ORTHOGONAL POLYNOMIALS OF DISCRETE VARIABLE AND BOUNDEDNESS OF DIRICHLET KERNEL

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Abstract. For orthogonal polynomials defined by compact Jacobi matrix with exponential decay of the coefficients, precise properties of orthogonality measure is determined. This allows showing uniform boundedness of partial sums of orthogonal expansions is shown with respect to L^∞ norm, which generalize analogous results obtained for little q-Legendre, little q-Jacobi and little q-Laguerre polynomials, by J. Obermaier and by J. Obermaier and the author.

1. Introduction

Let $s_n(f)$ denote the nth partial sum of the classical Fourier series of a continuous 2π periodic function $f(\theta)$. We know that the quantities $\|s_n(f)\|_\infty$ need not to be uniformly bounded since the Lebesgue numbers $\int_0^{2\pi} |D_n(\theta)|d\theta$ behave like constant multiple of $\log n$, where D_n denotes the Dirichlet kernel.

The question arises: Do there exist a measure space and an orthogonal system such that the partial sums are uniformly bounded in $\|\cdot\|_\infty$ norm ?

The answer is trivially positive if the measure space is finite. But the problem becomes nontrivial if the measure space is infinite, for instance of the form $\{q^n\}_{n=0}^\infty$ for some number $0 < q < 1$. There are examples of systems of orthogonal polynomials whose orthogonality measure is concentrated on the sequence $\{q^n\}_{n=0}^\infty$. Little q-Legendre polynomials, more generally q-Jacobi polynomials and little q-Laguerre polynomials are such. The uniform boundedness of $\|s_n(f)\|_\infty$ have been shown for these systems in [4, 5, 6]. The proof depended heavily on the precise knowledge of the orthogonality measure and pointwise estimates of these polynomials.

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In this paper we will generalize these results by allowing general orthogonal polynomials satisfying a three term recurrence relation

$$xp_n = -\lambda_n p_{n+1} + \beta_n p_n - \lambda_{n-1} p_{n-1},$$

where $\lambda_n > 0,\ \beta_n \in \mathbb{R}$. Assuming boundedness of these coefficients the orthogonality measure μ on the real line is determined uniquely. However finding this measure explicitly is a hopeless task in general and can be achieved in very few special cases. Nonetheless we are able sometimes to derive certain properties of this measure. We will use the well known fact that if J is the Jacobi matrix associated with the coefficients $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$, i.e.

$$J = \begin{pmatrix} \beta_0 & \lambda_1 & 0 & 0 & \cdots \\ \lambda_1 & \beta_1 & \lambda_2 & 0 & \cdots \\ 0 & \lambda_2 & \beta_2 & \lambda_3 & \ddots \\ 0 & 0 & \lambda_3 & \beta_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{1}$$

then the spectrum of J on $\ell^2(\mathbb{N}_0)$ coincides with the support of μ.

In this paper we impose conditions on the sequences $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ so that determining the behavior of the orthogonality measure is possible. In particular we will assume that these coefficients have exponential decay at infinity. The properties of the orthogonality measure will be sufficient for proving the uniform boundedness of the norms $\|s_n\|_{L^\infty \to L^\infty}$.

Throughout the paper we will be using certain classical results concerning orthogonal polynomials. In most such cases references will be given. In particular we will use the following well known property, whose proof follows immediately from orthogonality. If $\mu((a,b)) = 0$, where μ is an orthogonality measure, then the polynomial p_n may have at most one root in the interval $[a,b]$. Moreover, if $\mu((c,+\infty) = \mu((-\infty,d)) = 0$ then p_n does not vanish in either interval.

By $a_n \approx b_n$ we will mean that the ratio a_n/b_n has a positive limit, while by $a_n \sim b_n$ we will mean that the ratio a_n/b_n is positive, bounded and bounded away from zero.

