INHOMOGENEOUS JACOBI MATRICES ON TREES

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Abstract. We study Jacobi matrices on directed trees with one end at inifinity. We show that the defect indices cannot be greater than 1 and give criteria for essential self-adjointness. We construct certain polynomials associated with matrices which mimic orthogonal polynomials in the classical case. Nonnegativity of Jacobi matrices is studied as well.

1. Introduction

The aim of the paper is to study a special class of symmetric unbounded operators and their spectral properties. These are Jacobi operators defined on directed trees. They are immediate generalizations of classical Jacobi matrices which act on sequences $\{u_n\}_{n=0}^{\infty}$ by the rule

$$(Ju)_n = \lambda_n u_{n+1} + \beta_n u_n + \lambda_{n-1} u_{n-1}, \quad n \geq 0,$$

where $\{\lambda_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of positive and real numbers, respectively, with the convention $u_{-1} = \lambda_{-1} = 0$. These matrices are closely related with the set of polynomials defined recursively by

$$xp_n(x) = \lambda_n p_{n+1}(x) + \beta_n p_n(x) + \lambda_{n-1} p_{n-1}(x), \quad n \geq 0, \tag{1}$$

with $p_{-1} = 0,\ p_0 = 1$.

In case the coefficients of the matrix are bounded, the matrix J represents self-adjoint operator on $\ell^2(\mathbb{N}_0)$. If $E(x)$ denotes the resolution of identity associated to J, then the polynomials $p_n(x)$ are orthonormal with respect to the measure $d\mu(x) = d(E(x)e_0, e_0)$, where e_0 is the sequence taking value 1 at $n = 0$ and vanishing elsewhere and (u, v) denotes the standard inner product in $\ell^2(\mathbb{N}_0)$. The measure μ has bounded support.

When the coefficients are unbounded the operator J is well defined on the domain $D(J)$ consisting of sequences with finitely many nonzero terms. In that case, if this operator is essentially self-adjoint then again the polynomials p_n are orthonormal with respect to the measure $d\mu(x) = d(E(x)e_0, e_0)$, except that this measure has unbounded support. Moreover there is a unique orthogonality measure for polynomials p_n. By a classical theorem, if the operator J is not

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essentially self-adjoint, there are many measures μ on the real line so that the polynomials belong to $L^2(\mu)$, i.e.

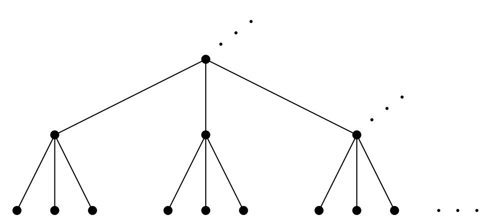
$$\int_{-\infty}^{\infty} x^{2n}\, d\mu(x) < \infty, \qquad n \in \mathbb{N}_0,$$

and the polynomials p_n are orthogonal with respect to the inner product

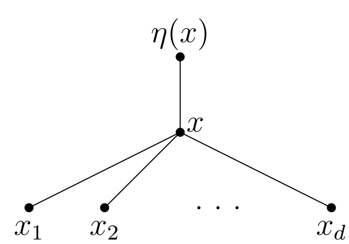
$$(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, d\mu(x).$$

Therefore essential self-adjointness is a crucial property that distinguishes between the so called determinate and indeterminate cases. Intuitively the unbounded matrix J is essentially self-adjoint when the coefficients have moderate growth. But the converse is not true in general. For the classical theory of Jacobi matrices, orthogonal polynomials and moment problems we address the reader to [1], [2], [6], and to [5] for a modern treatment.

In a recent paper [3] homogeneous Jacobi matrices on directed homogeneous trees were studied. Two types of homogeneous trees were considered. One of them was the tree with infinitely many origin points called leaves (on height 0) and one end at infinity.



