

Proof. Let g_n be any parameter sequence for a_n. By Lemma 1 we have $g_n \geqslant \frac{1}{2}$. Therefore

$$\begin{aligned} a_n - \tfrac{1}{4} &= g_n(1 - g_{n-1}) - \tfrac{1}{4} \\ &= \tfrac{1}{2}(g_n - g_{n-1}) - (\tfrac{1}{2} - g_{n-1})(\tfrac{1}{2} - g_n) \leqslant \tfrac{1}{2}(g_n - g_{n-1}). \end{aligned}$$

Next, adding up the terms and using Lemma 2 gives

$$\sum_{m=n}^{\infty}\left(a_m - \frac{1}{4}\right) \leqslant \frac{1}{2}\left(\frac{1}{2} - g_{n-1}\right) \leqslant \frac{1}{4n}.$$

This equality holds if and only if

$$(\tfrac{1}{2} - g_{m-1})(\tfrac{1}{2} - g_m) = 0 \qquad \text{for} \quad m \geqslant n$$

and $g_{n-1} = 1/2 - 1/2n$. Since by Lemma 1 the sequence g_n is nondecreasing we get $g_m = 1/2$ for $m \geqslant n$. Therefore

$$a_n = g_n(1 - g_{n-1}) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2n}\right) = \frac{1}{4} + \frac{1}{4n},$$

$$a_m = g_m(1 - g_{m-1}) = \frac{1}{4} \qquad \text{for} \quad m > n. \quad \blacksquare$$

THEOREM 2. *Let a_n be a chain sequence such that $a_n \searrow \frac{1}{4}$. Then*

$$0 \leqslant a_n - \frac{1}{4} \leqslant \frac{1}{4}\left(\tan\frac{\pi}{2(n+1)}\right)^2. \tag{5}$$

Proof. We will make use of the following lemma.

LEMMA 3. *Let*

$$f(x)=\frac{x-\varepsilon^2}{x+1}, \qquad \varepsilon=\tan\varphi.$$

Then denoting by f^m the mth iterate of f we have

$$f^m(x)=\frac{x-\varepsilon\tan m\varphi}{(\varepsilon^{-1}\tan m\varphi)x+1}.$$

In particular

$$f^m(1)=\frac{\tan\varphi}{\tan(m+1)\varphi}.$$

Lemma 3 can be proved by induction using the relation between $\tan(m+1)\varphi$ and $\tan m\varphi$. The more demanding reader may instead consider the corresponding 2×2 matrix

$$F=\begin{pmatrix}1 & -\varepsilon^2\\ 1 & 1\end{pmatrix}.$$

Its iterates can be computed by finding a basis of eigenvectors for F.

Assume that

$$a_n=\tfrac{1}{4}+\tfrac{1}{4}\varepsilon_n^2 \qquad \text{and} \qquad \varepsilon_n\searrow 0.$$

Let g_n be a parameter sequence for a_n. Write g_n in the form

$$g_n=\tfrac{1}{2}(1-\delta_n).$$

By Lemma 1 we have $0\leqslant\delta_n\leqslant 1$. Then using $a_n=g_n(1-g_{n-1})$ and $\varepsilon_m\leqslant\varepsilon_n$ gives

$$\delta_m=\frac{\delta_{m-1}-\varepsilon_m^2}{\delta_{m-1}+1}\leqslant\frac{\delta_{m-1}-\varepsilon_n^2}{\delta_{m-1}+1}, \qquad m\leqslant n.$$

Thus

$$\delta_m\leqslant f(\delta_{m-1}), \qquad \text{where} \quad f(x)=\frac{x-\varepsilon_n^2}{x+1}, \qquad \text{for} \quad m\leqslant n.$$

Since $f(x)$ is an increasing function for $x\geqslant 0$, by Lemma 3 we obtain

$$\delta_m\leqslant f^m(\delta_0)\leqslant f^m(1)=\frac{\tan\varphi}{\tan(m+1)\varphi} \tag{6}$$

for $m \leqslant n$, where $\varepsilon_n = \tan \varphi$. This implies that $\tan(m+1)\varphi \geqslant 0$ for $m \leqslant n$. Hence $(n+1)\varphi \leqslant \pi/2$. Thus

$$\varepsilon_n = \tan \varphi \leqslant \tan \frac{\pi}{2(n+1)}. \quad \blacksquare$$