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2. Orthogonality measure

Let $R_n(x)$ denote polynomials satisfying a three term recurrence relation

$$xR_n(x) = -\gamma_n R_{n+1}(x) + \beta_n R_n(x) - \alpha_n R_{n-1}(x), \tag{2}$$

where $\alpha_0 = 0$ and $R_0(x) \equiv 1$. We assume that $\gamma_n, \alpha_{n+1} > 0$ and

$$\beta_n = \alpha_n + \gamma_n. \tag{3}$$

In this way the polynomials are normalized at 0 so that

$$R_n(0) = 1. \tag{4}$$

Since the coefficient of the leading term of R_n is alternating, and the roots of R_n are distinct and real (see [1, Theorem I.5.2]), all these roots are positive in view of (4). Therefore (see [1, Proof of Theorem 2.1.1, for $\tau = 0$]) there is an orthogonality measure μ supported on half line $[0, +\infty)$. Let $h(0) = 1$ and

$$h(n) = \frac{\gamma_0\gamma_1 \cdots \gamma_{n-1}}{\alpha_1\alpha_2 \ldots \alpha_n}.$$

It can be easily computed that the polynomials

$$p_n(x) = \sqrt{h(n)}R_n(x) \tag{5}$$

are orthonormal and satisfy the recurrence relation

$$xp_n(x) = -\lambda_n p_{n+1}(x) + \beta_n p_n(x) - \lambda_{n-1}p_{n-1}(x), \tag{6}$$

where

$$\lambda_n = \sqrt{\alpha_{n+1}\gamma_n}. \tag{7}$$

We will consider polynomials with special properties such that the orthogonality measure is concentrated on a sequence of points ξ_n such that $\xi_n \searrow 0$ when $n \to \infty$. There are many instances of such behavior, e.g. little q-Jacobi polynomials, little q-Laguerre polynomials. Also we require that the polynomials satisfy nonnegative product linearization property, i.e. the coefficients in the expansions

$$R_n(x)R_m(x) = \sum_{k=|n-m|}^{n+m} g(n, m, k)R_k(x) \tag{8}$$

are all nonnegative. The above mentioned polynomials fulfill this property for certain values of parameters.

We will deal with general orthogonal polynomials satisfying the two above properties. In order to ensure the proper behaviour of the orthogonality measure as well as nonnegative linearization property we assume that there are constants q, κ, s, c and N such that

(9) $\alpha_n \approx q^n,\ \gamma_n \approx q^n,$ $\qquad 0 < q < 1,$

(10) $\alpha_n \le \kappa\gamma_n,$ $\qquad 1 \le \kappa < q^{-1} + q - 1,$

(11) $h(n) \sim s^n$ $\qquad s > 1,$

(12) $\lambda_n \le \beta_{n+1} - c\beta_{n+2},$ $\qquad \dfrac{1+q}{1+q^2} < c < \dfrac{1}{q}.$

(13) $\beta_1 \le \beta_0.$

(14) $\beta_n - c\beta_{n+1} \ge \beta_{n+1} - c\beta_{n+2}, \quad n \ge N.$

Remark. Assumption (10) is technical. In many cases, like little q-Jacobi polynomials, this assumption is satisfied with $\kappa = 1$. Actually it is natural to expect $\alpha_n \le \gamma_n$ (see (17)).

By assumptions (12) and (13) we obtain that β_n is a decreasing sequence and

(15) $$\lambda_n \le \beta_{n+1} - \beta_{n+2}, \quad n \ge 0.$$

Hence the assumptions of [3, Theorem 1] are satisfied. The fact that β_n is decreasing instead of being increasing follows from normalizing our polynomials in such a way that the sign of the leading coefficient is alternating, instead of being positive like in [3]. Therefore the polynomials $\{R_n\}_{n=0}^{\infty}$ admit nonnegative product linearization. This property implies that (see [7, (17), p. 166])

$$|R_n(x)| \le 1 \qquad x \in \mathrm{supp}\mu$$

or equivalently

(16) $$|p_n(x)| \le p_n(0) \qquad x \in \mathrm{supp}\mu.$$

By orthonormality and by (16) we have $p_n^2(0) \ge 1$. In particular

(17) $$h(n) = p_n^2(0) = \frac{\gamma_0\gamma_1 \ldots \gamma_{n-1}}{\alpha_1\alpha_2 \ldots \alpha_n} \ge 1.$$

In the next theorem we are going to describe the orthogonality measure μ for the polynomials $\{R_n\}_{n=0}^{\infty}$.

