arXiv:1405.7267v1 [math.CA] 28 May 2014

A determinant characterization of moment sequences with finitely many mass-points

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May 29, 2014

Abstract

To a sequence $(s_n)_{n\geq 0}$ of real numbers we associate the sequence of Hankel matrices $\mathcal{H}_n = (s_{i+j}), 0 \leq i, j \leq n$. We prove that if the corresponding sequence of Hankel determinants $D_n = \det \mathcal{H}_n$ satisfy $D_n > 0$ for $n < n_0$ while $D_n = 0$ for $n \geq n_0$, then all Hankel matrices are positive semi-definite, and in particular (s_n) is the sequence of moments of a discrete measure concentrated in n_0 points on the real line. We stress that the conditions $D_n \geq 0$ for all n do not imply the positive semi-definiteness of the Hankel matrices.

2000 *Mathematics Subject Classification*:
Primary 44A60; Secondary 15A15
Keywords: moment problems, Hankel determinants, sign pattern of leading principal minors.

1 Introduction and results

Given a sequence of real numbers $(s_n)_{n\geq 0}$, it was proved by Hamburger [3] that it can be represented as

$$s_n = \int_{-\infty}^{\infty} x^n \, d\mu(x), \quad n \geq 0 \tag{1}$$

with a positive measure μ on the real line, if and only if all the Hankel matrices $\mathcal{H}_n = (s_{i+j}), 0 \leq i, j \leq n$ are positive semi-definite. The sequences (1) are called *Hamburger moment sequences* or *positive definite sequences* on $\mathbb{N}_0 = \{0, 1, \ldots\}$ considered as an additive semigroup under addition, cf. [2].

*The first author has been supported by grant 10-083122 from *The Danish Council for Independent Research | Natural Sciences*

Given a Hamburger moment sequence it is clear that all the Hankel determinants $D_n = |\mathcal{H}_n|$ are non-negative. It is also easy to see (cf. Lemma 2.1 and its proof) that only two possibilities can occur: Either $D_n > 0$ for $n = 0, 1, \ldots$ and in this case any μ satisfying (1) has infinite support, or there exists n_0 such that $D_n > 0$ for $n \le n_0 - 1$ and $D_n = 0$ for $n \ge n_0$. In this latter case μ from (1) is uniquely determined and is a discrete measure concentrated in n_0 points on the real axis. (If $n_0 = 0$ and $D_n = 0$ for all n, then $\mu = 0$ is concentrated in the empty set.)

The purpose of the present paper is to prove the following converse result:

Theorem 1.1. *Let (s_n) be a real sequence and assume that the sequence of Hankel determinants $D_n = |\mathcal{H}_n|$ satisfy $D_n > 0, n \le n_0 - 1$, $D_n = 0, n \ge n_0$. Then (s_n) is a Hamburger moment sequence (and then necessarily the moments of a uniquely determined measure μ concentrated in n_0 points).*

Remark 1.2. It follows from a general theorem about real symmetric matrices, that if $D_n > 0$ for $n \le n_0$, then the Hankel matrix $\mathcal{H}_{n_0}$ is positive definite. For a proof see e.g. [2, p.70]. On the other hand, one cannot conclude that $\mathcal{H}_{n_0}$ is positive semi-definite, if it is just known that $D_n \ge 0$ for $n \le n_0$. For the sequence $1, 1, 1, 1, 0, 0, \ldots$ we have $D_0 = D_3 = 1, D_1 = D_2 = D_n = 0$ for $n \ge 4$, but the Hankel matrix $\mathcal{H}_2$ has a negative eigenvalue. It therefore seems to be of interest that Theorem 1.1 holds.[1]

Remark 1.3. It follows from the proof of Theorem 1.1 that the uniquely determined measure μ is concentrated in the zeros of the polynomial p_{n_0} given by (6).