Using $\tan x \leqslant 4x/\pi$, for $0 \leqslant x < \pi/4$, gives

COROLLARY 1. *Let a_n be a chain sequence such that $a_n \searrow \frac{1}{4}$. Then*

$$a_n - \frac{1}{4} \leqslant \frac{1}{(n+1)^2}.$$

The next theorem gives a characterization of the chain sequences with all terms greater than or equal to $\frac{1}{4}$. In view of Proposition 1 the constant chain sequence $a_n = \frac{1}{4}$ corresponds to the Jacobi matrix with $\lambda_n = \frac{1}{2}$, which is in turn associated with the Chebyshev polynomials of the second kind

$$U_n(\cos x) = \frac{\sin(n+1)x}{\sin x}.$$

THEOREM 3. *Let $a_n = \frac{1}{4}(1+\varepsilon_n)$, with $\varepsilon_n \geqslant 0$. Then $\{a_n\}$ is a chain sequence if and only if there exists a sequence $\{c_n\}$ of positive numbers such that*

$$\text{(i)} \qquad c_{n+1} \leqslant 2c_n, \qquad \textit{for} \quad n \geqslant 1.$$

$$\text{(ii)} \quad c_{n+1} - c_n \geqslant \sum_{m=n}^{\infty} c_m \varepsilon_m, \qquad \textit{for} \quad n \geqslant 1.$$

Proof. ($\Rightarrow$) Let $a_n = g_n(1 - g_{n-1})$ and $g_n = \frac{1}{2}(1 - \delta_n)$. Then

$$\varepsilon_n = \delta_{n-1} - \delta_n - \delta_{n-1}\,\delta_n.$$

Set $c_1 = 1$ and

$$\frac{c_{n+1}}{c_n} = 1 + \delta_{n-1}.$$

Then $c_{n+1} \leqslant 2c_n$, because $\delta_n \leqslant 1$. We have

$$c_{n+1} - c_n = c_n\,\delta_{n-1}. \tag{7}$$

Moreover

$$\varepsilon_n = \delta_{n-1} - \delta_n(1 + \delta_{n-1}) = \delta_{n-1} - \frac{c_{n+1}}{c_n}\,\delta_n.$$

Hence

$$c_n\varepsilon_n = c_n\,\delta_{n-1} - c_{n+1}\,\delta_n. \tag{8}$$

Thus the sequence $c_n\,\delta_{n-1}$ is nonincreasing; as such it has a limit $c_n\,\delta_{n-1} \to s \geqslant 0$.

Now summing up (8) and using (7) yields

$$\sum_{m=n}^{\infty} c_m\varepsilon_m = c_n\,\delta_{n-1} - s = c_{n+1} - c_n - s \leqslant c_{n+1} - c_n.$$

($\Leftarrow$)
Set

$$\delta_n = c_{n+1}^{-1} \sum_{m=n+1}^{\infty} c_m\varepsilon_m.$$

Then

$$\delta_n \leqslant \frac{c_{n+2}}{c_{n+1}} - 1.$$

Thus $\delta_n \leqslant 1$. Let $h_n = \frac{1}{2}(1-\delta_n)$. Then

$$\begin{aligned}
4h_n(1-h_{n-1}) = (1-\delta_n)(1+\delta_{n-1}) &= 1+\varepsilon_n + \frac{c_{n+1}-c_n}{c_n}\,\delta_n - \delta_{n-1}\,\delta_n \\
&\geqslant 1+\varepsilon_n + \frac{1}{c_n}\left(\sum_{m=n}^{\infty} c_m\varepsilon_m\right)\delta_n - \delta_{n-1}\,\delta_n \\
&= 1+\varepsilon_n = 4a_n.
\end{aligned}$$

