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AN ANALYTIC SERIES OF IRREDUCIBLE REPRESENTATIONS OF THE FREE GROUP

by Ryszard SZWARC

0. Introduction.

This paper is a natural continuation of [11], where a construction of an analytic family of uniformly bounded representations of the free group $\mathbb{F}$ on the Hilbert space $\ell^2(\mathbb{F})$ was presented. These representations are irreducible provided that $\mathbb{F}$ has infinitely many free generators. Here we deal with the case when the free group $\mathbb{F}_k$ is finitely generated (k-number of free generators).

The theory of representations of $\mathbb{F}_k$ involves a deep relationship between certain aspects of harmonic analysis on the free group and harmonic analysis on $SL(2,\mathbb{R})$. This analogy has been emphasized in the papers of P. Cartier [1] and A. Figà-Talamanca, M. A. Picardello [4], fundamental at present.

In analogy with $SL(2,\mathbb{R})$ a decomposition of the regular representation into irreducible ones was obtained due to the invention of the analogues of the positive definite spherical functions on $SL(2,\mathbb{R})$. In the paper of R. A. Kunze and E. M. Stein [7] a construction of an analytic series of the representations of the group $SL(2,\mathbb{R})$

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parametrized by spherical functions was given. There arises a problem of finding the theorem for a free group related to the theorem of Kunze and Stein. We refer to [4] for the classification of all spherical functions on a free group. In [8] there was constructed a uniformly bounded Hilbert space representation of a free group with an arbitrary spherical function as its matrix coefficient.

In this paper we are going to construct an analytic series of uniformly bounded representations Π_z, $\frac{1}{2k-1} < |z| < 1$, of the group $\mathbf{F}_k$, all of them acting on the same Hilbert space, such that

(i) $\Pi_z^*(x) = \Pi_{\bar{z}}(x)^{-1}$,

(ii) $\Pi_z = \Pi_u$ if $u = \frac{1}{(2k-1)z}$,

(iii) $\Pi_z(x) - \Pi_{z'}(x)$ has finite rank for any z, z' and $x \in \mathbf{F}_k$,

(iv) Π_z is an irreducible representation. The representations Π_z and $\Pi_{z'}$ are equivalent iff $z = z'$ or $z' = \frac{1}{(2k-1)z}$,

(v) For z unreal or $|z| \neq \frac{1}{\sqrt{2k-1}}$ the representation Π_z cannot be made unitary by introducing another equivalent inner product.

For $|z| = \frac{1}{\sqrt{2k-1}}$ or $z \in \mathbf{R}$ the representations Π_z correspond respectively to the principal or the complementary series of the representations defined in [4], where the irreducibility of these two series was proved.

Moreover it turns out that the matrix coefficients $\Psi_z(x) = < \Pi_z(x)\delta_e, \delta_e >$ is the collection of all spherical functions defined in [1] and in [4].

We also consider the norm closed algebra C_{Π_z} associated with the representation Π_z and we prove that there are nontrivial projections in C_{Π_z} whenever $|z| \neq \frac{1}{\sqrt{2k-1}}$. That distinguishes this algebra from $C^*_\lambda(\mathbf{F}_k)$ the projection-free algebra associated with the regular representation (cf. [9], [3]).

In Section 1 we include an easy proof (Corollary 2) of J.M. Cohen's result which states that the spectrum set of C^*-algebra $C^r_\lambda(\mathbf{F}_k)$ consisting of radial functions in $C^*_\lambda(\mathbf{F}_k)$, coincides with the interval $[-2\sqrt{2k-1}, 2\sqrt{2k-1}]$.

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1. Basic notations and definitions.

Here we will establish some notation relevant to the free group which we will need in the exposition. The notation we use appeared in a number of earlier papers ([5], [6], [11]). Especially important is the "cutting letter's" operator P introduced in [11] where a construction of an analytic family of free group representations was based on this notion.

Let F be a free group and E a fixed set of its free generators. When E is a finite set consisting of k elements then the group which E generates we denote by F_k. Each element x of F may be uniquely expressed as a reduced finite word which letters are from the set $E \cup E^{-1}$ provided that the letters a and a^{-1} do not follow each other. The number of letters of the word x we call the length of x and we denote by $|x|$, setting $|e| = 0$ for the empty word e. If $x \neq e$ then let $\overline{x}$ denote the word obtained from x by deleting its last letter. For n natural let E_n stand for the set of elements x in F_k such that the length $|x|$ equals n. The set E_n is finite and it consists of $2k(2k-1)^{n-1}$ words.

On the set $\mathcal{K}(\mathsf{F})$ of all complex functions finitely supported in F we introduce the convolution putting as usual $f * g = \sum_{x,y} f(x)g(y)\delta_{xy}$, where δ_x is the characteristic function of the one point set $\{x\}$.

We say that a function f on the group F_k is radial when its value $f(x)$ depends only on $|x|$ the length of the word x in F_k. It is easily seen that each radial function is represented as a series $\sum_{n=0}^{\infty} \alpha_n \chi_n$, $\alpha_n \in \mathbf{C}$, where χ_n denotes the characteristic function of the set E_n. The convolution of radial functions (if it exists, e.g. is absolutely convergent) is commutative and it leads to radial functions again. It follows from the recurrence ([2], Theorem 1) :

$$\text{(1)}\qquad \begin{aligned} \chi_1 * \chi_1 &= \chi_2 + 2k\,\chi_0 \\ \chi_1 * \chi_n &= \chi_{n+1} + (2k-1)\chi_{n-1} \quad \text{for} \quad n = 2,\ 3, \ldots . \end{aligned}$$

For $0 < p \leq \infty$ set $\ell^p_r(\mathsf{F}_k)$ to be the subspace of radial functions in $\ell^p(\mathsf{F}_k)$.

The process of deleting the last letter $x \mapsto \overline{x}$ lifts in the natural way to a linear operator $P : \mathcal{K}(\mathsf{F}_k) \to \mathcal{K}(\mathsf{F}_k)$ by putting $P\delta_x = \delta_{\overline{x}}$ if $x \neq e$ and $P\delta_e = 0$ ([11]). The operator P leaves the subspace of radial functions in $\mathcal{K}(\mathsf{F}_k)$ invariant for

$$\text{(2)}\qquad P\chi_1 = 2k\chi_0, \quad P\chi_n = (2k-1)\chi_{n-1} \quad \text{for} \quad n = 2,\ 3, \ldots .$$

For $a \in \mathsf{F}$ the symbols $\lambda(a)$ and $\rho(a)$ will denote the operations of the left and right translations of the functions of F given by $\lambda(a)f(x) = f(a^{-1}x)$ and $\rho(a)f(x) = f(xa)$.

2. Spectral properties of the operator P.

The properties of the operator P are crucial in deriving the properties of the family of the representations π_z, $|z| < 1$, introduced in [11]. If the group F has infinitely many free generators then the operator P is unbounded on $\ell^2(\mathsf{F})$, but the representations constructed are irreducible. The statement is not true when F is finitely generated. In this case the spectral properties of the operator P play the most important part. The section below is devoted to them.