The tree Γ consists of vertices with heights from zero to infinity. Every vertex x with height $n \geq 1$ is connected with a unique vertex $\eta(x)$, the parent, with height $n+1$, and d vertices $x_1, \ldots, x_d$ with height $n-1$, the children, like in the figure below:



The Jacobi matrices were defined on $\ell^2(\Gamma)$, where Γ denotes the set of all vertices of the tree. The formula is as follows

$$(Jv)(x) = \lambda_n v(\eta(x)) + \beta_n v(x) + \lambda_{n-1}[v(x_1) + v(x_2) + \ldots + v(x_d)],$$

where n denotes the height of the vertex x.

An interesting phenomenon occured. It turned out that the operator J defined on functions $\{v(x)\}_{x\in\Gamma}$, with finitely many nonzero terms, is always essentially self-adjoint, regardless of the growth of the coefficients λ_n and β_n. For example the operator J with coefficients $\lambda_n = (n+1)^2$ and $\beta_n = 0$ is not essentially self-adjoint when considered as the classical Jacobi matrix on $\ell^2(\mathbb{N}_0)$. But it is essentialy self-adjoint when it acts on $\ell^2(\Gamma)$.

Moreover its spectrum is discrete and consists of the zeros of all the polynomials p_n associated with classical Jacobi matrix with coefficients $\sqrt{d}\,\lambda_n$ and β_n, i.e. satisfying

$$xp_n(x) = \sqrt{d}\,\lambda_n p_{n+1}(x) + \beta_n p_n(x) + \sqrt{d}\,\lambda_{n-1}p_{n-1}(x), \quad n \geq 0.$$

Every eigenvalue is of infinite multiplicity.

Our aim is to study *inhomogeneous* Jacobi matrix on that tree. This means we do not require that the coefficients of the matrix depend only on the height of the vertex. With every vertex x we associate a positive number λ_x and a real number β_x. We are going to study operators of the form

$$Jv(x) = \lambda_x v(\eta(x)) + \beta_x v(x) + \lambda_{x_1} v(x_1) + \lambda_{x_2} v(x_2) + \ldots + \lambda_{x_d} v(x_d).$$

One of the main differences between the classical case and the case of the tree Γ is that the eigenvalue equation

$$zv(x) = \lambda_x v(\eta(x)) + \beta_x v(x) + \lambda_{x_1} v(x_1) + \lambda_{x_2} v(x_2) + \ldots + \lambda_{x_d} v(x_d) \tag{2}$$

cannot be solved recursively, unlike the equation

$$zv(n) = \lambda_n v(n+1) + \beta_n v(n) + \lambda_{n-1} v(n-1).$$

This not a coincidence as we are going to show that the equation (2) may not admit nonzero solutions for real values of z (cf. Proposition 5). But we will show the equation has a nonzero solution for every nonreal z (Corollary 5).

Actually, when we give up homogeneity of the matrix J, we can as well give up homogoneity of the tree. This means the number of descendants of vertices of Γ is not fixed, i.e. the quantities $\#\eta^{-1}(x)$ may vary.

The operator J is symmetric on $\ell^2(\Gamma)$ with respect to the natural inner product

$$(u, v) = \sum_{x\in\Gamma} u(x)\overline{v(x)}.$$

We are interested in studying the essential self-adjointness of the matrix J. It turns out that unlike in homogeneous case, the matrix J may not be essentially self-adjoint. However the defect indices cannot be greater than 1 (Corollary

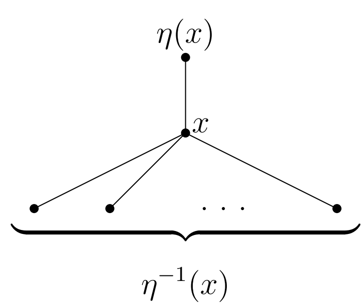
5). We derive certain criteria assuring essential self-adjointness. For example the analog of Carleman condition holds (see Theorem 21). Moreover we relate essential self-adjointness of J with essential self-adjointness of the classical Jacobi matrix J_0 obtained from J by restriction to an infinite path of the tree (see Theorem 13 and Remark following its proof).