Example 1.4. Let $a \ge 1$ and define $s_{2n} = s_{2n+1} = a^n, n = 0, 1, \ldots$. Then the Hankel determinants are $D_0 = 1, D_1 = a - 1, D_n = 0$, $n \ge 2$. Therefore (s_n) is a moment sequence of the measure

$$\mu = \frac{\sqrt{a} - 1}{2\sqrt{a}} \delta_{-\sqrt{a}} + \frac{\sqrt{a} + 1}{2\sqrt{a}} \delta_{\sqrt{a}}.$$

Similarly, for $0 \le a \le 1$, $s_0 = 1, s_{2n-1} = s_{2n} = a^n, n \ge 1$ is a moment sequence of the measure

$$\mu = \frac{1 - \sqrt{a}}{2} \delta_{-\sqrt{a}} + \frac{1 + \sqrt{a}}{2} \delta_{\sqrt{a}}.$$

2 Proofs

Consider a discrete measure

$$\mu = \sum_{j=1}^{n} m_j \delta_{x_j}, \tag{2}$$

[1]The authors thank Alan Sokal for having mentioned the question.

where $m_j > 0$ and $x_1 < x_2 < \ldots < x_n$ are n points on the real axis. Denote the moments

$$s_k = \int x^k \, d\mu(x) = \sum_{j=1}^{n} m_j x_j^k, \quad k = 0, 1, \ldots, \tag{3}$$

and let $\mathcal{H}_k, D_k$ denote the corresponding Hankel matrices and determinants. The following Lemma is well-known, but for the benefit of the reader we give a short proof.

Lemma 2.1. *The Hankel determinants D_k of the moment sequence* (3) *satisfy $D_k > 0$ for $k < n$ and $D_k = 0$ for $k \geq n$.*

Proof. Let

$$P(x) = \sum_{j=0}^{n} a_j x^j$$

be the monic polynomial (i.e., $a_n = 1$) of degree n with zeros $x_1, \ldots, x_n$. If $\mathbf{a} = (a_0, \ldots, a_n)$ then

$$\int P^2(x) \, d\mu(x) = \mathbf{a} \mathcal{H}_n \mathbf{a}^t = 0,$$

and it follows that $D_n = 0$. If $p \geq 1$ and $\mathbf{0}_p$ is the zero vector in $\mathbb{R}^p$, then also

$$(\mathbf{a}, \mathbf{0}_p) \mathcal{H}_{n+p} (\mathbf{a}, \mathbf{0}_p)^t = 0,$$

and it follows that $D_{n+p} = 0$ for all $p \geq 1$.

On the other hand, if a Hamburger moment sequence (1) has $D_k = 0$ for some k, then there exists $\mathbf{b} = (b_0, \ldots, b_k) \in \mathbb{R}^{k+1} \setminus \{\mathbf{0}\}$ such that $\mathbf{b} \mathcal{H}_k = \mathbf{0}$. Defining

$$Q(x) = \sum_{j=0}^{k} b_j x^j,$$

we find

$$0 = \mathbf{b} \mathcal{H}_k \mathbf{b}^t = \int Q^2(x) \, d\mu(x),$$

showing that μ is concentrated in the zeros of Q. Therefore μ is a discrete measure having at most k mass-points. This remark shows that the Hankel determinants of (3) satisfy $D_k > 0$ for $k < n$. □

Lemma 2.2. *Consider $n + 1$ non-negative integers $0 \leq c_1 < c_2 < \ldots < c_{n+1}$, let $p \geq 1$ be an integer and define the $(n+1) \times (n+p)$-matrix*

$$H_{n+1,n+p} = \begin{pmatrix} s_{c_1} & s_{c_1+1} & \cdots & s_{c_1+n+p-1} \\ s_{c_2} & s_{c_2+1} & \cdots & s_{c_2+n+p-1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{c_{n+1}} & s_{c_{n+1}+1} & \cdots & s_{c_{n+1}+n+p-1} \end{pmatrix}.$$

