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CONSTRUCTIVE APPROXIMATION

Orthogonal Polynomials of Discrete Variable and Boundedness of Dirichlet Kernel

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Abstract. For orthogonal polynomials defined by compact Jacobi matrix with exponential decay of the coefficients, precise properties of orthogonality measure is determined. This allows showing uniform boundedness of partial sums of orthogonal expansions with respect to L^∞ norm, which generalize analogous results obtained, for little q-Legendre, little q-Jacobi, and little q-Laguerre polynomials, by the authors.

1. Introduction

Let $s_n(f)$ denote the nth partial sum of the classical Fourier series of a continuous 2π periodic function $f(\theta)$. We know that the quantities $\|s_n(f)\|_\infty$ need not to be uniformly bounded since the Lebesgue numbers $\int_0^{2\pi} |D_n(\theta)|\, d\theta$ behave like constant multiples of $\log n$, where D_n denotes the Dirichlet kernel.

In principle, this is Faber's result [3] which shows that the system of trigonometric polynomials does not constitute a Schauder basis with respect to the set of continuous functions $C([0, 2\pi])$. Moreover, in the case of $C([-1, 1])$, Faber derived the analogous result regarding a system of algebraic polynomials with degrees increasingly passing through all positive integers. Let us recall that a sequence $\{\varphi_n\}_{n=0}^\infty$ in $C(S)$, where $S \subset \mathbf{R}$, is called a Schauder basis with respect to $C(S)$ if, for every $f \in C(S)$, there exists a unique sequence of numbers $\{a_n\}_{n=0}^\infty$ such that

$$f = \sum_{n=0}^{\infty} a_n \varphi_n. \tag{1}$$

Privalov [11] refined the result of Faber: If $\{P_n\}_{n=0}^\infty$ is a Schauder basis with respect to $C([a, b])$ consisting of algebraic polynomials, then there are $\varepsilon > 0$ and $m \in \mathbf{N}_0$ such that $\deg P_n \geq (1 + \varepsilon)n$ for all $n \geq m$. On the other hand, Privalov proved in [12] a remarkable result that for any $\varepsilon > 0$ there exists an algebraic polynomial Schauder basis $\{P_n\}_{n=0}^\infty$ with $\deg P_n \leq (1 + \varepsilon)n$. Such a basis is called a basis of optimal degree with respect to ε. Concerning the existence of an orthogonal polynomial Schauder basis of optimal degree there are two particular results we want to mention. The first gives

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an orthogonal basis with respect to the Tchebyshev weight of the first kind [5] and the second with respect to the Legendre weight [15]. The problem of construction or even of the existence of a minimal basis for general Jacobi weights seems still to be open and, more generally, it is open for an arbitrary positive measure concentrated on an interval.

There are reasons for having a polynomial basis $\{P_n\}_{n=0}^{\infty}$ with deg $P_n = n$. For instance this would imply that the partial sums $s_n(f)$ are converging toward f with the same order of magnitude as the elements of best approximation in $\mathcal{P}_n$ do [14], [19, Theorem 19.1], where $\mathcal{P}_n$ denotes the set of algebraic polynomials with degree less than or equal to n. With this in mind and due to the results above, we have to switch to spaces $C(S)$, where S differs from an interval.

The question arises: Do there exist a measure space and a corresponding orthogonal polynomial system $\{R_n\}_{n=0}^{\infty}$ with deg $R_n = n$ such that the partial sums of the Fourier series are uniformly bounded in $\|\cdot\|_\infty$ norm?

The situation is trivial if the support is finite. But the problem becomes nontrivial if the measure space is infinite, for instance, of the form $\{q^n\}_{n=0}^{\infty}$ for some number $0 < q < 1$. There are examples of systems of orthogonal polynomials whose orthogonality measure is concentrated on the sequence $\{q^n\}_{n=0}^{\infty}$. Little q-Legendre polynomials and, more generally, q-Jacobi polynomials and little q-Laguerre polynomials are such. The uniform boundedness of $\|s_n(f)\|_\infty$ has been shown for these systems in [8], [9], [10]. The proof depended heavily on the precise knowledge of the orthogonality measure and pointwise estimates of these polynomials.