Fix a natural number k. The operator P extends to a bounded operator on $\ell^2(\mathsf{F}_k)$, and also ([11], 1.4, Remark 1)

$$\text{(3)}\qquad \| P^n \| = \sqrt{2k(2k-1)^{n-1}} \quad \text{for} \quad n = 1, 2, \ldots .$$

Indeed, for $f \in \ell^2(\mathsf{F}_k)$ the value $(P^n f)(x)$ is equal to the sum $\sum_y f(xy)$ of the values of the function f where y runs through the words of length n such that xy is a reduced word, i.e. $|xy| = |x| + |y|$. The set of such y's has at most $2k(2k-1)^{n-1}$ elements (even less : $(2k-1)^n$ if $x \neq e$). By Schwarz inequality this gives $\|P^n\| \leq \sqrt{2k(2k-1)^{n-1}}$. The operator P^n attains its norm on the function χ_n, for $P^n\chi_n = 2k(2k-1)^{n-1}\delta_e$ and $\|\chi_n\|_2 = \sqrt{2k(2k-1)^{n-1}}$. Reasoning as above we also get

$$\|Pf\|_2 \leq \sqrt{2k-1}\|f\|_2 \quad \text{if} \quad (Pf)(e) = 0\,. \tag{4}$$

Note that the closed disc $\{z \in \mathbf{C} : |z| \leq \sqrt{2k-1}\}$ happens to be the spectrum of the operator P. In fact, by (3) this disc contains the spectrum of P. On the other hand each point z for which $|z| < \sqrt{2k-1}$ is an eigenvalue of P : the eigenfunction $h_z = 2k(2k-1)^{-1}\delta_e + \sum_{n=1}^{\infty}(2k-1)^{-n}z^n\chi_n$ belongs to $\ell^2(\mathbf{F}_k)$ for such z.

The adjoint operator P^* acts by "adding" letters. On the function δ_x it is given by $P^*\delta_x = \sum_a \delta_{xa}$ where a is from the set of the words of length 1 which do not cancel the word x, i.e. $|xa| = |x| + 1$. As we can treat the action of P on δ_x also as adding the letter, exactly this which cancels the word x, so the sum $P + P^*$ is the right convolution operator with the function χ_1 (the characteristic function of the set of words of length 1). Write it down as the formula

$$P + P^* = \rho(\chi_1). \tag{5}$$

We notice that $\delta_e - \frac{1}{2k}\rho(\chi_1)$ is considered as the analogous of the Laplace operator on $\mathbf{F}_k$. The operator P^* (just as P) leaves the subspace $\ell^2_r(\mathbf{F}_k)$ of radial functions invariant and in particular $P^*\chi_n = \chi_{n+1}$ for $n = 0, 1, \ldots$.

The operators P and P^* do not commute. Consider their composition PP^*. If $x \neq e$ then $P^*\delta_x$ is a sum of $2k-1$ terms δ_{xa}. So $PP^*\delta_x = (2k-1)\delta_x$. However $P^*\delta_e = \chi_1$ hence $PP^*\delta_e = 2k\delta_e$. If T denotes the orthogonal projection onto one-dimensional subspace $\mathbf{C}\delta_e$, then we have

$$PP^* = (2k-1)I + T\,. \tag{6}$$

Clearly the spectrum of the operator P^* also coincides with the closed disc of radius $\sqrt{2k-1}$.

We have already seen that for $|z| < \sqrt{2k-1}$ the operator $zI-P$ is noninvertible on $\ell^2(\mathbb{F}_k)$ (for it has nonzero kernel). Theorem 1 will let us know among other things, that in this case the image of $zI-P$ is nevertheless equal to $\ell^2(\mathbb{F}_k)$. Before that we need one more definition.

For $|z| < 1$ let T_z denote ([11], p. 10) the bounded invertible operator on $\ell^2(\mathbb{F}_k)$ given by

$$T_z = \sqrt{1-z^2}T + (I-T)\,,$$

where $\sqrt{1-z^2}$ denotes the main branch of the square root. For $|z| < 1$ the correspondence $z \mapsto T_z$ is analytic. The square of the operator T_z is equal to $I - z^2T$, so it is well defined with no condition on z.

THEOREM 1. — *Let $|z| > \frac{1}{\sqrt{2k-1}}$. The operator $\frac{1}{z}(I-zP)T_z^2(I-zP^*)$ is invertible on $\ell^2(\mathbb{F}_k)$ and commutes with left translations on $\mathbb{F}_k$. Its inverse is the right side convolution operator with the function $\frac{u}{1-u^2}u^{|x|}$, where $u = \frac{1}{(2k-1)z}$. Moreover*

$$\frac{1}{z}(I-zP)T_z^2(I-zP^*) = \frac{1}{u}(I-uP)T_u^2(I-uP^*). \tag{7}$$

Remark. — Let's note that none of the operators $I-zP$ and $I-zP^*$ is invertible on $\ell^2(\mathbb{F}_k)$ when $|z| > \frac{1}{\sqrt{2k-1}}$. So the statement of the theorem is nontrivial.

The proof of Theorem 1. — For every complex number $z \neq 0$

$$\frac{1}{z}(I-zP)T_z^2(I-zP^*) = \gamma(z)I - \rho(\chi_1),\ \gamma(z) = (2k-1)z + \frac{1}{z}\,. \tag{8}$$

This formula follows immediately from (5), (6) and the fact that $T_z^2 = I - z^2T$. Next (8) implies that the operator $\frac{1}{z}(I-zP)T_z^2(I-zP^*)$ is left translation invariant. Observe that $\gamma(z) = \gamma(u)$ with $u = \frac{1}{(2k-1)z}$. Combining this with (8) gives the formula (7).

Now if $|z| > \frac{1}{\sqrt{2k-1}}$ then $|u| < \frac{1}{\sqrt{2k-1}}$. Therefore the operator on the right hand side of (7) (denote it by A) is invertible for all its factors are invertible operators. The right side convolution with the function $A^{-1}\delta_e$ is the inverse operator for A (because A^{-1} also commutes with left translations). Finally as $(I-uP)^{-1} = \sum_{n=0}^{\infty} u^n P^n$ and $(I - uP^*)^{-1} = \sum_{n=0}^{\infty} u^n (P^*)^n$ for $|u| < \frac{1}{\sqrt{2k-1}}$ so $A^{-1}\delta_e = \frac{u}{1-u^2} u^{|x|}$.

Theorem 1 will be extremely helpful in Section 3. Here only the simplest corollaries will be derived.

COROLLARY 1. — *The operator $I - zP$ is a surjection if and only if $|z| \neq \frac{1}{\sqrt{2k-1}}$. Moreover if $|z| = \frac{1}{\sqrt{2k-1}}$ then $I - zP$ is an injection.*

Proof. — Theorem 1 implies that if $|z| \neq \frac{1}{\sqrt{2k-1}}$ then the operator $I - zP$ is a surjection. Observe that $\frac{1}{z}$ belongs to the spectrum of P if $|z| = \frac{1}{\sqrt{2k-1}}$. Therefore it suffices to show that $I - zP$ is an injection in this case. Assume $f \in \mathrm{Ker}(I - zP)$, then $f = zPf$. For $n = 0, 1, 2, \dots$ let $f_n = f\chi_n$. Thus $f_n = zPf_{n+1}$. By (4) $\|f_n\|_2 \leq \sqrt{2k-1}|z|\ \|f_{n+1}\|_2 = \|f_{n+1}\|_2$ for $n = 1, 2, \dots$. Since $\sum \|f_n\|_2^2 = \|f\|_2^2 < +\infty$, hence $f_1 = f_2 = \cdots = 0$. Moreover $f_0 = zPf_1 = 0$, so $f = 0$.