For any $(p-1)\times(n+p)$-matrix $A_{p-1,n+p}$ we have

$$D=\begin{vmatrix}H_{n+1,n+p}\\ A_{p-1,n+p}\end{vmatrix}=0.$$

Proof. By multilinearity of a determinant as function of the rows we have

$$D=\sum_{j_1,\ldots,j_{n+1}=1}^{n} m_{j_1}\cdots m_{j_{n+1}} x_{j_1}^{c_1}\cdots x_{j_{n+1}}^{c_{n+1}}\begin{vmatrix}J\\ A_{p-1,n+p}\end{vmatrix},$$

where J is the $(n+1)\times(n+p)$-matrix with rows

$$\left(1,x_{j_l},x_{j_l}^2,\ldots,x_{j_l}^{n+p-1}\right),\ l=1,2,\ldots,n+1,$$

and since there are n points $x_1,\ldots,x_n$, two of these rows will always be equal. This shows that each determinant in the sum vanishes and therefore $D=0$. □

With n,p as above we now consider a determinant of a matrix $(a_{i,j}), 0\le i,j\le n+p$ of size $n+p+1$ of the following special form

$$M_{n+p}=\begin{vmatrix}
s_0 & \cdots & s_{n-1} & s_n & \cdots & s_{n+p-1} & s_{n+p}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
s_{n-1} & \cdots & s_{2n-2} & s_{2n-1} & \cdots & s_{2n+p-2} & s_{2n+p-1}\\
s_n & \cdots & s_{2n-1} & s_{2n} & \cdots & s_{2n+p-1} & x_0\\
s_{n+1} & \cdots & s_{2n} & s_{2n+1} & \cdots & x_1 & a_{n+1,n+p}\\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots\\
s_{n+p} & \cdots & s_{2n+p-1} & x_p & \cdots & a_{n+p,n+p-1} & a_{n+p,n+p}
\end{vmatrix}$$

which has Hankel structure to begin with, i.e., $a_{i,j}=s_{i+j}$ for $i+j\le 2n+p-1$. For simplicity we have called $a_{n+j,n+p-j}=x_j, j=0,1,\ldots,p$.

Lemma 2.3.

$$M_{n+p}=(-1)^{p(p+1)/2}D_{n-1}\prod_{j=0}^{p}(x_j-s_{2n+p}).$$

In particular, the determinant is independent of $a_{i,j}$ with $i+j\ge 2n+p+1$.

Proof. We first observe that the determinant vanishes if we put $x_0=s_{2n+p}$, because then the first $n+1$ rows in M_{n+p} have the structure of the matrix of Lemma 2.2 with $c_j=j-1, j=1,\ldots,n+1$.

Next we develop the determinant after the last column leading to

$$M_{n+p}=\sum_{l=0}^{n+p}\pm\gamma_l A_l,$$

where γ_l are the elements in the last column and A_l are the corresponding minors, i.e., the determinants obtained by deleting row number $l+1$ and the last column. Notice that $A_l = 0$ for $l = n+1, \ldots, n+p$ because of Lemma 2.2. Therefore the numbers $a_{n+k,n+p}$ with $k = 1, \ldots, p$ do not contribute to the determinant.

For $l = 0, \ldots, n$ the determinant A_l has the form

$$\begin{vmatrix} s_{c_1} & \cdots & s_{c_1+n} & \cdots & s_{c_1+n+p-1} \\ s_{c_2} & \cdots & s_{c_2+n} & \cdots & s_{c_2+n+p-1} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{c_n} & \cdots & s_{c_n+n} & \cdots & s_{c_n+n+p-1} \\ s_{n+1} & \cdots & s_{2n+1} & \cdots & x_1 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ s_{n+p} & \cdots & x_p & \cdots & a_{n+p,n+p-1} \end{vmatrix}$$

for integers c_j satisfying $0 \le c_1 < \ldots < c_n \le n$.

Each of these determinants vanish for $x_1 = s_{2n+p}$ again by Lemma 2.2, so consequently M_{n+p} also vanishes for $x_1 = s_{2n+p}$. As above we see that the determinant does not depend on $a_{n+k,n+p-1}$ for $k = 2, \ldots, p$.