In this paper we will generalize considerably these results by allowing general orthogonal polynomials satisfying a three-term recurrence relation

$$xp_n = -\lambda_n p_{n+1} + \beta_n p_n - \lambda_{n-1} p_{n-1},$$

where $\lambda_n > 0$, $\beta_n \in \mathbf{R}$, with $\lim_n \lambda_n = \lim_n \beta_n = 0$. Since these coefficients are bounded, the orthogonality measure μ on the real line is determined uniquely, see [2, II-Theorem 5.6 and IV-Theorem 2.2]. However, finding this measure explicitly is a hopeless task in general and can be achieved in very few special cases. Nonetheless, we are able sometimes to derive certain properties of this measure. We will use the well-known fact that if J is the Jacobi matrix associated with the coefficients $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$, i.e.,

$$J = \begin{pmatrix} \beta_0 & \lambda_1 & 0 & 0 & \cdots \\ \lambda_1 & \beta_1 & \lambda_2 & 0 & \cdots \\ 0 & \lambda_2 & \beta_2 & \lambda_3 & \ddots \\ 0 & 0 & \lambda_3 & \beta_3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{2}$$

then the spectrum of J on $\ell^2(\mathbf{N}_0)$ coincides with the support of μ, see [1, Theorem 4.1.3].

In this paper we impose conditions on the sequences $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ so that determining the behavior of the orthogonality measure is possible. In particular, we will assume that these coefficients have exponential decay at infinity. The properties of the orthogonality measure will be sufficient to prove the uniform boundedness of the norms $\|s_n\|_{L^\infty \to L^\infty}$.

Throughout the paper we will be using certain classical results concerning orthogonal polynomials. In most such cases references will be given. In particular, we will use the following well-known property, whose proof follows immediately from orthogonality. If $\mu((a,b)) = 0$, where μ is an orthogonality measure, then the polynomial p_n may have at most one root in the interval $[a,b]$, see [2, II-Theorem 4.1]. Moreover, if $\mu((c,+\infty)) = \mu((-\infty,d)) = 0$, then p_n does not vanish in either interval, see [2, I-Theorem 5.2].

2. Orthogonality Measure

Let $R_n(x)$ denote polynomials satisfying a three-term recurrence relation

(3) $$xR_n(x) = -\gamma_n R_{n+1}(x) + \beta_n R_n(x) - \alpha_n R_{n-1}(x),$$

where $\alpha_0 = 0$ and $R_0(x) \equiv 1$. We assume that $\gamma_n, \alpha_{n+1} > 0$ and

(4) $$\beta_n = \alpha_n + \gamma_n.$$

In this way the polynomials are normalized at 0 so that

(5) $$R_n(0) = 1.$$

Since the coefficient of the leading term of R_n is alternating, and the roots of R_n are distinct and real (see [1, Theorem I.5.2]), all these roots are positive in view of (5). Therefore (see [1, Proof of Theorem 2.1.1, for $\tau = 0$]) there is an orthogonality measure μ supported on half-line $[0,+\infty)$. Let $h(0) = 1$ and

$$h(n) = \frac{\gamma_0\gamma_1\cdots\gamma_{n-1}}{\alpha_1\alpha_2\cdots\alpha_n}.$$

It can be easily computed that the polynomials

(6) $$p_n(x) = \sqrt{h(n)}R_n(x)$$

are orthonormal and satisfy the recurrence relation

(7) $$xp_n(x) = -\lambda_n p_{n+1}(x) + \beta_n p_n(x) - \lambda_{n-1} p_{n-1}(x),$$

where

(8) $$\lambda_n = \sqrt{\alpha_{n+1}\gamma_n}.$$

We will consider polynomials with special properties such that the orthogonality measure is concentrated on a sequence of points ξ_n such that $\xi_n \searrow 0$ when $n \to \infty$. There are many instances of such behavior, e.g., little q-Jacobi polynomials, little q-Laguerre polynomials. Also we require that the polynomials satisfy a nonnegative product linearization property, i.e., the coefficients in the expansions

(9) $$R_n(x)R_m(x) = \sum_{k=|n-m|}^{n+m} g(n,m,k)R_k(x)$$

are all nonnegative. The above-mentioned polynomials fulfill this property for certain generic values of parameters. Concerning little q-Jacobi polynomials, see [6] and [4], or [9], and concerning little q-Laguerre polynomials, see [10].