Thus $h_n(1-h_{n-1}) \geqslant a_n$. This implies that a_n is a chain sequence (see also [2, Theorem 5.7, p. 97]). ∎

The next Corollary can be found in [2, Problem 5.7, p. 100]

COROLLARY 2 (Chihara). *Let* $a_n = \frac{1}{4}(1+\varepsilon_n)$, *where* $\varepsilon_n \geqslant 0$ *and*

$$\sum_{m=1}^{\infty} m\varepsilon_m \leqslant 1.$$

Then $\{a_n\}$ *is a chain sequence.*

Proof. Apply Theorem 3 with $c_n = n$. ∎

COROLLARY 3. *Let $\{c_n\}$ be a concave nondecreasing sequence of positive numbers satisfying $2c_n \leqslant c_{n+1}$ for $n \geqslant 1$. Set*

$$\varepsilon_n = \frac{2c_{n+1} - c_n - c_{n+2}}{c_n}.$$

Then the sequence $a_n = \frac{1}{4}(1 + \varepsilon_n)$ is a chain sequence.

Proof. We have $c_{n+1} - c_n \searrow s \geqslant 0$. Thus

$$\sum_{m=n}^{\infty} \varepsilon_m c_m = c_{n+1} - c_n - s \leqslant c_{n+1} - c_n. \quad \blacksquare$$

4. MAXIMAL CHAIN SEQUENCES

A chain sequence $\{a_n\}$ is called *maximal* if there is no chain sequence $\{b_n\}$ such that $b_n \geqslant a_n$ and $\{a_n\} \neq \{b_n\}$.

Maximal chain sequences exist and moreover every chain sequence is bounded from above by a maximal one (see [7]).

The next proposition follows in part from Proposition 1.

PROPOSITION 2. *Let $\{a_n\}$ and $\{b_n\}$ be chain sequences. Then the sequence $\{c_n\}$ defined by*

$$\sqrt{c_n} = \lambda \sqrt{a_n} + (1 - \lambda) \sqrt{b_n}$$

is a chain sequence for any $0 < \lambda < 1$. Moreover if $\{a_n\} \neq \{b_n\}$ then $\{c_n\}$ is not a maximal chain sequence.

Proof. Let $a_n = g_n(1 - g_{n-1})$ and $b_n = h_n(1 - h_{n-1})$ for $0 \leqslant g_n \leqslant 1$ and $0 \leqslant h_n \leqslant 1$. Set $f_n = \lambda g_n + (1 - \lambda) h_n$. Then

$$\begin{aligned} f_n(1 - f_{n-1}) &= \{\lambda g_n + (1 - \lambda) h_n\}\{(1 - \lambda)(1 - h_{n-1}) + \lambda(1 - g_{n-1})\} \\ &= \lambda(1 - \lambda)\{g_n(1 - h_{n-1}) + h_n(1 - g_{n-1})\} + \lambda^2 a_n + (1 - \lambda)^2 b_n \\ &\geqslant 2\lambda(1 - \lambda) \sqrt{g_n(1 - g_{n-1}) h_n(1 - h_{n-1})} + \lambda^2 a_n + (1 - \lambda)^2 b_n \\ &= (\lambda \sqrt{a_n} + (1 - \lambda) \sqrt{b_n})^2 = c_n. \end{aligned}$$

Hence c_n is a chain sequence as it is bounded from above by the chain sequence $f_n(1 - f_{n-1})$.

If $\{c_n\}$ is a maximal chain sequence then both $\{a_n\}$ and $\{b_n\}$ are maximal chain sequences. In that case $g_0 = h_0 = 0$, because otherwise setting $g_0 = 0$ or $h_0 = 0$ leads to chain sequences which are greater than $\{a_n\}$ or $\{b_n\}$, respectively.

Also, if $\{c_n\}$ is a maximal chain sequence, the calculations performed above enforce

$$c_n = f_n(1 - f_{n-1}),$$
$$g_n(1 - h_{n-1}) = h_n(1 - g_{n-1}), \qquad n \geqslant 1.$$

Since $g_0 = h_0 = 0$, the last equation implies $g_n = h_n$ for $n \geqslant 0$. Hence $a_n = b_n$ for $n \geqslant 1$. ∎

We now turn to chain sequences such that $a_n \geqslant \frac{1}{4}$.

