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NONNEGATIVE LINEARIZATION OF ORTHOGONAL POLYNOMIALS

BY

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1. Introduction. We will deal with polynomials p_n orthogonal with respect to a probability measure on the real line. The product of two of these polynomials can be written as a sum of these polynomials:

$$p_n(x)p_m(x) = \sum_{k=|n-m|}^{n+m} c(n,m,k)p_k(x). \tag{1}$$

The constants $c(n,m,k)$ of this expansion are called the *linearization coefficients*. The nonnegativity of these coefficients leads to a Banach algebra structure associated with the polynomials which is analogous to the algebra of absolutely convergent Fourier series on a torus (see Askey [1], Askey–Wainger [2] and Igari–Uno [8] for examples).

In 1970 Richard Askey found a set of conditions that imply nonnegative linearization, i.e. $c(n,m,k) \ge 0$ for all n, m and k. His theorem was strong enough to include most of the classical orthogonal polynomials. However, for Jacobi polynomials his assumptions were satisfied only when $\alpha \ge \beta$ and $\alpha+\beta \ge 1$, despite the fact that by Gasper's result [6] the conditions $\alpha \ge \beta$ and $\alpha+\beta \ge -1$ were sufficient.

In 1992 in the two papers [12, 13] more general theorems were found. They imply nonnegative linearization for Jacobi polynomials with $\alpha \ge \beta$, $\alpha+\beta \ge -1$ as well as their associated polynomials. Assumed are certain monotonicity properties of the coefficients in the three-term recurrence relation. These assumptions are not always satisfied when we deal with basic orthogonal polynomials. In particular, the coefficients in the three-term recurrence relation for the continuous q-ultraspherical polynomials are oscillating about $1/2$ when q is negative. However, it is known from explicit formulas (see [3, (4.8)]) that the linearization coefficients are nonnegative in this case.

In this paper we will indicate new conditions which are sufficient for

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nonnegative linearization. These conditions admit coefficients oscillating about a certain value. In the proof we will always consider the polynomials normalized at the right endpoint of the support of the measure. We will show that this choice of normalization is in fact optimal. Examples are provided at the end of the paper.

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2. Normalization at the endpoint of the support.

Let p_n be polynomials orthogonal with respect to a measure μ symmetric about the origin. Then they satisfy the three-term formula

$$xp_n = \gamma_n p_{n+1} + \alpha_n p_{n-1} \tag{2}$$

for $n = 0, 1, \ldots$, where γ_n and α_n are positive coefficients, except for $\alpha_0 = 0$. When the polynomials p_n are orthonormal then the sequences γ_n and α_n are related by $\alpha_{n+1} = \gamma_n$.

For the sake of this paper we say that the polynomials p_n have *property* (A) if

$$\text{(A)}\qquad \begin{cases} \alpha_n \text{ is nondecreasing,} \\ \alpha_n + \gamma_n \text{ is nondecreasing,} \\ \forall n\ \alpha_n \le \gamma_n. \end{cases}$$

By [12, Theorem 1] property (A) implies that the p_n admit nonnegative linearization, i.e. the coefficients in (1) satisfy $c(n, m, k) \ge 0$. This property is independent of the choice of normalization of the polynomials p_n. In contrast, property (A) can fail if we change the normalization. For example, let us consider the symmetric Jacobi polynomials for $-1/2 \le \alpha < 1/2$. The orthonormalized polynomials do not satisfy (A); however, if we normalize the polynomials at $x = 1$, they do (see [12, Example]). It seems that the normalization at the rightmost end of the support of the corresponding measure is better than other normalizations. We are going to show that this is not a coincidence.

LEMMA 1. *Let polynomials p_n satisfy* (2) *and $p_n(x_0) > 0$ for $n \ge 0$. If the p_n have property* (A) *and either $\gamma_0 \ge x_0$ or p_n are orthonormal then the sequence $p_{n+1}(x_0)/p_n(x_0)$ is nonincreasing.*

Proof. Without loss of generality we may set $x_0 = 1$. For a contradiction assume that the sequence $c_n = p_{n+1}(0)/p_n(0)$ does not satisfy the conclusion. Let n be the first index where $c_{n+1} > c_n$. Then since $c_n \le c_{n-1}$