We will deal with general orthogonal polynomials satisfying the above two properties. In order to ensure the proper behavior of the orthogonality measure, as well as the nonnegative linearization property, we assume that there are constants q, κ, s, c, and N such that

$$\alpha_n \approx q^n, \qquad \gamma_n \approx q^n, \qquad 0 < q < 1, \tag{10}$$

$$\alpha_n \le \kappa\gamma_n, \qquad 1 \le \kappa < \frac{1-q+q^2}{q}, \tag{11}$$

$$h(n) \sim s^n, \qquad s > 1, \tag{12}$$

$$\lambda_n \le \beta_{n+1} - c\beta_{n+2}, \qquad \frac{1}{1-q+q^2} < c < \frac{1}{q}, \tag{13}$$

$$\beta_1 \le \beta_0, \tag{14}$$

$$\beta_n - c\beta_{n+1} \ge \beta_{n+1} - c\beta_{n+2}, \qquad n \ge N. \tag{15}$$

By $\boldsymbol{a_n \approx b_n}$ we will mean that the ratio a_n/b_n has a positive limit, while by $\boldsymbol{a_n \sim b_n}$ we will mean that the ratio a_n/b_n is positive, bounded, and bounded away from 1.

Remark 1. Assumption (11) is technical. In many cases, like little q-Jacobi polynomials, this assumption is satisfied with $\kappa = 1$. Actually, it is natural to expect $\alpha_n \le \gamma_n$ (see (18)).

By assumptions (13) and (14) we obtain that β_n is a decreasing sequence and

$$\lambda_n \le \beta_{n+1} - \beta_{n+2}, \qquad n \ge 0. \tag{16}$$

Hence the assumptions of [7, Theorem 1] are satisfied. The fact that β_n is decreasing instead of being increasing follows from normalizing our polynomials in such a way that the sign of the leading coefficient is alternating, instead of being positive as in [7]. Therefore, the polynomials $\{R_n\}_{n=0}^{\infty}$ admit nonnegative product linearization. This property implies that (see [13, (17), p. 166])

$$|R_n(x)| \le 1, \qquad x \in \operatorname{supp}\mu,$$

or, equivalently,

$$|p_n(x)| \le p_n(0), \qquad x \in \operatorname{supp}\mu. \tag{17}$$

By orthonormality and by (17) we have $p_n^2(0) \ge 1$. In particular,

$$h(n) = p_n^2(0) = \frac{\gamma_0\gamma_1\cdots\gamma_{n-1}}{\alpha_1\alpha_2\cdots\alpha_n} \ge 1. \tag{18}$$

In the next theorem we are going to describe the orthogonality measure μ for the polynomials $\{R_n\}_{n=0}^{\infty}$.

Theorem 1. *Assume the orthogonal polynomial sequence $\{R_n\}_{n=0}^{\infty}$ is defined by*

$$xR_n(x) = -\gamma_n R_{n+1}(x) + \beta_n R_n(x) - \alpha_n R_{n-1}(x),$$

where $\alpha_0 = 0$, $R_0(x) \equiv 1$, $\gamma_n, \alpha_{n+1} > 0$, *and* $\beta_n = \alpha_n + \gamma_n$, *and conditions* (10)–(15) *are satisfied. Then the orthogonality measure* μ *is concentrated on the decreasing sequence* $\{\xi_n\}_{n=1}^{\infty}$, *where* $\xi_n \sim q^n$, *the quantity* $1 - \xi_{n+1}/\xi_n$ *is bounded away from* 0, *and* $\mu([0, \xi_n]) \sim s^{-n}$.

Remark 2. We conjecture that the conclusion of the theorem cannot be strengthened to $\mu(\{\xi_n\}) \sim s^{-n}$. Indeed, consider the probability measure

$$\mu = \frac{3}{2} \sum_{n=0}^{\infty} \frac{1}{4^{n+1}} \delta_{2^{-2n}} + \frac{7}{2} \sum_{n=0}^{\infty} \frac{1}{8^{n+1}} \delta_{2^{-(2n+1)}}.$$

Then $\xi_n \sim 2^{-n}$ and $\mu([0, \xi_n]) \sim 2^{-n}$ but $\mu(\{\xi_n\}) \not\sim 2^{-n}$. Of course, we cannot guarantee that the polynomials orthogonal with respect to this measure satisfy nonnegative product linearization.