Theorem 1 combined with Corollary 1 imply that the operator $\frac{1}{z}(I - zP)T_z^2(I - zP^*)$ is invertible if and only if $|z| \neq \frac{1}{\sqrt{2k-1}}$. Then by (8) the complex number α is in the spectrum of the operator $\rho(\chi_1)$ exactly when it is of the form $\alpha = \gamma(z)$ for some $z \in \mathbb{C}$ with $|z| = \frac{1}{\sqrt{2k-1}}$, i.e. α belongs to the interval $[-2\sqrt{2k-1},\ 2\sqrt{2k-1}]$. Moreover in this case in view of (8), Corollary 1 and the fact that $I - zP^*$ is a bijection for every z, the operator $\alpha I - \rho(\chi_1)$ is one-to-one. In this way we have proved the following :

COROLLARY 2 ([2], *Theorem* 4). — *The spectrum of the operator*

$\rho(\chi_1)$ (the right side convolution operator with the function χ_1) on $\ell^2(\mathsf{F}_k)$ coincides with the interval $[-2\sqrt{2k-1},\ 2\sqrt{2k-1}]$. Moreover $\rho(\chi_1)$ has no eigenvectors in $\ell^2(\mathsf{F}_k)$.

3. The representations π_z and their decomposition into the direct sum.

This section is devoted to study the properties of the family of representations π_z, introduced in [11], in the case when the free group has finitely many free generators. For a reader's convenient we recall the definition of the family π_z and the main theorem concerning it.

For any complex number z the operator $I - zP$ is invertible on the set $\mathcal{K}(\mathsf{F})$ of functions with finite support in F, for the series $\sum_{n=0}^{\infty} z^n P^n f$ has only finitely many nonzero terms. Generally speaking the representation π_z is obtained by the conjugation of the regular representation λ with the operator $(I - zP)T_z$. For z, $|z| < 1$, $x \in \mathsf{F}$ and $f \in \mathcal{K}(\mathsf{F})$ set

$$\pi_z(x)f = T_z^{-1}(I - zP)^{-1}\lambda(x)(I - zP)T_z f \ . \tag{9}$$

In [11] was shown that π_z extends to a uniformly bounded representation of the group F on the whole space $\ell^2(\mathsf{F})$.

THEOREM 2 ([11], *Theorem* 1). — *Let F be a free group on arbitrary many generators. The representations π_z, $z \in D = \{z \in \mathbb{C} : |z| < 1\}$, form an analytic family of uniformly bounded representations of F on the Hilbert space $\ell^2(\mathsf{F})$. Moreover*

(i) $\|\pi_z(a)\| \le 2\dfrac{|1 - z^2|}{1 - |z|}$,

(ii) $\pi_z^*(a) = \pi_{\bar{z}}(a)^{-1}$,

(iii) $\pi_z(a) - \lambda(a)$ *has finite rank,*

(iv) $< \pi_z(x)\delta_e \ , \ \delta_e >= z^{|x|}$,

(v) *The representations π_z are cyclic with δ_e as the cyclic vector,*

(vi) *If the group* F *has infinitely many generators, then any representation* $\pi_z, z \neq 0$, *has no nontrivial closed invariant subspace. Any two different* π_z*'s are topologically inequivalent.*

The question arises whether the representations π_z are irreducible if the group F has finitely many generators. It is not true when $|z| < \frac{1}{\sqrt{2k-1}}$, because then the representation π_z is equivalent to the regular representation. This is due to the fact that the spectrum of P is the disc $|z| \leq \sqrt{2k-1}$. The conjecture that it is not true for all other z comes from the fact that π_z weakly contains the regular representation. The aim of this section is to show that for z with $\frac{1}{\sqrt{2k-1}} < |z| < 1$ the regular representation can be split off the representation π_z as its direct component. In the ensuing sections only the complementary component of the representation π_z will be of interest.

Since now we use the notation : $u = \frac{1}{(2k-1)z}$ for $z \neq 0$ and $\gamma(z) = (2k-1)z + \frac{1}{z}$.

For $z \in \mathbb{C}$, $\frac{1}{\sqrt{2k-1}} < |z| < 1$ define the operators U_z and R_z by

$$\begin{aligned} U_z &= \sqrt{\frac{u}{z}} T_u^{-1}(I - uP)^{-1}(I - zP)T_z \\ R_z &= I - U_{\bar{z}}^* U_z. \end{aligned} \tag{10}$$

The square root $\sqrt{\frac{u}{z}}$ is defined as $\sqrt{\frac{u}{z}} = \sqrt{2k-1}u$. The fundamental properties of operators U_z and R_z are listed below.

PROPOSITION 1. — *Let* $\frac{1}{\sqrt{2k-1}} < |z| < 1$. *Then*

(i) $U_z U_{\bar{z}}^* = I$,

(ii) R_z *is a projection and* $R_z^* = R_{\bar{z}}$,

(iii) $U_z \pi_z(x) = \pi_u(x) U_z$ *and* $\pi_z(x) U_{\bar{z}}^* = U_{\bar{z}}^* \pi_u(x)$,

(iv) $\operatorname{Ker} U_z = \operatorname{Ker} (I - zP)T_z$ *and* $\operatorname{Im} U_{\bar{z}}^* = \operatorname{Im} T_z(I - zP^*)$.

(v) $R_z \pi_z(x) = \pi_z(x) R_z$.

Proof. — Point (i) is the simple consequence of (7). Next (i) yields (ii). The first equality of (iii) follows from (9), then the second one is obtained by conjugation and by Theorem 2 (ii). Point (iv) holds for both $I-uP$ and $I-uP^*$ are invertible. Finally (iii) implies (v).

THEOREM 3. — *Assume* $\frac{1}{\sqrt{2k-1}} < |z| < 1$. *The subspace* $\mathrm{Im}T_z(I-zP^*)$ *is a closed subspace of* $\ell^2(\mathbb{F}_k)$ *and* Ker $(I-zP)T_z$ *is a complementary subspace not necessarily orthogonal i.e.* $\ell^2(\mathbb{F}_k) = \mathrm{Im}T_z(I-zP^*) \oplus \mathrm{Ker}(I-zP)T_z$. *Both the subspaces are invariant for the representations* π_z, *so* π_z *decomposes into the direct sum of two subrepresentations.*

The proof of Theorem 3 is derived from Proposition 1 and the following simple lemma which we state without proof.

LEMMA 1. — *Let* A *and* B *be bounded linear operators on a Hilbert space* $\mathcal{H}$ *such that their product* AB *is an invertible operator. Then :*

(i) *The subspace* $\mathrm{Im}B$ *is closed and the space* $\mathcal{H}$ *decomposes into the direct sum of the subspaces* $\mathrm{Im}B$ *and* Ker A.

(ii) *The operator* $B(AB)^{-1}A$ *is a projection (not always orthogonal) onto the space* $\mathrm{Im}B$ *along* Ker A.

(iii) *A linear operator* C *on the space* $\mathcal{H}$ *preserves both of the subspaces* $\mathrm{Im}B$ *and* Ker A *iff* C *commutes with the projection* $B(AB)^{-1}A$.