(if $m = 0$ set $c_{-1} = c_0$), we get by (2),

$$1 = \gamma_n c_n + \frac{\alpha_n}{c_{n-1}} \le \gamma_n c_n + \frac{\alpha_n}{c_n},$$
$$1 = \gamma_{n+1} c_{n+1} + \frac{\alpha_{n+1}}{c_n} > \gamma_{n+1} c_n + \frac{\alpha_{n+1}}{c_n}$$

Thus

$$0 < \left(\gamma_n c_n + \frac{\alpha_n}{c_n}\right) - \left(\gamma_{n+1} c_n + \frac{\alpha_{n+1}}{c_n}\right) = (\gamma_n - \gamma_{n+1}) c_n - (\alpha_{n+1} - \alpha_n)\frac{1}{c_n}. \tag{3}$$

If $\gamma_0 \ge 0$ then $c_0 = p_1(1)/p_0(1) = 1/\gamma_0 \le 1$ and so $c_n \le 1$. Thus (3) implies

$$0 < (\gamma_n - \gamma_{n+1}) c_n - (\alpha_{n+1} - \alpha_n) c_n = [\alpha_n + \gamma_n - (\alpha_{n+1} + \gamma_{n+1})] c_n \ \le\ 0.$$

On the other hand, when p_n are orthonormal and satisfy (A) then $\gamma_n = \alpha_{n+1}$ and both sequences are nondecreasing. Hence we get a contradiction in (3). ∎

THEOREM 1. *Assume that p_n are orthogonal polynomials with respect to a symmetric probability measure μ whose support is contained in the interval $[-x_0, x_0]$ and $\pm x_0 \in \operatorname{supp}\mu$. If the p_n have property* (A) *and either $\gamma_0 \ge x_0$ or p_n are orthonormal, then the polynomials $R_n = p_n/p_n(x_0)$ also have property* (A).

Proof. We may assume that $x_0 = 1$. Observe that by (2) we have

$$\begin{aligned} &xR_n = \widetilde{\gamma}_n R_{n+1} + \widetilde{\alpha}_n R_{n-1} \\ &\widetilde{\alpha}_n = \alpha_n \frac{p_{n-1}(1)}{p_n(1)}, \qquad \widetilde{\gamma}_n = \gamma_n \frac{p_{n+1}(1)}{p_n(1)}. \end{aligned} \tag{4}$$

Since $\operatorname{supp}\mu \subset [-1,1]$, the polynomials p_n have constant sign in $[1, +\infty)$ (see [11, Theorem 3.3.1]). As they have positive leading coefficients we get $p_n(1) > 0$. Thus Lemma 1 implies $\tilde{\alpha}_n$ is nondecreasing. Evaluating (4) at $x = 1$ gives $\widetilde{\alpha}_n + \widetilde{\gamma}_n = 1$. It remains to show that $\widetilde{\alpha}_n \le \widetilde{\gamma}_n$.

As $\widetilde{\alpha}_n$ is nondecreasing it has a limit $\widetilde{\alpha}$. Thus $\widetilde{\gamma}_n$ is nonincreasing and converges to $\widetilde{\gamma} = 1 - \widetilde{\alpha}$. In that case by Blumenthal's theorem (see [4, p. 121]) the support of μ consists of the interval $I = [-2\sqrt{\widetilde{\alpha}\widetilde{\gamma}}, 2\sqrt{\widetilde{\alpha}\widetilde{\gamma}}]$ and a denumerable set of points that can accumulate at the endpoints of this interval. We are going to show that $\widetilde{\alpha} = \widetilde{\gamma} = 1/2$. Assume the opposite. Then the interval I is strictly contained in $[-1, 1]$. As 1 belongs to $\operatorname{supp}\mu$ it has to be a mass point of μ. On the other hand, since the polynomials p_n satisfy (A) they have nonnegative linearization. Therefore by [10, Theorem 6(iii)] the measure μ cannot have an atom at the rightmost end point of $\operatorname{supp}\mu$. This gives a contradiction. In this way we have proved that $\widetilde{\alpha} =$

$\widetilde{\gamma} = 1/2$. This implies

$$\widetilde{\alpha_n} \le \widetilde{\alpha} = \widetilde{\gamma} \le \widetilde{\gamma}_n. \ \blacksquare$$

Remark. We do not know if the assumption $\gamma_0 \ge x_0$ is essential for the theorem to hold. However, we think it is not unnatural. For example if $\operatorname{supp}\mu \subset [-x_0, x_0]$ and the polynomials p_n are normalized at a point $x_1 \ge x_0$, then $\gamma_0 = x_1 \ge x_0$. Also if $\operatorname{supp}\mu \subset [-1,1]$ and p_n are monic, i.e. normalized so that the leading coefficient of p_n is 1, then $\gamma_0 = 1$.