The converse assertion is not true, that is, there are orthogonality measures concentrated on a set with the above properties, but conditions (10)–(15) do not hold. For instance, there are parameters for little q-Laguerre polynomials which do not admit nonnegative product linearization, see [10]. Although the orthogonality measure is concentrated on $\{q^n\}_{n=1}^{\infty}$, in such cases (13) or (14) fails to be true.

Remark 3. Theorem 1 cannot be applied for little q-Legendre polynomials. This is because we have to impose strong assumptions on the coefficients λ_n and β_n so as to achieve nonnegative linearization and proper behavior of the ratio ξ_{n+1}/ξ_n. For little q-Legendre polynomials, as well as for little q-Jacobi polynomials the orthogonality measure is given explicitly and one can check by inspection that the statement of Theorem 1 is satisfied in these cases. Also nonnegative product linearization for these particular polynomials has been proved in [6] and in [4]. In turn, only the statement of Theorem 1 is used to prove Theorem 2 from Section 3.

Proof. Let J denote the Jacobi matrix associated with the polynomials p_n (see (2)). By assumption, J is a compact operator on $\ell^2(\mathbf{N}_0)$. Moreover, J is semipositive definite because by (8) we have $J = S^*S$, where

$$S = \begin{pmatrix} \sqrt{\gamma_0} & \sqrt{\alpha_1} & 0 & 0 & \cdots \\ 0 & \sqrt{\gamma_1} & \sqrt{\alpha_2} & 0 & \cdots \\ 0 & 0 & \sqrt{\gamma_2} & \sqrt{\alpha_3} & \ddots \\ 0 & 0 & 0 & \sqrt{\gamma_3} & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Hence the spectrum of J consists of 0 and a decreasing sequence of points $\{\xi_n\}_{n=1}^{\infty}$ accumulating at zero. As we have mentioned in the Introduction, the support of μ coincides with the spectrum of J. First we will show that $\mu(\{0\}) = 0$. Indeed, by [1, Theorem 2.5.3], we have

$$\mu(\{0\})^{-1} = \sum_{n=0}^{\infty} p_n^2(0).$$

We know that $p_n^2(0) \geq 1$ (see (18)). Hence $\mu(\{0\})^{-1} = \infty$.

Now we turn to determining the behavior of ξ_n. Let $\{x_{jn}\}_{j=1}^{n}$ denote the zeros of the polynomial $p_n(x)$ arranged in increasing order. It is well known (see [2, Exercise I.4.12]) that this set coincides with the set of eigenvalues of the truncated Jacobi matrix J_n, where

$$J_n = \begin{pmatrix} \beta_0 & \lambda_0 & 0 & \cdots & 0 & 0 \\ \lambda_0 & \beta_1 & \lambda_1 & \cdots & 0 & 0 \\ 0 & \lambda_1 & \beta_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \beta_{n-2} & \lambda_{n-2} \\ 0 & 0 & 0 & \cdots & \lambda_{n-2} & \beta_{n-1} \end{pmatrix}.$$

By (13) and (14) we have, for $n \geq 2$,

$$\begin{aligned} \lambda_0 &\leq \beta_1 - c\beta_2 \leq \beta_0 - c\beta_n, \\ \lambda_{i-1} + \lambda_i &\leq \beta_i - c\beta_{i+2} \leq \beta_i - c\beta_n, \qquad 1 \leq i \leq n-2, \\ \lambda_{n-2} &\leq \beta_{n-1} - c\beta_n. \end{aligned}$$

These inequalities imply

$$J_n \geq c\beta_n I_n,$$

where I_n denotes the identity matrix of rank n. Therefore $x_{1n} \geq c\beta_n$. On the other hand, by orthogonality the polynomial $p_n(x)$ cannot change sign more than once between two consecutive points of supp μ and it cannot change sign in the interval $[\xi_1, +\infty)$. Therefore, $\xi_n \geq x_{1n}$ and, consequently,

$$\xi_n \geq c\beta_n. \tag{19}$$

For the upper estimate we will use the minimax theorem. Let $(\cdot, \cdot)$ denote the standard inner product in the real Hilbert space $\ell^2(\mathbf{N}_0)$ and let $\{\delta_n\}_{n=0}^{\infty}$ denote the standard orthogonal basis in this space. We have