THEOREM 4. *Let $a_n = \frac{1}{4}(1 + \varepsilon_n)$, where $\varepsilon \geqslant 0$. Then $\{a_n\}$ is a maximal chain sequence if and only if there exists a unique sequence $\{c_n\}$ of positive numbers such that $c_1 = 1$ and*

$$\text{(i)} \qquad 2c_n \geqslant c_{n+1}, \qquad \textit{for} \quad n \geqslant 1.$$

$$\text{(ii)} \quad c_{n+1} - c_n \geqslant \sum_{m=n}^{\infty} c_m \varepsilon_m, \qquad \textit{for} \quad n \geqslant 1.$$

Proof. Let $\{a_n\}$ be a maximal chain sequence. By Theorem 3 a sequence $\{c_n\}$ exists. Let g_n be a unique parameter sequence for $\{a_n\}$. Set $g_n = \frac{1}{2}(1 - \delta_n)$. Analyzing the second part of the proof of Theorem 3 we get

$$c_{n+1} - c_n = \sum_{m=n}^{\infty} c_m \varepsilon_m = c_n \, \delta_{n-1}, \qquad n \geqslant 1.$$

Since $c_1 = 1$ and δ_n is uniquely determined by g_n we conclude that c_n is also uniquely determined.

Assume $\{a_n\}$ is not maximal. By [7] it has two different parameter sequences. Hence there exist $\{g_n\}$ and $\{h_n\}$ such that

$$a_n = g_n(1 - g_{n-1}) = h_n(1 - h_{n-1}), \qquad n \geqslant 1,$$

and $g_0 < h_0$. Define c_n and d_n by $c_1 = d_1 = 1$ and $c_{n+1} = 2(1 - g_{n-1})\, c_n$ and $d_{n+1} = 2(1 - h_{n-1})\, d_n$. This leads to two different sequences satisfying Theorem 3. ∎

DEFINITION 1. The Jacobi matrix associated with the sequence $\{\lambda_n\}$ will be called *maximal* if $\|J\| \leqslant 1$ and for each Jacobi matrix J' associated with the sequence $\{\lambda'_n\} \neq \{\lambda_n\}$ such that $|\lambda_n| \leqslant |\lambda'_n|$ we have $\|J'\| > 1$.

In view of Proposition 1 J is a maximal Jacobi matrix if and only if $\{\lambda_n^2\}$ is a maximal parameter sequence. By [7] a sequence $\{a_n\}$ is a maximal chain sequence if and only if the continued fraction

$$\frac{1|}{|1} - \frac{a_1|}{|1} - \frac{a_2|}{|1} - \frac{a_3|}{|1} - \cdots$$

tends to 0.

Favard's Theorem states that for each Jacobi matrix of the form (2) there exists a probability measure μ symmetric about the origin, $\operatorname{supp}\mu \subset [-\|J\|, \|J\|]$, such that it is the orthogonality measure for the polynomials p_n given recursively by (1).

The next theorem collects facts that can be deduced from [7] and [1, Theorem 1]. Our setting is a bit different, so we provide a short independent proof.

THEOREM 5. *Let J be a Jacobi matrix associated with the sequence $\{\lambda_n\}$. Assume that $\|J\| \leqslant 1$ and that μ is the associated orthogonality measure. The following conditions are equivalent.*

(i) *J is a maximal Jacobi matrix.*

(ii) *$\sum_{n=0}^{\infty} m_{2n} = +\infty$, where $m_n = \int_{-\infty}^{\infty} x^n \, d\mu(x)$.*

(iii) *The continued fraction*

$$1 - \frac{\lambda_1^2|}{|1} - \frac{\lambda_2^2|}{|1} - \frac{\lambda_3^2|}{|1} - \cdots$$

tends to 0.