Theorem 1. *Assume (2-14) are satisfied. Then the orthogonality measure μ is concentrated on decreasing sequence $\{\xi_n\}_{n=1}^{\infty}$, where $\xi_n \sim q^n$, the quantity $1 - \xi_{n+1}/\xi_n$ is bounded away from 0, and $\mu([0, \xi_n]) \sim s^{-n}$.*

Remark 1. The conclusion of the theorem cannot be strengthened to $\mu(\{\xi_n\} \sim s^n$. Indeed, consider the probability measure

$$\mu = \frac{3}{2}\sum_{n=0}^{\infty} \frac{1}{4^{n+1}}\delta_{2^{-2n}} + \frac{7}{2}\sum_{n=0}^{\infty} \frac{1}{8^{n+1}}\delta_{2^{-(2n+1)}}.$$

Then $\xi_n \sim 2^{-n}$ and $\mu([0,\xi_n]) \sim 2^{-n}$ but $\mu(\{\xi_n\}) \not\sim 2^{-n}$. Of course we cannot guarantee that the polynomials orthogonal with respect to this measure satisfy nonnegative product linearization.

Proof. Let J denote the Jacobi matrix associated with the polynomials p_n (see (1)). By assumptions J is a compact operator on $\ell^2(\mathbb{N}_0)$. Moreover J is semipositive definite because by (7) we have $J = S^*S$, where

$$S = \begin{pmatrix} \sqrt{\gamma_0} & \sqrt{\alpha_1} & 0 & 0 & \cdots \\ 0 & \sqrt{\gamma_1} & \sqrt{\alpha_2} & 0 & \cdots \\ 0 & 0 & \sqrt{\gamma_2} & \sqrt{\alpha_3} & \ddots \\ 0 & 0 & 0 & \sqrt{\gamma_3} & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Hence the spectrum of J consists of 0 and a decreasing sequence of points $\{\xi_n\}_{n=1}^\infty$ accumulating at zero. As we have mentioned in the introduction, the support of μ coincides with the spectrum of J. First we will show that $\mu(\{0\}) = 0$. Indeed, by [1, Theorem 2.5.3] we have

$$\mu(\{0\})^{-1} = \sum_{n=0}^\infty p_n^2(0).$$

We know that $p_n^2(0) \ge 1$ (see (17)). Hence $\mu(\{0\})^{-1} = \infty$.

Now we turn to determining the behavior of ξ_n. Let $\{x_{jn}\}_{j=1}^n$ denote the zeros of the polynomial $p_n(x)$ arranged in the increasing order. It is well known (see [2, Exercise I.4.12]) that this set coincides with the set of eigenvalues of the truncated Jacobi matrix J_n, where

$$J_n = \begin{pmatrix} \beta_0 & \lambda_0 & 0 & \cdots & 0 & 0 \\ \lambda_0 & \beta_1 & \lambda_1 & \cdots & 0 & 0 \\ 0 & \lambda_1 & \beta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{n-2} & \lambda_{n-2} \\ 0 & 0 & 0 & \cdots & \lambda_{n-2} & \beta_{n-1} \end{pmatrix}$$

By (12) and by (13) we have for $n \ge 2$

$$\begin{aligned} &\lambda_0 \le \beta_1 - c\beta_2 \le \beta_0 - c\beta_n, \\ &\lambda_{i-1} + \lambda_i \le \beta_i - c\beta_{i+2} \le \beta_i - c\beta_n, \quad 1 \le i \le n-2 \\ &\lambda_{n-2} \le \beta_{n-1} - c\beta_n. \end{aligned}$$