$$\xi_n = \min_{v_1,\ldots,v_{n-1}} \max_{w \perp v_1,\ldots,v_{n-1}} \frac{(Jw, w)}{(w, w)} \leq \max_{w \perp \delta_0,\ldots,\delta_{n-2}} \frac{(Jw, w)}{(w, w)} = \|A_n\|,$$

where

$$A_n = \begin{pmatrix} \beta_{n-1} & \lambda_{n-1} & 0 & 0 & \cdots \\ \lambda_{n-1} & \beta_n & \lambda_n & 0 & \cdots \\ 0 & \lambda_n & \beta_{n+1} & \lambda_{n+1} & \ddots \\ 0 & 0 & \lambda_{n+1} & \beta_{n+2} & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Therefore,

$$\|A_n\| \leq \max\{\beta_{n-1} + \lambda_{n-1}, \max\{\lambda_{i-1} + \beta_i + \lambda_i \ : \ i \geq n\}\}.$$

By (13) we obtain

$$\begin{aligned} \beta_{n-1} + \lambda_{n-1} &\le \beta_{n-1} + \beta_n - c\beta_{n+1}, \\ \lambda_{i-1} + \beta_i + \lambda_i &\le 2\beta_i - c\beta_{i+2}, \qquad i \ge n. \end{aligned}$$

By (15) and the fact that β_n is decreasing we may conclude that

$$\|A_n\| \le \beta_{n-1} + \beta_n - c\beta_{n+2}$$

for $n \ge N$.

Summarizing we proved that

$$c\beta_n \le \xi_n \le \beta_{n-1} + \beta_n - c\beta_{n+2}, \qquad n \ge N, \tag{20}$$

which shows that $\xi_n \sim q^n$ because $\beta_n = \alpha_n + \gamma_n \approx q^n$. For $n \ge N$ we have

$$\xi_{n+1} \le \beta_n + \beta_{n+1} - c\beta_{n+3}.$$

Thus

$$\frac{\xi_{n+1}}{\xi_n} \le \frac{\beta_n + \beta_{n+1} - c\beta_{n+3}}{c\beta_n}.$$

By $\beta_n \approx q^n$ and by the second part of (13) we obtain

$$\limsup_{n\to\infty} \frac{\xi_{n+1}}{\xi_n} = \frac{1 + q - cq^3}{c} < 1.$$

Since $\xi_{n+1} < \xi_n$ for any n, the quantity ξ_{n+1}/ξ_n is bounded away from zero.

Concerning the second part we will estimate from above the quantities

$$\mu(\{\xi_n\}) = \left(\sum_{j=0}^{\infty} p_j(\xi_n)^2 \right)^{-1}.$$

There is a positive constant C such that

$$\frac{\xi_n}{\gamma_j} \le Cq^{n-j}. \tag{21}$$

By the second part of (11) there exists a positive integer t such that

$$q^t \le \frac{1-q}{C}\left(1 - \frac{\kappa q}{1 - q + q^2}\right). \tag{22}$$

We are going to show that for $j \le n - t$ there holds $R_{j-1}(\xi_n) > 0$ and

$$\frac{R_j(\xi_n)}{R_{j-1}(\xi_n)} = 1 - \varepsilon_j, \tag{23}$$

$$0 \le \varepsilon_j \le \frac{Cq^{n-j}}{1-q}. \tag{24}$$

The proof will go by induction on $j \le n-t$. By (3) and (21) we have, for $j = 1$,

$$\frac{R_1(\xi_n)}{R_0(\xi_n)} = R_1(\xi_n) = 1 - \frac{\xi_n}{\gamma_0},$$

and

$$\varepsilon_1 = \frac{\xi_n}{\gamma_0} \le Cq^n \le \frac{Cq^{n-1}}{1-q}.$$

Assume that (23) and (24) hold for j, where $1 \le j < n-t$. Hence, by (22), we obtain

$$\varepsilon_j \le \frac{Cq^{n-j}}{1-q} \le \frac{Cq^t}{1-q} \le 1 - \frac{\kappa q}{q^2-q+1} < 1,$$

which by (23) implies $R_j(\xi_n) > 0$. By virtue of (3) and $\beta_j = \alpha_j + \gamma_j$, we have

$$\gamma_j \frac{R_{j+1}(\xi_n)}{R_j(\xi_n)} + \alpha_j \frac{R_{j-1}(\xi_n)}{R_j(\xi_n)} = \alpha_j + \gamma_j - \xi_n.$$