The argument can now be repeated and we see that M_{n+p} vanishes for $x_k = s_{2n+p}$ when $k = 0, \ldots, p$.

This implies that

$$M_{n+p} = K \prod_{j=0}^{p} (x_j - s_{2n+p}),$$

where K is the coefficient to $x_0 x_1 \ldots x_p$, when the determinant is written as

$$M_{n+p} = \sum_{\sigma} \operatorname{sign}(\sigma) \prod_{j=0}^{n+p} a_{j,\sigma(j)},$$

and the sum is over all permutations σ of $0, 1, \ldots, n+p$.

The terms containing the product $x_0 x_1 \ldots x_p$ requires the permutations σ involved to satisfy $\sigma(n+l) = n+p-l, l = 0, \ldots, p$. This yields a permutation of $n, n+1, \ldots, n+p$ onto itself reversing the order hence of sign $(-1)^{p(p+1)/2}$, while σ yields an arbitrary permutation of $0, 1, \ldots, n-1$. This shows that $K = (-1)^{p(p+1)/2} D_{n-1}$.

$\square$

Proof of Theorem 1.1.

The proof of Theorem 1.1 is obvious if $n_0 = 0$, and if $n_0 = 1$ the proof is more elementary than in the general case, so we think it is worth giving it separately. Without loss of generality we assume $s_0 = D_0 = 1$, and call $s_1 = a$. From $D_1 = 0$ we then get that $s_2 = a^2$, and we have to prove that $s_n = a^n$ for $n \ge 3$.

Suppose now that it has been established that $s_k = a^k$ for $k \le n$, where $n \ge 2$. By assumption we have

$$0 = D_n = \begin{vmatrix} 1 & a & \cdots & a^{n-1} & a^n \\ a & a^2 & \cdots & a^n & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{n-1} & a^n & \cdots & s_{2n-2} & s_{2n-1} \\ a^n & s_{n+1} & \cdots & s_{2n-1} & s_{2n} \end{vmatrix}. \tag{4}$$

Developing the determinant after the last column, we notice that only the first two terms will appear because the minors for the elements $s_{n+j}, j = 2, \ldots, n$ have two proportional rows $(1, a, \ldots, a^{n-1})$ and $(a, a^2, \ldots, a^n)$. Therefore

$$D_n = (-1)^{n+2} a^n \begin{vmatrix} a & a^2 & \cdots & a^n \\ a^2 & a^3 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a^n & s_{n+1} & \cdots & s_{2n-1} \end{vmatrix} + (-1)^{n+3} s_{n+1} \begin{vmatrix} 1 & a & \cdots & a^{n-1} \\ a^2 & a^3 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a^n & s_{n+1} & \cdots & s_{2n-1} \end{vmatrix},$$

hence

$$D_n = (-1)^n (a^{n+1} - s_{n+1}) \begin{vmatrix} 1 & a & \cdots & a^{n-1} \\ a^2 & a^3 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a^n & s_{n+1} & \cdots & s_{2n-1} \end{vmatrix}.$$

The last $n \times n$-determinant is developed after the last column and the same procedure as before leads to

$$D_n = (-1)^{n+(n-1)} \left(a^{n+1} - s_{n+1}\right)^2 \begin{vmatrix} 1 & a & \cdots & a^{n-2} \\ a^3 & a^4 & \cdots & s_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a^n & s_{n+1} & \cdots & s_{2n-2} \end{vmatrix}.$$

Going on like this we finally get

$$D_n = (-1)^{n+(n-1)+\cdots+2} \left(a^{n+1} - s_{n+1}\right)^{n-1} \begin{vmatrix} 1 & a \\ a^n & s_{n+1} \end{vmatrix} = (-1)^{n(n+1)/2} \left(a^{n+1} - s_{n+1}\right)^n,$$

and since $D_n = 0$ we obtain that $s_{n+1} = a^{n+1}$.

We now go to the general case, where $n_0 \ge 2$ is arbitrary.