These inequalities imply

$$J_n \ge c\beta_n I_n,$$

where I_n denotes the identity matrix of rank n. Therefore $x_{1n} \ge c\beta_n$. On the other hand by orthogonality the polynomial $p_n(x)$ cannot change sign more than once between two consecutive points of supp μ and it cannot change sign in the interval $[\xi_1, +\infty)$. Therefore $\xi_n \geq x_{1n}$ and consequently

$$\xi_n \geq c\beta_n. \tag{18}$$

For the upper estimate we will use the minimax theorem. Let $(\cdot,\cdot)$ denote the standard inner product in the real Hilbert space $\ell^2(\mathbb{N}_0)$ and $\{\delta_n\}_{n=0}^{\infty}$ denote the standard orthogonal basis in this space. We have

$$\xi_n = \min_{v_1,\ldots,v_{n-1}} \max_{w\perp v_1,\ldots,v_{n-1}} \frac{(Jw,w)}{(w,w)} \leq \max_{w\perp \delta_0,\ldots,\delta_{n-2}} \frac{(Jw,w)}{(w,w)} = \|A_n\|,$$

where

$$A_n = \begin{pmatrix} \beta_{n-1} & \lambda_{n-1} & 0 & 0 & \cdots \\ \lambda_{n-1} & \beta_n & \lambda_n & 0 & \cdots \\ 0 & \lambda_n & \beta_{n+1} & \lambda_{n+1} & \ddots \\ 0 & 0 & \lambda_{n+1} & \beta_{n+2} & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

Therefore

$$\|A_n\| \leq \max\{\beta_{n-1} + \lambda_{n-1}, \max\{\lambda_{i-1} + \beta_i + \lambda_i \,:\, i \geq n\}\}.$$

By (12) we obtain

$$\begin{aligned} \beta_{n-1} + \lambda_{n-1} &\leq \beta_{n-1} + \beta_n - c\beta_{n+1}, \\ \lambda_{i-1} + \beta_i + \lambda_i &\leq 2\beta_i - c\beta_{i+1}, \qquad\qquad i \geq n. \end{aligned}$$

By (14) and the fact that β_n is decreasing we may conclude that

$$\|A_n\| \leq \beta_{n-1} + \beta_n - c\beta_{n+1},$$

for $n \geq N$.

Summarizing we proved that

$$c\beta_n \leq \xi_n \leq \beta_{n-1} + \beta_n - c\beta_{n+1}, \quad n \geq N, \tag{19}$$

which shows that $\xi_n \sim q^n$ because $\beta_n = \alpha_n + \gamma_n \approx q^n$. For $n \geq N$ we have

$$\xi_{n+1} \leq \beta_n + \beta_{n+1} - c\beta_{n+2}.$$

Thus

$$\frac{\xi_{n+1}}{\xi_n} \leq \frac{\beta_n + \beta_{n+1} - c\beta_{n+2}}{c\beta_n}.$$

By $\beta_n \approx q^n$. and by the second part of (12) we obtain

$$\limsup_{n\to\infty} \frac{\xi_{n+1}}{\xi_n} = \frac{1 + q - cq^2}{c} < 1.$$

Since $\xi_{n+1} < \xi_n$ for any n, the quantity ξ_{n+1}/ξ_n is bounded away from zero.

Concerning the second part we will estimate from above the quantities

$$\mu(\{\xi_n\}) = \left(\sum_{j=0}^{\infty} p_j(\xi_n)^2\right)^{-1}.$$

There is a positive constant C such that

$$\frac{\xi_n}{\gamma_j} \le Cq^{n-j}. \tag{20}$$

By the second part of (10) there exists a positive integer t such that

$$q^t \le \frac{1-q}{C}\left(1 - \frac{\kappa q}{1-q+q^2}\right) \tag{21}$$

We are going to show that for $j \le n-t$ there holds $R_{j-1}(\xi_n) > 0$ and

$$\frac{R_j(\xi_n)}{R_{j-1}(\xi_n)} = 1 - \varepsilon_j, \tag{22}$$

$$0 \le \varepsilon_j \le \frac{Cq^{n-j}}{1-q}. \tag{23}$$

The proof will go by induction on $j \le n-t$. By (2) and by (20) we have for $j = 1$

$$\frac{R_1(\xi_n)}{R_0(\xi_n)} = R_1(\xi_n) = 1 - \frac{\xi_n}{\gamma_0},$$

and

$$\varepsilon_1 = \frac{\xi_n}{\gamma_0} \le Cq^n \le \frac{Cq^{n-1}}{1-q}.$$

Assume that (22) and (23) hold for j, where $0 \le j < n-t$. Hence by (21) we obtain

$$\varepsilon_j \le \frac{Cq^{n-j}}{1-q} \le \frac{Cq^t}{1-q} \le 1 - \frac{\kappa q}{q^2-q+1} < 1.$$

which by (22) implies $R_j(\xi_n) > 0$. By virtue of (2) and $\beta_j = \alpha_j + \gamma_j$ we have

$$\gamma_j \frac{R_{j+1}(\xi_n)}{R_j(\xi_n)} + \alpha_j \frac{R_{j-1}(\xi_n)}{R_j(\xi_n)} = \alpha_j + \gamma_j - \xi_n.$$