Proof of Theorem 3. — By Proposition 1 (i) the operators $A = U_z$ and $B = U^*_{\bar{z}}$ satisfy the hypotheses of Lemma 1. So the first part of the theorem follows from Lemma 1 (i) and Proposition 1 (iv). Next Proposition 1 (iii), (iv) gives the invariance of these two subspaces.

Remark 1. — Theorem 2 implies that both subrepresentations $\pi_z \,|_{\mathrm{Ker}(I-zP)T_z}$ and $\pi_z \,|_{\mathrm{Im}T_z(I-zP^*)}$ are uniformly bounded by $2\frac{|1-z^2|}{1-|z|}$. When z is a real number the decomposition of the space $\ell^2(\mathbb{F}_k)$ is orthogonal and both subrepresentations are unitary.

Remark 2. — In view of Lemma 1 and Proposition 1 the operator R_z is the projection onto $\mathrm{Ker}(I - zP)T_z$ along $\mathrm{Im}\, T_z(I - zP^*)$ which commutes with the action of the representation π_z. Applying Theorem 1 gives the explicit formula

$$R_z = I - \frac{(2k-1)u^2}{1-u^2} T_z(I - zP^*)\rho(u^{|x|})(I - zP)T_z \ . \tag{11}$$

Remark 3. — As δ_e is a cyclic vector for the representation π_z so $R_z\delta_e$ is a cyclic vector for the representation π_z restricted to the invariant subspace $\mathrm{Ker}(I - zP)T_z$.

Remark 4. — Now we can explain why in Section 1 we separated the subspace $\ell^2_r(\mathbb{F}_k)$ of radial functions in $\ell^2(\mathbb{F}_k)$. This subspace is invariant for the collection of the operators P, P^*, T_z, $\lambda(\chi_1)$ thus also for $\pi_z(\chi_1)$ and the projection R_z. All just listed operators preserve also the orthogonal complement $(\ell^2_r)^\perp$ to the subspace $\ell^2_r(\mathbb{F}_k)$.

THEOREM 4. — *Assume* $\frac{1}{\sqrt{2k-1}} < |z| < 1$. *Then the representation obtained by the restriction of the representation* π_z *to the subspace* $\mathrm{Im}\, T_z(I - zP^*)$ *is equivalent to the regular representation* λ.

Proof. — Proposition 1 (i), (iv) implies that the operator $U_{\bar{z}}$ maps $\ell^2(\mathbb{F}_k)$ onto $\mathrm{Im}\, T_z(I - zP^*)$ isomorphically. Furthermore by Proposition 1 (iii) $U_{\bar{z}}$ intertwines the representation $\pi_z \mid \mathrm{Im}\, T_z(I - zP^*)$ and the representation π_u, the latter being equivalent to the regular representation for $|u| < \frac{1}{\sqrt{2k-1}}$.

As we mentioned earlier from now on we will discuss only the second subrepresentation which occured in the decomposition in Theorem 3, namely the representation $\pi_z \mid_{\mathrm{Ker}(I-zP)T_z}$. In Section 4 we will show that this representation is irreducible. We will prove also that for nonreal z it cannot be made unitary.

To simplify the notation : $\mathcal{H}_z$ is the subspace $\mathrm{Ker}(I - zP)T_z$ and π'_z is the subrepresentation π_z restricted to $\mathcal{H}_z$.

4. Irreducibility of the representations π'_z.

In order to prove the irreducibility of π'_z we show that the orthogonal projection onto the cyclic vector of π'_z belongs to the norm closed algebra of operators generated by π'_z. Remind that M. Pimsner and D. Voiculescu ([9]) proved, solving an old problem of Kadison, that $C^*_\lambda(\mathbb{F}_k)$ the algebra associated with the regular representation contains no nontrivial projections. This fact extends to representations weakly contained in the regular representation. The existence of nontrivial projections clearly distinguishes the behaviour of π'_z from the behaviour of the regular representation.

The projection will be constructed from the operator $\pi_z(\chi_1)$ by analytic functional calculus. So first we have to indicate the spectrum of $\pi_z(\chi_1)$.

LEMMA 2. — *Let* $\frac{1}{\sqrt{2k-1}} < |z| < 1$. *The function* f_z

$$f_z = \delta_e + \frac{2k-1}{2k}\sqrt{1-z^2}\sum_{n=1}^{\infty} u^n \chi_n\ , \quad u = \frac{1}{(2k-1)z} \tag{12}$$

is the unique, up to the constant multiple, radial function in $\mathcal{H}_z =$ $\mathrm{Ker}(I - zP)T_z$.

Remark. — Since the function $R_z\delta_e$ lies in $\mathcal{H}_z$ and is radial (cf. Remarks 2, 4 following Theorem 3) hence $R_z = c_z f_z$ for some constant c_z. It means that f_z is a cyclic vector for π'_z.

LEMMA 3. — *Let* $\frac{1}{\sqrt{2k-1}} < |z| < 1$. *The function* f_z *is an eigenfunction of the operator* $\pi_z(\chi_1)$ *corresponding to the eigenvalue* $\gamma(z) = (2k-1)z + \frac{1}{z}$.

Proof. — Theorem 3 and Remark 4 following it implies that $\pi_z(\chi_1)f_z$ is a radial function which lies in $\mathcal{H}_z$. Now applying Lemma 2 gives $\pi_z(\chi_1)f_z = \alpha f_z$ for some complex number α. In order to determine α it remains to evaluate $\pi_z(\chi_1)f_z$ on a point x in $\mathbb{F}_k$, e.g. on e. Taking into account the identities $\pi_z^*(x) = \pi_{\overline{z}}(x^{-1})$ and $\pi_z(x)\delta_e = z\delta_e + \sqrt{1-z^2}\delta_x$ for $|x| = 1$ (compare [11], p. 12, (5)) we have

$$\alpha =< \pi_z(\chi_1)f_z, \delta_e >=< f_z, \pi_{\overline{z}}(\chi_1)\delta_e >=$$
$$< f_z, 2k\overline{z}\delta_e + \sqrt{1-\overline{z}^2}\chi_1 >= (2k-1)z + \frac{1}{z}\,.$$

Let's define an operator Q which acts similarly to P but on the left side, i.e. if $x_1x_2\ldots x_n$ is a reduced word then put $Q\delta_{x_1x_2\ldots x_n} = \delta_{x_2\ldots x_n}$ and $Q\delta_e = 0$. The operator Q has the same spectral properties as P and its adjoint operator Q^* acts like P^* but on the left side. In particular certain versions of results of Section 2 remain valid with Q instead of P. Analogously the formulas (5), (6), (7) and (8) are interchanged by

$$(13)\qquad Q + Q^* = \lambda(\chi_1),$$
$$(14)\qquad QQ^* = (2k-1)I + T,$$
$$(15)\qquad \frac{1}{z}(I - zQ)T_z^2(I - zQ^*) = \frac{1}{u}(I - uQ)T_u^2(I - uQ^*),$$
$$(16)\qquad \frac{1}{z}(I - zQ)T_z^2(I - zQ^*) = \gamma(z)I - \lambda(\chi_1).$$