On the other hand, when the polynomials p_n are orthonormal and $\operatorname{supp}\mu \subset [-x_0, x_0]$ then

$$\gamma_0^2 = \int_{-x_0}^{x_0} x^2\, d\mu(x) \le x_0^2.$$

That is why we treated this case separately.

3. Chain sequences. From now on we will consider polynomials orthogonal with respect to a symmetric probability measure whose support is contained in the interval $[-1,1]$. The polynomials are usually given by a three-term recurrence formula and the measure μ is not shown explicitly. It can be very cumbersome to determine when $\operatorname{supp}\mu \subset [-1,1]$. We will discuss conditions on the three-term recurrence formula that imply $\operatorname{supp}\mu \subset [-1,1]$.

Let p_n be polynomials orthonormal with respect to μ. Then they satisfy a three-term recurrence relation of the form

$$xp_n = \lambda_{n+1}p_{n+1} + \lambda_n p_{n-1},$$

where λ_n are positive coefficients for $n \ge 0$ and $\lambda_0 = 0$. The condition $\operatorname{supp}\mu \subset [-1,1]$ is equivalent to the fact that the polynomials p_n take positive values at $x = 1$. Therefore we can normalize them at $x = 1$ and get the polynomials $R_n = p_n/p_n(1)$, which satisfy

$$\begin{gathered} xR_n = \gamma_n R_{n+1} + \alpha_n R_{n-1}, \\ \alpha_n = \lambda_n \frac{p_{n-1}(1)}{p_n(1)}, \qquad \gamma_n = \lambda_{n+1}\frac{p_{n+1}(1)}{p_n(1)}. \end{gathered} \tag{5}$$

We also have $\alpha_n + \gamma_n = 1$ and

$$\alpha_0 = 0, \quad \alpha_n(1-\alpha_{n-1}) = \lambda_n^2, \quad n \ge 1. \tag{6}$$

By [4, Defs. 5.1,5.2, pp. 92–93] this means that λ_n^2 is a chain sequence with α_n as its minimal parameter sequence. By the reasoning above or by [4, Theorem 2.1, p. 108] the converse is also true, i.e. if λ_n^2 is a chain sequence then $\operatorname{supp}\mu \subset [-1,1]$ and $p_n(1) > 0$ for $n \ge 0$. There are few criteria for being a chain sequence that proved useful. One of them states (see [7] or [4, Exercise 6.5, p. 106]) that if $\lambda_n + \lambda_{n+1} \le 1$ for $n \ge 1$, then λ_n^2 is a chain sequence. For our purposes we will need another result.

THEOREM 2. *Let λ_n satisfy at least one of the following two conditions.*

(a) $\lambda_{2n-1}^2 + \lambda_{2n}^2 \le 1/2$ *for* $n \ge 1$.
(b) $\lambda_{2n}^2 + \lambda_{2n+1}^2 \le 1/2$ *for* $n \ge 0$.

Then λ_n^2 is a chain sequence.

Proof. We will show part (a) only. Let $\alpha_0 = 0$ and $\alpha_n = \lambda_n^2(1-\alpha_{n-1})^{-1}$ for $n \ge 1$. We have to show that $0 < \alpha_n < 1$. Instead we will prove by induction that $0 < \alpha_{2n} < 1/2$ and $0 < \alpha_{2n+1} < 1$. Assume the latter holds. Then

$$\alpha_{2n+2} = \lambda_{2n+2}^2(1-\alpha_{2n+1})^{-1} = \lambda_{2n+2}^2\left(1 - \frac{\lambda_{2n+1}^2}{1-\alpha_{2n}}\right)^{-1}$$
$$< \lambda_{2n+2}^2(1-2\lambda_{2n+1}^2)^{-1} \le \frac{1}{2}.$$

Consequently,

$$0 < \alpha_{2n+3} = \lambda_{2n+3}^2(1-\alpha_{2n+2})^{-1} < 2\lambda_{2n+3}^2 \le 1. \ \blacksquare$$