Therefore

$$\varepsilon_{j+1} = \frac{\xi_n}{\gamma_j} + \frac{\alpha_j}{\gamma_j}\frac{\varepsilon_j}{1-\varepsilon_j}. \tag{24}$$

By induction hypothesis, in view of (10) and (20), we get

$$\begin{aligned}\varepsilon_{j+1} \le Cq^{n-j} + \kappa\frac{Cq^{n-j}}{1-q-Cq^{n-j}} \\ = \frac{Cq^{n-j-1}}{1-q}\left[q(1-q) + \frac{\kappa q(1-q)}{1-q-Cq^{n-j}}\right] \\ \le \frac{Cq^{n-j-1}}{1-q}\left[q(1-q) + \frac{\kappa q(1-q)}{1-q-Cq^{t}}\right],\end{aligned}$$

because $n-j \ge t$. Condition (21) implies

$$1-q-Cq^t \ge \frac{\kappa q(1-q)}{1-q+q^2}.$$

Therefore

$$q(1-q) + \frac{\kappa q(1-q)}{1-q-Cq^t} \le q(1-q) + 1 - q + q^2 = 1.$$

Therefore

$$\varepsilon_{j+1} \le \frac{Cq^{n-j-1}}{1-q}.$$

The assumption (21) and $\kappa \ge 1$ imply

$$q^t \le \frac{1}{C}(1-q)^2.$$

Now (22) and (23) yield that for $j \le n-t$ there holds

$$\begin{aligned}R_j(\xi_n) = (1-\varepsilon_1)(1-\varepsilon_1)\ldots(1-\varepsilon_j) \ge 1 - \sum_{i=1}^{j}\varepsilon_i \\ \ge 1 - \sum_{i=1}^{j}\frac{Cq^{n-i}}{1-q} \ge 1 - \frac{Cq^{n-j}}{(1-q)^2} \ge 1 - \frac{Cq^t}{(1-q)^2} > 0.\end{aligned}$$

Let $\eta = 1 - \frac{Cq^t}{(1-q)^2}$. Then $R_j(\xi_n) \ge \eta$, for $j \le n-t$. In view of $R_j(x) = p_j(x)/p_j(0)$ we get

$$p_j(\xi_n) \ge \eta p_j(0), \qquad j \le n-t.$$

Therefore

$$\mu(\xi_n)^{-1} = \sum_{j=0}^{\infty} p_j^2(\xi_n) \ge \sum_{j=0}^{n-t} p_j^2(\xi_n) \ge \eta^2 \sum_{j=0}^{n-t} p_j(0)^2.$$

By (11) and (17) we have $p_j^2(0) \sim s^j$ for $s > 1$. Hence

$$\mu(\xi_n) \le Ds^{-n}$$

for some constant D. This implies

$$\mu([0,\xi_n]) = \mu((0,\xi_n]) = \sum_{k=n}^{\infty} \mu(\xi_k) \le \frac{D}{s-1}\, s^{-n-1}.$$