The operators P and Q commute. P and Q^* do not commute however their commutator is finite dimensional, namely

$$(17)\qquad [P, Q^*] = (2k)T - J, \text{ where } Jf = \sum_{|x|=1} f(x^{-1})\delta_x.$$

LEMMA 4. — *Let $|z| < 1$. On the subspace $(\ell_r^2)^\perp$ the operator $\pi_z(\chi_1)$ is expressed as*

$$(18)\qquad \pi_z(\chi_1) = Q + Q^* - zJ.$$

Proof. — If $f \in (\ell_r^2)^\perp$ then $f(e) = 0$ and so $T_zf = f$. Thus by (13) $\pi_z(\chi_1) = (I - zP)^{-1}(Q + Q^*)(I - zP)$ on $(\ell_r^2)^\perp$. Since P and Q commute hence $(I - zP)^{-1}Q(I - zP) = Q$. On the other hand $(I - zP)^{-1}Q^*(I - zP) = Q^* + z(I - zP)^{-1}[P, Q^*]$ thus by (17) $(I - zP)^{-1}Q^*(I - zP) = Q^* - z(I - zP)^{-1}J$. As $PJ = 0$ on

$(\ell^2_r)^\perp$ so $(I - zP)^{-1}J = J$. Collecting all above yields the desired $\pi_z(\chi_1) = Q + Q^* - zJ$.

THEOREM 5. — *Let* $\frac{1}{\sqrt{2k-1}} < |z| < 1$. *The spectrum of the operator* $\pi_z(\chi_1)$ *consists of the interval* $[-2\sqrt{2k-1},\ 2\sqrt{2k-1}]$ *and the simple eigenvalue* $\gamma(z) = (2k-1)z + \frac{1}{z}\cdot\gamma(z)$ *is the only eigenvalue of* $\pi_z(\chi_1)$.

Proof. — In virtue of Theorem 4 the spectrum of $\pi_z(\chi_1)$ on the subspace Im $T_z(I - zP^*)$ coincides with the spectrum of $\lambda(\chi_1)$ on $\ell^2(\mathbb{F}_k)$. Therefore by Lemma 3 and by J.M. Cohen's theorem (cf. Corollary 2, Section 2) the set mentioned in the theorem is contained in the spectrum of $\pi_z(\chi_1)$.

In order to prove the containment in opposite direction we decompose $\ell^2(\mathbb{F}_k)$ into the direct sum of three subspaces invariant under $\pi_z(\chi_1)$, and then we examine the spectrum of $\pi_z(\chi_1)$ on each of them separately. Namely $\ell^2(\mathbb{F}_k) = \mathbb{C}f_z \oplus (\ell^2_r)^\perp \oplus \mathcal{M}$ where $\mathcal{M} = \ell^2_r \cap \mathrm{Im}\, T_z(I - zP^*)$. By Theorem 4 the spectrum of $\pi_z(\chi_1)$ on $\mathcal{M}$ is contained in $[-2\sqrt{2k-1}, 2\sqrt{2k-1}]$. Moreover by Corollary 2 $\pi_z(\chi_1)$ has no eigenvectors in $\mathcal{M}$. In view of lemma 3 it remains to consider $\pi_z(\chi_1)$ restricted to $(\ell^2_r)^\perp$. We claim that the spectrum of $\pi_z(\chi_1)$ on $(\ell^2_r)^\perp$ lies in $[-2\sqrt{2k-1}, 2\sqrt{2k-1}]$, too.

Every complex number α is of the form $\alpha = \gamma(z')$, where $|z'| \le \frac{1}{\sqrt{2k-1}}$. Then by (13), (16) and (18) there holds

$$\begin{aligned} \alpha I - \pi_z(\chi_1) &= \gamma(z')I - (Q + Q^*) + zJ \\ &= \frac{1}{z'}(I - z'Q)(I - z'Q^*) + zJ \\ &= \frac{1}{z'}(I - z'Q)(I + zz'J)(I - z'Q^*) \end{aligned} \tag{19}$$

on $(\ell^2_r)^\perp$. In calculation above we used also the formulas $QJ = JQ^* = 0$ valid on $(\ell^2_r)^\perp$. If α is outside $[-2\sqrt{2k-1}, 2\sqrt{2k-1}]$ then $|z'| < \frac{1}{\sqrt{2k-1}}$ and the operators $I - z'Q$ and $I - z'Q^*$ are invertible. $I + zz'J$ is also invertible for $|zz'| < |z| < 1$ and J is a contraction. This implies that $\alpha I - \pi_z(\chi_1)$ is invertible on $(\ell^2_r)^\perp$ and consequently the spectrum of $\pi_z(\chi_1)$ is contained in the interval mentioned above.

If α belongs to $[-2\sqrt{2k-1}, 2\sqrt{2k-1}]$ then $|z'| = \frac{1}{\sqrt{2k-1}}$. In this case by Corollary 3 (applied to Q) all the factors representing $\alpha I - \pi_z(\chi_1)$ in (19) are injections ($I + zz'J$ remains invertible). It means $\pi_z(\chi_1)$ has no eigenvectors in $(\ell_r^2)^\perp$. This completes the proof.

Remark 1. — Collecting results of the preceeding proof gives that the whole space $\ell^2(\mathbb{F}_k)$ decomposes into the direct sum of two subspaces invariant under $\pi_z(\chi_1)$: the one-dimensional subspace spanned by the eigenvector f_z and the subspace $(\ell_r^2)^\perp \oplus (\ell_r^2 \cap \mathrm{Im} T_z (I - zP^*))$. The spectrum of $\pi_z(\chi_1)$ on the second space coincides with $[-2\sqrt{2k-1}, 2\sqrt{2k-1}]$.

Remark 2. — Theorem 5 yields that the spectrum of $\pi'_z(\chi_1)$ is contained in $[-2\sqrt{2k-1}, 2\sqrt{2k-1}] \cup \{\gamma(z)\}$ and $\gamma(z)$ is a unique and simple eigenvalue of $\pi'_z(\chi_1)$. Actually the spectrum of $\pi'_z(\chi_1)$ contains entire interval. To see this it suffices (cf. (19)) to check that the image of $I - z'Q, |z'| = \frac{1}{\sqrt{2k-1}}$ does not contain $\mathcal{H}_z = \mathrm{Ker}(I - zP)T_z$.

THEOREM 6. — *Let* $\frac{1}{\sqrt{2k-1}} < |z| < 1$. *The subspace* $\mathcal{H}_z = \mathrm{Ker}(I - zP)T_z$ *contains no nontrivial closed invariant subspace of the representation* π_z. *It means the representation* π'_z *obtained by the restriction of* π_z *to the subspace* $\mathcal{H}_z$ *is irreducible. The representations* π'_z *are mutally inequivalent.*

Proof. — Let C be the circle centered at $\gamma(z)$ with radius so small that the interval $[-2\sqrt{2k-1}, 2\sqrt{2k-1}]$ lies outside C. Define the operator A on $\mathcal{H}_z$ by

$$A = \frac{1}{2\pi i}\int_C (\zeta - \pi'_z(\chi_1))^{-1} d\zeta.$$

By Remark 1 the operator A is the projection onto $\mathbb{C}f_z$ along $(\ell_r^2)^\perp \cap \mathcal{H}_z$. Actually A is an orthogonal projection for f_z is orthogonal to $(\ell_r^2)^\perp$. So the projection onto the cyclic vector of π'_z (cf. Remark following Lemma 2) belongs to $C_{\pi'_z}$ the norm closed algebra generated by π'_z. Now we may argue in a routine way. Let $\mathcal{M}$ be a nonzero closed subspace of $\mathcal{H}_z$ invariant under π'_z. Then $A\mathcal{M} \subset \mathcal{M}$ and there are two cases to consider : $A\mathcal{M} = \mathbb{C}f_z$ or $A\mathcal{M} = 0$.