4. Criteria for nonnegative linearization. We say that the polynomials p_n satisfying (2) have *property* (B) if

$$\text{(B)}\qquad \begin{cases} \alpha_{2n} \text{ and } \alpha_{2n+1} \text{ are nondecreasing,} \\ \alpha_{2n}+\gamma_{2n} \text{ and } \alpha_{2n+1}+\gamma_{2n+1} \text{ are nondecreasing,} \\ \forall n\ \alpha_n \le \gamma_n. \end{cases}$$

Property (B) is weaker than (A). Nonetheless, by [13, Theorem 1] it implies that the p_n have nonnegative linearization. We are going to derive new criteria for nonnegative linearization basing on (B).

Let p_n be polynomials orthonormal with respect to a symmetric measure μ. Then they satisfy a three-term recurrence relation of the form

$$(7)\qquad xp_n = \lambda_{n+1}p_{n+1} + \lambda_n p_{n-1}, \qquad n \ge 0,$$

with positive coefficients λ_n, $n \ge 0$. For $n \ge 0$ let

$$\Delta_n = (1+\lambda_n+\lambda_{n+1})(1+\lambda_n-\lambda_{n+1})(1-\lambda_n+\lambda_{n+1})(1-\lambda_n-\lambda_{n+1}),$$
$$r_n = \frac{1}{2}(1-\lambda_n^2+\lambda_{n+1}^2-\sqrt{\Delta_n}).$$

THEOREM 3. *Let p_n be polynomials orthonormal with respect to a measure μ and satisfy*

$$xp_n = \lambda_{n+1}p_{n+1} + \lambda_n p_{n-1}.$$

Let λ_n converge to $1/2$. Then the p_n admit nonnegative linearization if one of the following four conditions is satisfied.

(i) $\lambda_n + \lambda_{n+1} \ge 1$ *for* $n \ge 1$, *and* $\operatorname{supp}\mu \subset [-1,1]$.
(ii) $\lambda_n + \lambda_{n+1} \le 1$ *for* $n \ge 1$, *and* $r_n \le r_{n+2}$ *for* $n \ge 0$.

(iii) $\lambda_{2n}+\lambda_{2n+1}\le 1,\ \lambda_{2n-1}+\lambda_{2n}\ge 1$ *for* $n\ge 1$, *and* $r_{2n}\le r_{2n+2}$ *for* $n\ge 0$.

(iv) $\lambda_{2n}+\lambda_{2n+1}\ge 1,\ \lambda_{2n-1}+\lambda_{2n}\le 1$ *for* $n\ge 1$, *and* $r_{2n-1}\le r_{2n+1}$ *for* $n\ge 1$.

Proof. First we will show that λ_n^2 is a chain sequence whose minimal parameter sequence α_n satisfies

(8) $$\alpha_{n+2}\ge\alpha_n.$$

Let α_n be defined by

(9) $$\alpha_0=0,\qquad \alpha_n(1-\alpha_{n-1})=\lambda_n^2.$$

It will also be convenient to define $\alpha_{-1}=\lambda_0=0$. Then (9) is also satisfied for $n=0$. We get

(10) $$\alpha_{n+2}=f_n(\alpha_n),\qquad f_n(x)=\frac{\lambda_{n+1}^2(1-x)}{1-\lambda_n^2-x}.$$

It can be easily computed that if $\lambda_n+\lambda_{n+1}\le 1$, then the equation $f_n(x)=x$ has real roots, the smaller one being r_n. Moreover, if $0<x<\min\{r_n,1-\lambda_n^2\}$ then $f_n(x)>x$.

In case $\lambda_n+\lambda_{n+1}\ge 1$, the equation $f_n(x)=x$ has at most one real root. Also if $x<1-\lambda_n^2$ then $f_n(x)>x$.

Consider (i). Since $\operatorname{supp}\mu\subset[-1,1]$, the sequence λ_n^2 is a chain sequence (see Sec. 3). Since α_n is the minimal parameter sequence we have $0\le\alpha_n<1$ for $n\ge 0$. Therefore $\alpha_{n+2}=f_n(\alpha_n)>\alpha_n$ for $n\ge 0$.