Therefore,

$$\varepsilon_{j+1} = \frac{\xi_n}{\gamma_j} + \frac{\alpha_j}{\gamma_j}\frac{\varepsilon_j}{1-\varepsilon_j}. \tag{25}$$

By induction hypothesis, in view of (11) and (21), we get

$$\begin{aligned} \varepsilon_{j+1} &\le Cq^{n-j} + \kappa \frac{Cq^{n-j}}{1-q-Cq^{n-j}} \\ &= \frac{Cq^{n-j-1}}{1-q}\left[q(1-q) + \frac{\kappa q(1-q)}{1-q-Cq^{n-j}}\right] \\ &\le \frac{Cq^{n-j-1}}{1-q}\left[q(1-q) + \frac{\kappa q(1-q)}{1-q-Cq^{t}}\right], \end{aligned}$$

because $n-j \ge t$. Condition (22) implies

$$1-q-Cq^t \ge \frac{\kappa q(1-q)}{1-q+q^2}.$$

Therefore,

$$q(1-q) + \frac{\kappa q(1-q)}{1-q-Cq^t} \le q(1-q) + 1 - q + q^2 = 1.$$

Hence,

$$\varepsilon_{j+1} \le \frac{Cq^{n-j-1}}{1-q}.$$

Assumption (22) and $\kappa \ge 1$ imply

$$q^t \le \frac{1}{C}(1-q)^2.$$

Now (23) and (24) yield that for $j \le n - t$ there holds

$$R_j(\xi_n) = (1-\varepsilon_1)(1-\varepsilon_2)\dots(1-\varepsilon_j) \ge 1 - \sum_{i=1}^{j} \varepsilon_i$$

$$\ge 1 - \sum_{i=1}^{j} \frac{Cq^{n-i}}{1-q} \ge 1 - \frac{Cq^{n-j}}{(1-q)^2} \ge 1 - \frac{Cq^t}{(1-q)^2} > 0.$$

Let $\eta = 1 - Cq^t/(1-q)^2$. Then $R_j(\xi_n) \ge \eta$ for $j \le n-t$. In view of $R_j(x) = p_j(x)/p_j(0)$, we get

$$p_j(\xi_n) \ge \eta p_j(0), \qquad 0 \le j \le n-t.$$

Therefore, in the case of $n \ge t$, we get

$$\mu(\xi_n)^{-1} = \sum_{j=0}^{\infty} p_j^2(\xi_n) \ge \sum_{j=0}^{n-t} p_j^2(\xi_n) \ge \eta^2 \sum_{j=0}^{n-t} p_j^2(0).$$

By (12) and (18) we have $p_j^2(0) \sim s^j$ for $s > 1$. Hence

$$\mu(\xi_n) \le Ds^{-n}$$

for some constant D. This implies

$$\mu([0,\xi_n]) = \mu((0,\xi_n]) = \sum_{k=n}^{\infty} \mu(\xi_k) \le \frac{Ds}{s-1}\, s^{-n}.$$

It remains to show that $\mu([0,\xi_n]) \ge ds^{-n}$ for some constant d. To this end, we will use Tchebyshev inequalities. Let $\{x_{ni}\}_{i=1}^n$ denote the zeros of the polynomial p_n arranged in increasing order. Let

$$\mu_{ni} = \left(\sum_{j=0}^{n-1} p_j^2(x_{ni}) \right)^{-1}.$$

By [16, Theorem 3.41.1] we have

$$\mu_{n1} \le \mu([0, x_{n2})).$$

Since $|p_j(x_{n1})| \le p_j(0)$ (see (17)) we have

$$\mu([0, x_{n2})) \ge \left(\sum_{j=0}^{n-1} p_j^2(0) \right)^{-1} \ge ds^{-n}$$

for some $d > 0$. By orthogonality no two consecutive points of $\{x_{ni}\}_{i=1}^n$ may lie between two consecutive points of $\{\xi_m\}_{m=1}^{\infty}$. Also $x_{nn} < \xi_1$. Therefore, $x_{n2} < \xi_{n-1}$. This gives

$$\mu([0,\xi_n]) = \mu([0,\xi_{n-1})) \ge \mu([0,x_{n2})) \ge ds^{-n}. \qquad \blacksquare$$

3. Boundedness of Dirichlet Kernel

Consider orthogonal polynomials defined by (3). Let μ denote the corresponding orthogonality measure. Let $S = \operatorname{supp}\mu$.