The first case implies $\mathcal{M} = \mathcal{H}_z$ for f_z is a cyclic vector of π'_z. The second case implies $\mathcal{M} = 0$. Indeed, observe that $Ag = g(e)f_z$ for $g \in \mathcal{H}_z$. So if $A\mathcal{M} = 0$ then $g(e) = 0$ for each $g \in \mathcal{M}$. It means that $0 = < \pi_z(x)g, \delta_e > = < g, \pi_{\bar{z}}(x^{-1})\delta_e >$ for each $g \in \mathcal{M}$ and $x \in \mathbb{F}_k$. Taking into account that δ_e is a cyclic vector of $\pi_{\bar{z}}$ gives $g = 0$.

As regards to inequivalence if $z \neq z'$ then the spectrum of $\pi'_z(\chi_1)$ does not coincide with $\pi'_{z'}(\chi_1)$ for $\gamma(z) \neq \gamma(z')$. Therefore the representations π'_z and $\pi'_{z'}$ cannot be equivalent.

PROPOSITION 2. — *Assume* $\frac{1}{\sqrt{2k-1}} < |z| < 1$ *and* $z \notin \mathbb{R}$. *Then the representation* π'_z, *as well as* π_z, *cannot be equivalent to any unitary representation.*

Proof. — The point is that the spectrum of the operator $\pi'_z(\chi_1)$ is unreal, while the function χ_1 is hermitian.

Remark. — Since $C_{\pi'_z}$ contains the orthogonal projection onto the cyclic vector of π'_z hence it contains all compact operators.

5. An analytic series of irreducible representations.

In this section we are going to show that the representations π'_z, $\frac{1}{\sqrt{2k-1}} < |z| < 1$, can be settled on a common Hilbert space, on which they form an analytic family of representations. Concerning the related group $SL(2,\mathbb{R})$ its irreducible representations of the principal series together with its analytic continuation, work on a common Hilbert space (see [7]). So it is reasonable to expect that the theorem like this would hold in the case of free group.

One may proceed in the following manner : to fix a number z_0, and to try to map all subspaces $\mathcal{H}_z$ onto $\mathcal{H}_{z_0}$. However no one of z with $\frac{1}{\sqrt{2k-1}} < |z| < 1$ does not distinguish from others in a natural way. Let us observe that the nonradial parts (i.e. the parts orthogonal to ℓ^2_r) of $\mathcal{H}_z = \mathrm{Ker}(I - zP)T_z$ and $\mathrm{Ker}(I - zP)$ are equal (the radial parts of these subspaces are one-dimensional, cf. Lemma 2). So instead of $\mathcal{H}_z$ consider the subspaces $\mathrm{Ker}(I - zP)$, now with

$|z| > \frac{1}{\sqrt{2k-1}}$. Next since $\mathrm{Ker}(I - zP) = \mathrm{Ker}(P - \frac{1}{z}I)$ hence if we let z tend to infinity we will "get" Ker P. This is the space on which we are going to settle the representations π'_z.

Denote $\mathcal{H}_\infty = \mathrm{Ker}P$. Observe that δ_e is the only radial function in $\mathcal{H}_\infty$. We will transform the subspaces $\mathcal{H}_z$ onto $\mathcal{H}_\infty$. We will do that separately for the radial and nonradial parts of $\mathcal{H}_z$ and $\mathcal{H}_\infty$. Fortunately there are no difficulties with the radial parts for these are one-dimensional.

If $\mathcal{M} \subset \ell^2(\mathbb{F}_k)$ then $\mathcal{M}^0$ will denote the nonradial part of $\mathcal{M}$, i.e. $\mathcal{M}^0 = \mathcal{M} \cap (\ell^2_r)^\perp$.

PROPOSITION 3. — *Let* $|z| > \frac{1}{2k-1}$. *The operator* $(I - uP^*)^{-1}$ *maps* $\mathcal{H}^0_\infty$ *onto* $\mathcal{H}^0_z$ *isomorphically, where* $u = \frac{1}{(2k-1)z}$.

Proof. — The statement follows immediately from the formulas

$$\text{(20)} \qquad \begin{aligned} P(I - uP^*) &= -\frac{1}{z}(I - zP) \\ (I - zP)(I - uP^*)^{-1} &= -zP \end{aligned}$$

valid on $(\ell^2_r)^\perp$. As regards to the first equality :

$$\begin{aligned} P(I - uP^*) = P - uPP^* &= P - (2k-1)uI \\ &= P - \frac{1}{z}I = -\frac{1}{z}(I - zP). \end{aligned}$$

Now the second equality follows from the first one.

LEMMA 4. — *Let* $|z| > \frac{1}{2k-1}$. *Then*

$$\text{(21)} \qquad R_z(I - uP^*) = \frac{z-u}{z}R_z$$

$$\text{(22)} \qquad R_z(I - uP)(I - uP^*)R_z = \frac{z-u}{z}R_z$$

on the subspace $(\ell^2_r)^\perp$.

Proof. — First let us note that in all the formulas we may omit the operator T_z for $T_z f = f$ whenever $f \in (\ell^2_r)^\perp$. Then by Remark

2 following Theorem 3 $R_z(I - zP^*) = 0$ and so $R_z P^* = \frac{1}{z} R_z$. This implies (21). Next using (20) and $(I - zP)R_z = 0$ gives

$$R_z(I - uP)(I - uP^*)R_z = R_z(I - uP^*)R_z - uR_z P(I - uP^*)R_z = \frac{z-u}{z} R_z - \frac{u}{z} R_z(I - zP)R_z = \frac{z-u}{z} R_z.$$

PROPOSITION 4. — *Let $|z| > \frac{1}{\sqrt{2k-1}}$. Then $R_z = \frac{z-u}{z} (I - uP^*)^{-1}$ on $\mathcal{H}^0_\infty$. The operator $\sqrt{\frac{z}{z-u}} R_z$ maps $\mathcal{H}^0_\infty$ onto $\mathcal{H}^0_z$ isomorphically. If z is real then this mapping is an isometry.*

Proof. — (21) implies $R_z(I - uP^*) = \frac{z-u}{z} I$ on $\mathcal{H}^0_z$. Then by Proposition 3

$$(I - uP^*)R_z = \frac{z-u}{z} I \text{ on } \mathcal{H}^0_\infty. \tag{23}$$

The above means $R_z = \frac{z-u}{z}(I - uP^*)^{-1}$ on $\mathcal{H}^0_\infty$. So by Proposition 3 the operator $\sqrt{\frac{z}{z-u}} R_z$ maps $\mathcal{H}^0_\infty$ onto $\mathcal{H}^0_z$ isomorphically.