Proof. Assume J is not maximal. In view of Proposition 1 the sequence $a_n = \lambda_n^2$ is not a maximal chain sequence. Then there exists a chain sequence $\{b_n\}$ such that $a_n \leqslant b_n$ and $\{a_n\} \neq \{b_n\}$. Let $\{h_n\}$ be a parameter sequence for $\{b_n\}$ and N be the smallest index such that $a_N < b_N = h_N(1 - h_{N-1})$. Set $g_N = h_N$ and define g_n recursively by $a_n = g_n(1 - g_{n-1})$. Then it is immediate that $g_n > h_n$ for $n < N$ and $g_n \leqslant h_n$ for $n > N$. In particular we have that $g_0 > h_0 \geqslant 0$.

Let $r_n(x)$ be the polynomials defined recursively by

$$x r_n(x) = g_{n-1} r_{n+1}(x) + (1 - g_{n-1})\, r_{n-1}(x), \qquad n \geqslant 1, \tag{9}$$

with initial conditions $r_0(x)=1, r_1(x)=x$. Thus $r_n(1)=1$ and $r_n(-1)=(-1)^n$. Hence $r_{n+2}(x)-r_n(x)$ is divisible by x^2-1. Consider the polynomials $q_n(x)$ defined by

$$q_n(x)=\frac{r_{n+2}(x)-r_n(x)}{x^2-1}. \tag{10}$$

Then by (9) we obtain

$$xq_n(x)=g_{n+1}q_{n+1}(x)+(1-g_{n-1})\,q_{n-1}(x), \qquad n\geqslant 1,$$

and $q_0(x)=g_0^{-1}, q_1(x)=(g_0g_1)^{-1}x$. Set

$$p_n(x)=g_0\sqrt{\frac{g_1g_2\cdots g_n}{(1-g_0)(1-g_1)\cdots(1-g_{n-1})}}\,g_n(x). \tag{11}$$

Then the polynomials p_n satisfy

$$xp_n(x)=\sqrt{g_{n+1}(1-g_n)}\,p_{n+1}(x)+\sqrt{g_n(1-g_{n-1})}\,p_{n-1}(x), \qquad n\geqslant 1, \tag{12}$$

with $p_0(x)=1$ and $p_1(x)=c^{-1}x$, where $c=\sqrt{g_1(1-g_0)}$. Then taking into account $\lambda_n^2=a_n=g_n(1-g_{n-1})$ implies

$$xp_n(x)=\lambda_{n+1}p_{n+1}(x)+\lambda_n p_n(x), \qquad n\geqslant 1, \tag{13}$$

with $p_0(x)=1$ and $p_1(x)=\lambda_1^{-1}x$. Let ν be a probability measure associated with the polynomials r_n, which exists by Favard's Theorem. Recall that by (13) μ is the orthogonality measure for the p_ns and it is supported in $[-1,1]$. By (10) and (11) the measures ν and μ are related by

$$d\mu(x)=c^{-1}(1-x^2)\,d\nu(x),$$

where $c=\int_{-1}^{1}(1-x^2)\,d\nu(x)$. Thus

$$\sum_{n=0}^{\infty} m_{2n}=\int_{-1}^{1}\frac{d\mu(x)}{1-x^2}<+\infty. \tag{14}$$

This completes the proof of (ii) $\Rightarrow$ (i).

Assume $\sum_{n=0}^{\infty} m_{2n}<+\infty$. By (14) the measure

$$d\nu(x)=c^{-1}\frac{d\mu(x)}{1-x^2}, \qquad \text{where} \quad c=\sum_{n=0}^{\infty} m_{2n},$$

has total mass equal to 1 and its support is contained in $[-1, 1]$. Hence the zeros of the corresponding orthogonal polynomials belong to $(-1, 1)$. Let r_n be these polynomials normalized at $x=1$, i.e., $r_n(1)=1$. There exists a sequence $\{g_n\}_{n=0}^{\infty}$ such that $g_0>0$ and

$$xr_n(x)=g_{n-1}r_{n+1}(x)+(1-g_{n-1})\,r_{n-1}(x), \qquad n\geqslant 1,$$

where $r_0(x)=1$ and $r_1(x)=x$. Define the polynomials q_n and p_n by (10) and (11), respectively. Then by the relation between μ and ν the polynomials p_n are orthonormal relative to μ. Hence combining (12) and (13) gives

$$\lambda_n^2=g_n(1-g_{n-1}).$$

Since $g_0>0$ the sequence $\{\lambda_n^2\}$ is not a maximal chain sequence, which in turn implies that J is not a maximal Jacobi matrix. This shows (i) $\Rightarrow$ (ii).