For functions $f \in C(S)$ and $k \in \mathbf{N}_0$ the generalized Fourier coefficients $a_k(f)$ of f are defined by

$$a_k(f) = \int_S f(y) R_k(y)\, d\mu(y). \tag{26}$$

$s_n(f)$ denotes the partial sum of the generalized Fourier series of f, i.e.,

$$s_n(f, x) = \sum_{k=0}^{n} a_k(f) R_k(x) h(k). \tag{27}$$

Theorem 2. *Assume the orthogonal polynomial sequence $\{R_n\}_{n=0}^{\infty}$ is defined by*

$$x R_n(x) = -\gamma_n R_{n+1}(x) + \beta_n R_n(x) - \alpha_n R_{n-1}(x),$$

where $\alpha_0 = 0$, $R_0(x) \equiv 1$, $\gamma_n, \alpha_{n+1} > 0$, and $\beta_n = \alpha_n + \gamma_n$, and conditions (10)–(15) *are satisfied. Then for any $f \in C(S)$ the partial sums $s_n(f, x)$ are convergent to f uniformly on S.*

Proof. By orthogonality we have that $s_n(R_m, x) = R_m(x)$ for $n \geq m$. Therefore for any polynomial $p(x)$ there holds $s_n(p, x) = p(x)$ for $n \geq \deg p$. Since the polynomials are dense in $C(S)$ (as S is a compact subset of the real line) it suffices to show that partial sums are uniformly bounded in L^∞ norm, i.e., there exists a constant c such that

$$\|s_n(f, x)\|_{L^\infty} \leq c \|f\|_{L^\infty}. \tag{28}$$

The proof of this estimate will go roughly along the lines of [8], [9], except that we have to overcome technical difficulties arising from the fact that orthogonality measure is not given explicitly. By (6) we get

$$s_n(f, x) = \int_S f(y) \sum_{k=0}^{n} R_k(x) R_k(y) h(k)\, d\mu(y) = \int_S f(y) \sum_{k=0}^{n} p_k(x) p_k(y)\, d\mu(y).$$

Define the generalized Dirichlet kernel $K_n(x, y)$ by

$$K_n(x, y) = \sum_{k=0}^{n} p_k(x) p_k(y). \tag{29}$$

Then

$$\begin{aligned} \|s_n(f, x)\|_{L^\infty} &= \sup_{x \in S} \left| \int_S f(y) K_n(x, y)\, d\mu(y) \right| \\ &\leq \|f\|_{L^\infty} \sup_{x \in S} \int_S |K_n(x, y)|\, d\mu(y). \end{aligned}$$

The proof will be finished if we show that

$$\sup_n \sup_{x\in S} \int_S |K_n(x, y)|\, d\mu(y) < +\infty. \tag{30}$$

For this purpose we will use the conclusion of Theorem 1 which implies, in particular, that $S = \{0\} \cup \{\xi_k\}_{k=1}^{\infty}$ and $\xi_n \sim q^n$. Since $S \subset [0, \xi_1]$ we obtain

$$\int_S |K_n(x, y)|\, d\mu(y) = \int_{[0,\xi_n]} |K_n(x, y)|\, d\mu(y) + \int_{(\xi_n,\xi_1]} |K_n(x, y)|\, d\mu(y). \tag{31}$$

Combining (12), (17), and (18) yields

$$\int_{[0,\xi_n]} |K_n(x, y)|\, d\mu(y) \le \mu([0, \xi_n]) \sum_{k=0}^{n} p_k^2(0) \le c,$$

for some constant independent of n. It remains to estimate uniformly the second integral of the right-hand side of (31) for $x \in S = \{0\} \cup \{\xi_k\}_{k=1}^{\infty}$. We split this integral into an upper bound

$$K_n(x, x)\mu(x) + \int_{(\xi_n,\xi_1],\, y\neq x} |K_n(x, y)|\, d\mu(y).$$

The first term is less than 1, because

$$\mu(x)^{-1} = \sum_{k=0}^{\infty} p_k^2(x) \ge \sum_{k=0}^{n} p_k^2(x) = K_n(x, x).$$