Let $f,\ g \in (\ell^2_r)^\perp$. Then by (22)

$$< (I - uP^*)R_z f, (I - \bar{u}P^*)R_{\bar{z}} g > = < R_z(I - uP)(I - uP^*)R_z f, g > = \frac{z-u}{z} < R_z f, R_{\bar{z}} g > .$$

Assume $f,\ g \in \mathcal{H}^0_\infty$. Then by (23)

$$\left(\frac{z-u}{z}\right)^2 < f, g > = \frac{z-u}{z} < R_z f, R_{\bar{z}} g > .$$

This gives the last part of Proposition 4.

For $\frac{1}{\sqrt{2k-1}} < |z| < 1$ define the operator $V_z : \mathcal{H}_\infty \to \mathcal{H}_z$ by the rule

$$V_z f = \sqrt{\frac{z}{z-u}} R_z f \qquad \text{if } f \in \mathcal{H}^o_\infty$$

$$V_z \delta_e = c_u \sqrt{\frac{z}{z-u}} R_z \delta_e, \text{ where } c_u = \sqrt{\frac{2k-1}{2k}} \sqrt{1-u^2}. \tag{24}$$

The constant c_u is chosen to satisfy $< V_z\delta_e, V_{\bar{z}}\delta_e >= 1$.

THEOREM 7. — *Let* $\frac{1}{\sqrt{2k-1}} < |z| < 1$. *The operator* V_z *maps the space* $\mathcal{H}_\infty$ *onto the space* $\mathcal{H}_z$ *isomorphically. If* z *is real* V_z *is an isometry. Moreover*

$$< V_z f, V_{\bar{z}} g >=< f, g >, \quad f,\ g \in \mathcal{H}_\infty. \tag{25}$$

Proof. — Except (25) the theorem follows from Proposition 4 and from (24). By Proposition 2 and the remark following (24) the formula (25) holds for real z. Then by the analycity of the function $z \mapsto < V_z f, V_{\bar{z}} g >$ this formula extends on other z.

With the aid of the isomorphisms V_z we can move the representations π'_z to the space $\mathcal{H}_\infty$, this way obtaining the representations $V_z^{-1}\pi'_z(x)V_z$. In order to find an explicit formula for the action of the representations on the new space we look at the matrix coefficients.

Let $f,\ g \in \mathcal{H}^0_\infty$. Combining (25), (24), Proposition 1 and (10) gives

$$\begin{aligned}
&< V_z^{-1}\pi'_z(x)V_z f, g >=< \pi'_z(x)V_z f, V_{\bar{z}} g > \\
&\quad = \frac{z}{z-u} < \pi'_z(x)R_z f, R_{\bar{z}} g >= \frac{z}{z-u} < R_z\pi_z(x)f, g > \\
&\quad = \frac{z}{z-u} < \pi_z(x)f, g > - \frac{z}{z-u} < U^*_{\bar{z}}U_z\pi_z(x)f, g > \\
&\quad = \frac{z}{z-u} < \pi_z(x)f, g > - \frac{z}{z-u} < \pi_u(x)U_z f, U_{\bar{z}} g > .
\end{aligned}$$

Since $U_z f = \sqrt{\frac{u}{z}} f$ and $U_{\bar{z}} g = \sqrt{\frac{\bar{u}}{\bar{z}}} g$, if $f,\ g \in \mathcal{H}^0_\infty$, hence

$$< V_z^{-1}\pi'_z(x)V_z f, g >= \frac{1}{z-u} < [z\pi_z(x) - u\pi_u(x)]f, g > . \tag{26}$$

In the same way we derive the remaining formulas

$$\begin{aligned}
&< V_z^{-1}\pi'_z(x)V_z\delta_e, g >= \\
&\qquad \frac{\sqrt{c(z)}}{z-u} < \left[\frac{z}{\sqrt{1-z^2}}\pi_z(x) - \frac{u}{\sqrt{1-u^2}}\pi_u(x)\right]\delta_e, g >
\end{aligned}$$

(27)

$$< V_z^{-1}\pi'_z(x)V_z\delta_e, \delta_e >= \frac{c(z)}{z-u} < \left[\frac{z}{1-z^2}\pi_z(x) - \frac{u}{1-u^2}\pi_u(x)\right]\delta_e, \delta_e >$$

for $g \in \mathcal{H}^0_\infty$, where $c(z) = \frac{2k-1}{2k}(1-z^2)(1-u^2)$.

THEOREM 8. — *Let* $\mathbf{F}_k$ *be a free group on* k *generators. There exists an analytic series of uniformly bounded representations* $\Pi_z, \frac{1}{2k-1} < |z| < 1$, *of the group* $\mathbf{F}_k$ *on the Hilbert space* $\mathcal{H}_\infty = \operatorname{Ker} P$, *such that*

(i) *If* $\frac{1}{\sqrt{2k-1}} < |z| < 1$ *then* Π_z *is equivalent to* π'_z.

(ii) $\Pi_z = \Pi_u$, *where* $u = \frac{1}{(2k-1)z}$.

(iii) $\Pi^*_z(x) = \Pi_{\bar{z}}(x)^{-1}$.

(iv) $\Pi_z(x) - \Pi_{z'}(x)$ *has finite rank.*

(v) *Any representation* Π_z *is irreducible. The representations* Π_z *and* $\Pi_{z'}$ *are equivalent iff* $z = z'$ *or* $z' = \frac{1}{(2k-1)z}$.

(vi) Π_z *is a unitary representation if* $|z| = \frac{1}{\sqrt{2k-1}}$ *or* z *is real. In other case the representation* Π_z *cannot be made unitary.*

Proof. — By (26) and (27) the family $V_z^{-1}\pi'_z(x)V_z, \frac{1}{\sqrt{2k-1}} < |z| < 1$, extends to the analytic family $\Pi_z, \frac{1}{2k-1} < |z| < 1$, satysfying (i) and (ii). Theorem 2 (iii) and (26) imply (iv). Next for real z the representation Π_z is unitary for in that case π'_z is a unitary representation and V_z is an isometry. Then (iii) holds for real z, therefore by analyticity it remains valid for other z. Consider (vi) : by (iii) if z is real then Π_z is a unitary representation. Furthermore if $|z| = \frac{1}{\sqrt{2k-1}}$ then $u = \bar{z}$. Hence by (ii) and (iii)

$$\Pi^*_z(x) = \Pi_{\bar{z}}(x)^{-1} = \Pi_u(x)^{-1} = \Pi_z(x)^{-1}.$$

It means Π_z is a unitary representation. Observe that by Lemma 3 and by (24) δ_e is an eigenvector of $\Pi_z(\chi_1)$ corresponding to the eigenvalue $\gamma(z)$. But $\gamma(z)$ is a real number if and only if z is real or $|z| = \frac{1}{\sqrt{2k-1}}$. In other cases $\Pi_z(\chi_1)$ is nonselfadjoint hence Π_z cannot be equivalent to any unitary representation.

It remains to show (v). In view of (i) and (ii) we have to discuss the case $|z| = \frac{1}{\sqrt{2k-1}}$ only. The following lemma is a key one.

LEMMA 5. — *Let* $|z| = \frac{1}{\sqrt{2k-1}}$. *Then* δ_e *is the only eigenvector of the operator* $\Pi_z(\chi_1)$. *So* $\gamma(z)$ *is the unique eigenvalue of* $\Pi_z(\chi_1)$.