The equivalence (ii) $\Leftrightarrow$ (iii) follows from the formula

$$\int_{-1}^{1}\frac{d\mu(x)}{y-x}=\frac{1|}{|y}-\frac{\lambda_1^2|}{|y}-\frac{\lambda_2^2|}{|y}-\cdots, \tag{15}$$

(see [3, p. 46]) and the fact that since the measure μ is symmetric about $x=0$

$$\int_{-1}^{1}\frac{d\mu(x)}{1-x^2}=\int_{-1}^{1}\frac{d\mu(x)}{1-x}.$$

The formula (15) holds for $y\notin[-1, 1]$. We get the desired result by taking the limit when $y\to 1^+$. ∎

5. MAXIMAL PARAMETER SEQUENCES

Wall [7] observed that a chain sequence $\{a_n\}$ is maximal if and only if it admits a unique parameter sequence. Other chain sequences admit more parameter sequences. Among them there exists a maximal parameter sequence (see [7, Theorem 19.2; 2, Theorem III.5.3]). Wall proved that maximal parameter sequences are exactly those sequences $\{g_n\}$ for which

$$\sum_{n=1}^{\infty}\frac{g_1g_2\cdots g_n}{(1-g_1)(1-g_2)\cdots(1-g_n)}=+\infty. \tag{16}$$

(see [7, (19.10), p. 82; 2, Theorem III.6.1]). For chain sequences $\{a_n\}$, with terms greater than $\frac{1}{4}$ we have the following.

Proposition 3. *Let $\{a_n\}$ be a chain sequence such that $a_n \geqslant \frac{1}{4}$ for $n \geqslant 1$. Let $\{a_n\}$ be a parameter sequence and set $g_n = \frac{1}{2}(1-\delta_n)$. Then $\{g_n\}$ is the maximal parameter sequence if and only if*

$$\sum_{n=1}^{\infty} \exp\left(-2\sum_{k=1}^{n} \delta_k\right) = \infty.$$

Proof. Observe that

$$\frac{g_1 g_2 \cdots g_n}{(1-g_1)(1-g_2)\cdots(1-g_n)} = \frac{(1-\delta_1^2)(1-\delta_2^2)\cdots(1-\delta_n^2)}{(1+\delta_1)^2\,(1+\delta_2)^2\cdots(1+\delta_n)^2}. \tag{17}$$

By Lemma 2 we have $\delta_n \leqslant 1/(n+1)$. Thus

$$\frac{n+1}{2n} \leqslant (1-\delta_1^2)(1-\delta_2^2)\cdots(1-\delta_n^2) \leqslant 1. \tag{18}$$

Again by using Lemma 2 and the fact that the function $x \mapsto e^x/(1+x)$ is increasing we obtain

$$\begin{aligned} 1 &\leqslant \frac{e^{\delta_1}}{1+\delta_1}\,\frac{e^{\delta_2}}{1+\delta_2}\cdots\frac{e^{\delta_n}}{1+\delta_n} \\ &\leqslant \frac{\exp(1+\frac{1}{2}+\cdots+\frac{1}{n})}{(1+\frac{1}{2})(1+\frac{1}{3})\cdots(1+\frac{1}{n})} \\ &= \frac{2}{n+1}\exp\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) \leqslant 2. \end{aligned} \tag{19}$$

Combining (17), (18), and (19) gives the conclusion. ∎

Corollary 4. *Using the notation of Proposition 3, if*

$$\sum_{n=1}^{\infty} \delta_n < +\infty$$

then the sequence $g_n = \frac{1}{2}(1-\delta_n)$ is the maximal parameter sequence.

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