It remains to show that $\mu([0,\xi_n]) \ge ds^{-n}$ for some constant d. To this end we will use Tchebyshev inequalities. Let $\{x_{ni}\}_{i=1}^{n}$ denote the zeros of the polynomial p_n arranged in the increasing order. Let

$$\mu_{ni} = \left(\sum_{j=0}^{n-1} p_j^2(x_{ni}) \right)^{-1}.$$

By [8, Thm. 3.41.1] we have

$$\mu_{n1} \le \mu([0, x_{n2})).$$

Since $|p_j(x_{n1})| \le p_j(0)$ (see (16)) we have

$$\mu([0, x_{n2})) \ge \left(\sum_{j=0}^{n-1} p_j^2(0) \right)^{-1} \ge ds^{-n}$$

for some $d > 0$. By orthogonality no two consecutive points of $\{x_{ni}\}_{i=1}^{n}$ may lie between two consecutive points of $\{\xi_m\}_{m=1}^{\infty}$. Also $x_{nn} < \xi_1$. Therefore $x_{n2} < \xi_{n-1}$. This gives

$$\mu([0,\xi_n]) = \mu([0,\xi_{n-1})) \ge \mu([0, x_{n2})) \ge ds^{-n}.$$

□

3. Boundedness of Dirichlet kernel

Consider orthogonal polynomials defined by (2). Let μ denote the corresponding orthogonality measure. Let $S = \operatorname{supp} \mu$.

For functions $f \in C(S)$ and $n \in \mathbb{N}$ the generalized Fourier coefficients $a_k(f)$ of f are defined by

$$a_k(f) = \int_S f(x) R_k(y)\, d\mu(y). \tag{25}$$

$s_n(f)$ denote the partial sum of the generalized Fourier series of f, i.e.

$$s_n(f,x) = \sum_{i=0}^{n} a_n(f) R_n(x) h(n). \tag{26}$$

Theorem 2. *Let $\{R_n\}_{n=0}^{\infty}$ be orthogonal polynomials satisfying (2-14). Then for any $f \in C(S)$ the partial sums $s_n(f,x)$ are convergent to f uniformly on S.*

Proof. By orthogonality we have that $s_n(R_m, x) = R_m(x)$ for $n \geq m$. Therefore for any polynomial $p(x)$ there holds $s_n(p, x) = p(x)$ for $n \geq \deg p$. Since the polynomials are dense in $C(S)$ (as S is a compact subset of the real line) it suffices to show that partial sums are uniformly bounded in L^∞ norm, i.e. there exists a constant c such that

$$\|s_n(f,x)\|_{L^\infty} \leq c\|f\|_{L^\infty}. \tag{27}$$

The proof of this estimate will go roughly along the lines of [4, 5], except that we have to overcome technical difficulties arising from the fact that orthogonality measure is not given explicitly. By (5) we get

$$s_n(f,x) = \int_S f(y) \sum_{k=0}^{n} R_k(x)R_k(y)\,d\mu(y) = \int_S f(y) \sum_{k=0}^{n} p_k(x)p_k(y)\,d\mu(y).$$

Define the generalized Dirichlet kernel $K_n(x, y)$ by

$$K_n(x,y) = \sum_{k=0}^{n} p_k(x)p_k(y). \tag{28}$$

Then

$$\|s_n(f,x)\|_{L^\infty} = \sup_{x\in S} \left| \int_S f(y)K_n(x,y)\,d\mu(y) \right| \leq \|f\|_{L^\infty} \sup_{x\in S} \int_S |K_n(x,y)|\,d\mu(y).$$

The proof will be finished if we show that

$$\sup_{x\in S} \int_S |K_n(x,y)|\,d\mu(y) < +\infty. \tag{29}$$

For this purpose we will use the conclusion of Theorem 1 which implies in particular that $S = \{0\} \cup \{\xi_k\}_{k=1}^\infty$ and $\xi_n \sim q^n$. Since $S \subset [0, \xi_1]$ we obtain

$$\int_S |K_n(x,y)|\,d\mu(y) = \int_{[0,\xi_n]} |K_n(x,y)|\,d\mu(y) + \int_{(\xi_n,\xi_1]} |K_n(x,y)|\,d\mu(y) \tag{30}$$

Combining (11), (16) and (17) yields

$$\int_{[0,\xi_n]} |K_n(x,y)|\,d\mu(y) \leq \mu([0,\xi_n]) \sum_{k=0}^{n} p_k^2(0) \leq c,$$

for some constant independent of n. It remains to estimate uniformly the second integral of the right hand side of (30) for $y \in S = \{0\} \cup \{\xi_k\}_{k=1}^\infty$. We split this integral into

$$K_n(x,x)\mu(x) + \int_{(\xi_n,\xi_1],\,y\neq x} |K_n(x,y)|d\mu(y).$$

The first term is less than 1, because

$$\mu(x)^{-1} = \sum_{k=0}^{\infty} p_k^2(x) \geq \sum_{k=0}^{n} p_k^2(x) = K_n(x,x).$$