We have already remarked that the Hankel matrix $\mathcal{H}_{n_0-1}$ is positive definite, and we claim that $\mathcal{H}_{n_0}$ is positive semi-definite. In fact, if for $\varepsilon > 0$ we define

$$s_k(\varepsilon) = s_k,\ k \ne 2n_0, \quad s_{2n_0}(\varepsilon) = s_{2n_0} + \varepsilon, \tag{5}$$

and denote the corresponding Hankel matrices and determinants $\mathcal{H}_k(\varepsilon), D_k(\varepsilon)$, then

$$\mathcal{H}_k(\varepsilon) = \mathcal{H}_k,\ 0 \le k \le n_0 - 1, D_{n_0}(\varepsilon) = D_{n_0} + \varepsilon D_{n_0-1} = \varepsilon D_{n_0-1} > 0.$$

This shows that $\mathcal{H}_{n_0}(\varepsilon)$ is positive definite and letting ε tend to 0 we obtain that $\mathcal{H}_{n_0}$ is positive semi-definite.

The positive semi-definiteness of the Hankel matrix $\mathcal{H}_{n_0}$ makes it possible to define a semi-inner product on the vector space Π_{n_0} of polynomials of degree $\le n_0$ by defining $\langle x^j, x^k\rangle = s_{j+k}$, $0 \le j,k \le n_0$. The restriction of $\langle\cdot,\cdot\rangle$ to Π_{n_0-1} is an ordinary inner product and the formulas

$$p_0(x) = 1,\ p_n(x) = \begin{vmatrix} s_0 & s_1 & \cdots & s_n \\ \vdots & \vdots & \ddots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}, \quad 1 \le n \le n_0 \tag{6}$$

define orthogonal polynomials, cf. [1, Ch. 1]. While $p_n(x)/\sqrt{D_{n-1}D_n}$ are orthonormal polynomials for $n < n_0$, it is not possible to normalize p_{n_0} since $D_{n_0} = 0$. The theory of Gaussian quadratures remain valid for the polynomials $p_n, n \le n_0$, cf. [1, Ch.1], so p_{n_0} has n_0 simple real zeros and there is a discrete measure μ concentrated in these zeros such that

$$s_k = \int x^k\, d\mu(x), \quad 0 \le k \le 2n_0 - 1. \tag{7}$$

To finish the proof of Theorem 1.1 we introduce the moments

$$\tilde{s}_k = \int x^k\, d\mu(x), \quad k \ge 0 \tag{8}$$

of μ and shall prove that $s_k = \tilde{s}_k$ for all $k \ge 0$. We already know this for $k < 2n_0$, and we shall now prove that $s_{2n_0} = \tilde{s}_{2n_0}$. Since μ is concentrated in the zeros of p_{n_0} we get

$$\int p_{n_0}^2(x)\, d\mu(x) = 0. \tag{9}$$

If $(\tilde{D}_k)$ denotes the sequence of Hankel determinants of the moment sequence $(\tilde{s}_k)$, we get from Lemma 2.1 that $\tilde{D}_k = 0$ for $k \ge n_0$.

Developing the determinants D_{n_0} and $\tilde{D}_{n_0}$ after the last column and using that they are both equal to 0, we get

$$s_{2n_0} D_{n_0-1} = \tilde{s}_{2n_0} D_{n_0-1},$$

hence $s_{2n_0} = \tilde{s}_{2n_0}$.

Assume now that $s_k = \tilde{s}_k$ for $k \leq 2n_0 + p - 1$ for some $p \geq 1$, and let us prove that $s_{2n_0+p} = \tilde{s}_{2n_0+p}$.

The Hankel determinant D_{n_0+p} is then a special case of the determinant M_{n_0+p} of Lemma 2.3, and it follows that

$$D_{n_0+p} = (-1)^{p(p+1)/2} D_{n_0-1} \left(s_{2n_0+p} - \tilde{s}_{2n_0+p}\right)^{p+1}.$$

Since $D_{n_0+p} = 0$ by hypothesis, we conclude that $s_{2n_0+p} = \tilde{s}_{2n_0+p}$. $\square$

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