By the Christoffel–Darboux formula [1, 1.17] we have

$$K_n(x, y) = \lambda_n \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x - y}.$$

Moreover, since ξ_{k+1}/ξ_k is bounded away from 1 there exists a constant d such that

$$|x - y| \ge dy, \qquad x \neq y, \quad x, y \in \{\xi_i\}_{i=1}^{\infty}.$$

Therefore, by using $|p_k(x)| \le p_k(0)$ for $x \in S$, we obtain

$$\begin{aligned}&\int_{(\xi_n,\xi_1],\, y\neq x} |K_n(x, y)|\, d\mu(y)\\ &\qquad \le \frac{\lambda_n p_{n+1}(0)}{d} \int_{(\xi_n,\xi_1]} \frac{|p_n(y)|}{y}\, d\mu(y) + \frac{\lambda_n p_n(0)}{d} \int_{(\xi_n,\xi_1]} \frac{|p_{n+1}(y)|}{y}\, d\mu(y).\end{aligned}$$

In view of $\lambda_n = \sqrt{\alpha_{n+1}\gamma_n} \approx q^n$ and $p_n(0) \sim s^{n/2}$ (see (12) and (18)) it suffices to show that

$$\int_{(\xi_n,\xi_1]} \frac{|p_n(y)|}{y}\, d\mu(y) = O(q^{-n}s^{-n/2}). \tag{32}$$

Fix a nonnegative integer l such that $q^{2l+2} < s^{-1}$. Then we have

$$\left(\int_{(\xi_n,\xi_1]} \frac{|p_n(y)|}{y}\, d\mu(y)\right)^2 = \left(\int_{(\xi_n,\xi_1]} \frac{y^l |p_n(y)|}{y^{l+1}}\, d\mu(y)\right)^2$$
$$\leq \int_S y^{2l} p_n^2(y)\, d\mu(y) \int_{(\xi_n,\xi_1]} \frac{1}{y^{2l+2}}\, d\mu(y).$$

Then we apply the recurrence relation (7) $2l$ times, and use orthonormality and the fact that $\beta_n \approx q^n$, $\lambda_n \approx q^n$, to get

$$\int_S y^{2l} p_n^2(y)\, d\mu(y) = O(q^{2nl}).$$

On the other hand, by Theorem 1 we have $\xi_k^{-1} \leq Cq^{-k}$ and $\mu(\{\xi_k\}) \leq Cs^{-k}$ for some constant C. Thus

$$\int_{(\xi_n,\xi_1]} \frac{1}{y^{2l+2}}\, d\mu(y) = \sum_{k=1}^{n-1} \xi_k^{-(2l+2)} \mu(\{\xi_k\})$$
$$\leq C^{2l+3} \sum_{k=1}^{n-1} q^{-k(2l+2)} s^{-k} = O(q^{-n(2l+2)} s^{-n}).$$

Therefore,

$$\left(\int_{(\xi_n,\xi_1]} \frac{|p_n(y)|}{y}\, d\mu(y)\right)^2 = O(q^{-2n} s^{-n}),$$

as we required in (32). ■

Example. Fix $0 < a < 1$ and $0 < q < 1$. Let $\alpha_n = a^2 q^n$ and $\gamma_n = q^n$. Then

$$\beta_n = (1 + a^2) q^n, \qquad \lambda_n = a q^{1/2} q^n.$$

It can be checked easily that assumptions (10)–(15) are satisfied with $s = a^{-2}$, $\kappa = 1$, $N = 1$, i.e., there exists c satisfying (13) and (15), if

$$\frac{a}{1 + a^2} < q^{1/2} \frac{(1 - q)^2}{1 - q + q^2}.$$

Therefore, for orthonormal polynomials associated with the recurrence relation

$$x p_n = -\lambda_n p_{n+1} + \beta_n p_n - \lambda_{n-1} p_{n-1}$$

the conclusion of Theorem 2 holds. Moreover, these polynomials admit nonnegative product linearization.

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