Let us turn to (iii). First observe that since $\lambda_{2n}^2+\lambda_{2n+1}^2\le 1$ we have

$$r_{2n}\le\frac{1+\lambda_{2n+1}^2-\lambda_{2n}^2}{2}\le 1-\lambda_{2n}^2.$$

Now we will prove by induction that $a_{2n}\le r_{2n}$. We have $\alpha_0=0\le r_0$. Assume that $\alpha_{2n}\le r_{2n}$. Then since f_{2n} is increasing in the interval $[0,r_{2n}]$ we obtain

$$\alpha_{2n+2}=f_{2n}(\alpha_{2n})\le f_{2n}(r_{2n})=r_{2n}\le r_{2n+2}.$$

By the first part of the proof we get $\alpha_{2n}\le\alpha_{2n+2}$. We also get

$$0<\alpha_{2n}\le r_{2n}\le 1-\lambda_{2n}^2<1.$$

Hence

$$\alpha_{2n+1}=\frac{\lambda_{2n+1}^2}{1-\alpha_{2n}}>0.$$

Thus $\alpha_n\ge 0$ for $n\ge 0$, and consequently λ_{n-1}^2 is a chain sequence. As $\lambda_{2n-1}+\lambda_{2n}\ge 1$, we can show exactly as in the proof of (i) that $\alpha_{2n-1}\le\alpha_{2n+1}$. Finally, we get $\alpha_n\le\alpha_{n+2}$.

Much in the same way we can show that (ii) and (iv) each imply $\alpha_n\le\alpha_{n+2}$.

Let

$$\lim_{n\to\infty} \alpha_{2n} = \alpha, \qquad \lim_{n\to\infty} \alpha_{2n} = \alpha'.$$

Then by (9),

$$\alpha(1-\alpha') = \alpha'(1-\alpha) = \tfrac{1}{4}.$$

Thus $\alpha = \alpha' = 1/2$. We also see that $\alpha_n \le 1/2$. By the first part of Sec. 3 we get

$$\alpha_n = \lambda_n \frac{p_{n-1}(1)}{p_n(1)}.$$

Thus $p_n(1) > 0$ for $n \ge 0$. Consider the renormalized polynomials $R_n(x) = p_n(x)/p_n(1)$. By (5) they satisfy

$$xR_n = \gamma_n R_{n+1} + \alpha_n R_{n-1},$$

where $\gamma_n = 1-\alpha_n$. We have $\gamma_n \ge 1/2 \ge \alpha_n$. Thus the polynomials R_n have property (B), which implies nonnegative linearization. ■

In the following examples the linearization coefficients are known explicitly from the papers by Rogers [9] and by Dougall [5](see [3, (4.8)] and [1, (5.7)]). We will show the positivity of the linearization basing on Theorem 3.

Example 1. Consider the polynomials C_n satisfying

$$2xC_n(x) = (1-q^{n+1})C_{n+1}(x) + C_{n-1}(x), \qquad n \ge 0, \tag{11}$$

where $C_{-1}(x) = 0$ and $C_1(x) = 1$. According to [3, p. 20] these are continuous q-ultraspherical polynomials with $\beta = 0$. Their orthonormal versions $\widehat{C}_n$ satisfy

$$x\widehat{C}_n(x) = \lambda_{n+1}\widehat{C}_{n+1}(x) + \lambda_n\widehat{C}_{n-1}(x),$$

where

$$\lambda_n^2 = \tfrac{1}{4}(1-q^n).$$

For $-1 < q < 0$ it can be computed that assumption (iii) of Theorem 3 is satisfied while for $0 \le q < 1$ we can apply part (iv) or Askey's criterion [1].

Example 2. The symmetric Jacobi polynomials $\widehat{p}_n^{(\alpha,\alpha)}$ are orthonormal with respect to the measure $d\mu^{(\alpha,\alpha)}(x) = c_\alpha(1-x^2)_+^\alpha dx$ and satisfy

$$x\widehat{p}_n^{(\alpha,\alpha)}(x) = \lambda_{n+1}\widehat{p}_{n+1}^{(\alpha,\alpha)}(x) + \lambda_n\widehat{p}_{n-1}^{(\alpha,\alpha)}(x),$$

where

$$\lambda_n^2 = \frac{n(n+2\alpha)}{(2n+2\alpha-1)(2n+2\alpha+1).}$$

It can be easily verified that $\lambda_n + \lambda_{n+1} \ge 1$ for $-1/2 \le \alpha \le 1/2$. Thus they admit nonnegative linearization.

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