Proof. — Suppose $\Pi_z(\chi_1)f = \lambda f$, $f \in \mathcal{H}_\infty$. We may assume that f is orthogonal to δ_e, i.e. $f \in \mathcal{H}_\infty^0$. Then by (27)

$$0 =< [\lambda I - \Pi_z(\chi_1)]f, g >=< \left[\lambda I - \frac{z\pi_z(\chi_1) - u\pi_u(\chi_1)}{z-u}\right] f, g >$$

for each $g \in \mathcal{H}_\infty^0$. It means $\left[\lambda I - \frac{z\pi_z(\chi_1) - u\pi_u(\chi_1)}{z-u}\right] f = P^*h$ for some $h \in (\ell_r^2)^\perp$. We may write λ as $\lambda = \gamma(z')$, with $|z'| \le \frac{1}{\sqrt{2k-1}}$. Next using Lemma 4 and repeating the transformations as in (19) leads to

$$\frac{1}{z'}(I - z'Q)[I + z'(z+u)J](I - z'Q^*)f = P^*h. \tag{28}$$

Applying the operator P to both sides of (28) and using the identities $PJ = QJ = 0$, $Q^*P - PQ^* = J$ and $PP^* = (2k-1)I$ gives

$$Jf = (2k-1)h\ . \tag{29}$$

It means h is supported by the words of length 1. This implies $Q^2P^*h = 0$ for $h \in (\ell_r^2)^\perp$, so $P^*h = (I - z'Q)(I + z'Q)P^*h$. As $I - z'Q$ is a bijection (cf. Proposition 1) so the latter and (28) imply

$$\frac{1}{z'}[I + z'(z+u)J](I - z'Q^*)f = (I + z'Q)P^*h = P^*h - z'Jh.$$

Next applying J to the above identity and using $JP^* = JQ^* = 0$ yields

$$\frac{1}{z'}Jf + (z+u)J^2f = -z'J^2h = -z'h.$$

By (29) the above gives $Jh = -\frac{\gamma(z'^{-1})}{\gamma(z)}h$. As $J^3 = J$ so the spectrum of J consists of -1, 0 and 1. The assumption $h \neq 0$ would imply $\gamma(z'^{-1}) = \pm\gamma(z)$. However it cannot hold for $|z'^{-1}| \geq \sqrt{2k-1} > 1$ and $\frac{1}{2k-1} < |z| < 1$. Therefore $h = 0$ and by (28)

$$(I - z'Q)[I + z'(z+u)J](I - z'Q^*)f = 0 .$$

As (cf. the proof of Theorem 5) the factors $I - z'Q$, $I - z'Q^*$ and $I + z'(z+u)J$ are injections for $|z'| \leq \frac{1}{\sqrt{2k-1}}$ and $|z'(z+u)| < 1$, so $f = 0$. This completes the proof of Lemma 5.

Return to the proof of Theorem 8(v). First we show that δ_e is a cyclic vector of Π_z. Assume $< \Pi_z(x)\delta_e, f >= 0$ for each x in $\mathbb{F}_k$ and some f in $\mathcal{H}_\infty$. Then $f \in \mathcal{H}_\infty^0$ for $< \delta_e, f >= 0$. We are going to show that $f(x) = 0$ for each x, by induction on the length $|x|$. Suppose $f(x) = 0$ for $|x| < n$. Let $|z| = n$, then by (27)

$$< \left[\frac{z}{\sqrt{1-z^2}}\pi_z(x) - \frac{u}{\sqrt{1-u^2}}\pi_u(x)\right]\delta_e, f >= 0 .$$

Applying $\pi_z(x)\delta_e = z^{|x|}\delta_e + \sqrt{1-z^2}\sum_{n=0}^{|x|-1} z^n P^n \delta_x$ yields $< \delta_x, f >= 0$ (cf. [11], (5)). Hence $f = 0$ and δ_e is a cyclic vector.

By Lemma 5 the orthogonal projection T onto δ_e belongs to the von Neumann algebra generated by Π_z. Indeed, if E_λ denotes the spectral resolution of the identity corresponding to the selfadjoint operator $\Pi_z(\chi_1)$ and $c = \gamma(z)$ then by theorem of Lorch $T = \lim_{\varepsilon \to 0+}(E_c - E_{c-\varepsilon})$. Now repeating the routine arguments (cf. the proof of Theorem 5) implies Π_z is irreducible.

The inequivalence follows from Theorem 5 and Lemma 5 for $\gamma(z) = \gamma(z')$ if and only if $z = z'$ or $z' = \frac{1}{(2k-1)z}$. This completes the proof of Theorem 8.

Remark. — Define the function $\psi_z(x) =< \Pi_z(x)\delta_e, \delta_e >$, $x \in \mathbb{F}_k$, $\frac{1}{2k-1} < |z| < 1$. By (27) and by Theorem 2 (iv) ψ_z is a radial function. Moreover the property $\Pi_z(\chi_1)\delta_e = \gamma(z)\delta_e$ implies

$$\chi_1 * \psi_z = \gamma(z)\psi_z .$$

So ψ_z is a spherical function in the sense of Cartier [1] (cf. also [4]). By (27) we can derive an explicit formula

$$\psi_z(x) = \frac{2k-1}{2k}\left[\frac{z(1-u^2)}{z-u}z^{|x|} - \frac{u(1-z^2)}{z-u}u^{|x|}\right].$$

If z is a real number or $|z| = \frac{1}{\sqrt{2k-1}}$ then ψ_z is positive definite and it determines the representation Π_z up to the equivalence.

In [4] the classification of all spherical functions is presented. They are denoted by φ_z, where z belongs to the rectangle $S = \{x + iy : 0 \leq x \leq 1,\ 0 \leq y < \frac{2\pi}{\log(2k-1)}\}$. Consider the analytic mapping $h(z) = (2k-1)^{-z}$ on S. The function $h(z)$ is single-valued and maps the rectangle S onto the annulus $\frac{1}{2k-1} < |z| < 1$. In particular the segment [0,1] is mapped onto the segment $\left[\frac{1}{2k-1}, 1\right]$, the segment $\left\{z : 0 \leq x \leq 1,\ y = \frac{\pi}{\log(2k-1)}\right\}$ – onto $\left[-1, -\frac{1}{2k-1}\right]$, the vertical segment $\left\{z \in S : x = \frac{1}{2}\right\}$ – onto the inner circle $\left\{z : |z| = \frac{1}{\sqrt{2k-1}}\right\}$. It is easy to see that in the notation we apply we have $\varphi_z = \psi_{h(z)}$, $z \in S$. In the papers [4], [8] for an arbitrary $z \in S$ the construction of a representation with the matrix coefficient equal to φ_z, is given. It is shown there ([4]) that for real z or $z = \frac{1}{2} + iy$ the representation constructed are irreducible. Since they have equal associate positive definite functions, it turns out that our irreducible unitary representations Π_z are unitary equivalent to the principal and complementary series in [4] and [8].

Added in proof. After submission of the manuscript A. M. Mantero, T. Pytlik and A. Zappa proved that the family Π_z of the present paper is isomorphic to the family of uniformly bounded representations of [8]. This was announced during Conference on Harmonic Analysis in Karpacz (Poland) in January 1987.

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