By the Christoffel-Darboux formula ([1, 1.17] we have

$$K_n(x,y) = \lambda_n \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}.$$

Moreover since ξ_{k+1}/ξ_k is bounded away from 1 there exists a constant d such that

$$|x-y| \geq dy, \quad x \neq y,\ x,y \in \{\xi_i\}_{i=1}^{\infty}.$$

Therefore by using $|p_k(x)| \leq p_k(0)$ for $x \in S$ we obtain

$$\int_{(\xi_n,\xi_1],y\neq x} |K_n(x,y)|d\mu(y) \leq$$
$$\frac{\lambda_n p_{n+1}(0)}{d} \int_{(\xi_n,1\xi_1]} \frac{|p_n(y)|}{y}\, d\mu(y) + \frac{\lambda_n p_n(0)}{d} \int_{(\xi_n,\xi_1]} \frac{|p_{n+1}(y)|}{y}\, d\mu(y).$$

In view of $\lambda_n = \sqrt{\alpha_{n+1}\gamma_n} \approx q^n$ and $p_n(0) \sim s^{n/2}$ (see (11) and (17)) it suffices to show that

$$\int_{(\xi_n,\xi_1]} \frac{|p_n(y)|}{y}\, d\mu(y) = O(q^{-n}s^{-n/2}). \tag{31}$$

Fix a nonnegative integer l such that $q^{2l+2} < s^{-1}$. Then we have

$$\left(\int_{(\xi_n,\xi_1]} \frac{|p_n(y)|}{y}\, d\mu(y)\right)^2 = \left(\int_{(\xi_n,\xi_1]} \frac{y^l|p_n(y)|}{y^{l+1}}\, d\mu(y)\right)^2$$
$$\leq \int_S y^{2l}p_n^2(y)\, d\mu(y) \ \int_{(\xi_n,\xi_1]} \frac{1}{y^{2l+2}}\, d\mu(y).$$

Then we apply the recurrence relation (6) $2l$ times, and use orthonormality and the fact that $\beta_n \approx q^n$, $\lambda_n \approx q^n$, to get

$$\int_S y^{2l}p_n^2(y)\, d\mu(y) = O(q^{2nl}).$$

On the other hand by Theorem 1 we have $\xi_k \leq Cq^k$ and $\mu(\{\xi_k\}) \leq Cs^{-k}$ for some constant C. Thus

$$\int_{(\xi_n,\xi_1]} \frac{1}{y^{2l+2}}\, d\mu(y) = \sum_{k=1}^{n-1} \xi_k^{-(2l+2)}\mu(\{\xi_k\})$$
$$\leq C^2 \sum_{k=1}^{n-1} q^{-k(2l+2)}s^{-k} = O(q^{-n(2l+2)}s^{-n}).$$

Therefore

$$\left(\int_{(\xi_n,\xi_1]} \frac{|p_n(y)|}{y}\, d\mu(y)\right)^2 = O(q^{-2n}s^{-n}),$$

as we required in (31). □

Example. Fix $0 < a < 1$ and $0 < q < 1$. Let $\alpha_n = a^2q^n$ and $\gamma_n = q^n$. Then

$$\beta_n = (1 + a^2)q^n, \qquad \lambda_n = aq^{1/2}q^n.$$

It can be checked easily that the assumptions (9)-(14) are satisfied with $s = a^{-2}$, $\kappa = 1$, $N = 1$, i.e. there exists c satisfying (12) and (14), if

$$\frac{a}{1 + a^2} < q^{1/2}\frac{1 - q}{1 + q^2}.$$

Therefore for orthonormal polynomials associated with the recurrence relation

$$xp_n = -\lambda_n p_{n+1} + \beta_n p_n - \lambda_{n-1}p_{n-1}$$

the conclusion of Theorem 2 holds. Moreover these polynomials admit nonnegative product linearization.

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