Classical Jacobi matrices are associated with orthogonal polynomials through the formula (1). In case of the tree Γ there is no natural way of defining polynomials associated with Jacobi matrices on Γ, since (as was mentioned above) the eigenvalue equation may be not solvable. In Section 3 we define certain polynomials associated with J. We prove that they have real and simple zeros. Also we show interlacing property for roots of two consecutive polynomials. However, unlike in the classical case, there is no natural orthogonality relation between these polynomials. The most important reason to study these objects is the fact that the roots of these polynomials describe the eigenvalues of the restriction of J to finite subtrees of Γ. However, unlike in the classical case, it may occur that these eigenvalues are multiple.

In Section 5 we give a criterion for nonnegativity of the Jacobi matrix J on Γ. In the classical case the Jacobi matrix J is positive definite if and only if $(-1)^n p_n(0) > 0$ for every n, where p_n are the orthogonal polynomials associated with J. In case of tree Γ we do not have solutions of eigenvalue problem at our disposal or orthogonal polynomials. Therefore we had to find another way of getting the result. The nonnegativity of the matrix J proved to be a major tool in construction of a Jacobi matrix on Γ for which the eigenvalue equation (2) does not admit solutions for some real values.

2. Definitions and basic properties

We will consider a tree Γ with one end at infinity. Its vertices are located on heights from zero to infinity. Every vertex x with height $n \geq 0$ is directly connected with a unique vertex $\eta(x)$ with height $n+1$, the parent. When $n \geq 1$ the vertex x is thus directly connected with a finite number of vertices y on height $n-1$, called its children. The set of children of x will be denoted by $\eta^{-1}(x)$. The number of vertices in $\eta^{-1}(x)$ may vary with x. Let $l(x)$ denote the height of the vertex x.



For a given vertex x let Γ_x denote the finite subtree containing the vertex x together with all its descendants, i.e. vertices y such that $\eta^k(y) = x$ for some k. Thus $l(y) = l(x) - k$.

Define $\mathcal{F}(\Gamma)$ to be the set of all complex valued functions with finite support on Γ. Let

$$\delta_x(y) = \begin{cases} 1 & y = x, \\ 0 & y \neq x. \end{cases}$$

Consider the operator J acting on $\mathcal{F}(\Gamma)$ according to the rule

$$J\delta_x = \lambda_x \delta_{\eta(x)} + \beta_x \delta_x + \sum_{y \in \eta^{-1}(x)} \lambda_y \delta_y, \qquad l(x) > 1, \tag{3}$$

$$J\delta_x = \lambda_x \delta_{\eta(x)} + \beta_x \delta_x, \qquad l(x) = 0, \tag{4}$$

where λ_x are positive constants while β_x are real ones. Let S be the operator acting by the rule

$$S\delta_x = \lambda_x \delta_{\eta(x)}.$$

Then the adjoint operator S^* is given by

$$S^*\delta_x = \begin{cases} \displaystyle\sum_{y \in \eta^{-1}(x)} \lambda_y \delta_y, & l(x) > 0, \\ 0, & l(x) = 0. \end{cases}$$

The operators S and S^* are straightforward generalizations of weighted shift and backward weighted shift operators usually acting on $\ell^2(\mathbb{N}_0)$. Let M be a multiplication operator defined by

$$M\delta_x = \beta_x \delta_x.$$

Then

$$J = S + S^* + M. \tag{5}$$

In particular J is a symmetric linear operator.

We will study formal eigenfunctions of the operator J, i.e. functions v defined on Γ and satisfying

$$Jv = zv.$$

Evaluation at the vertex x gives that equivalently we have the recurrence relation

$$zv(x) = \begin{cases} \lambda_x v(\eta(x)) + \beta_x v(x) + \displaystyle\sum_{y \in \eta^{-1}(x)} \lambda_y v(y) & l(x) \geq 1, \\ \lambda_x v(\eta(x)) + \beta_x v(x) & l(x) = 0. \end{cases} \tag{6}$$

Since $\eta^{-1}(x)=\emptyset$ for $l(x)=0$ we may simplify the notation and (6) takes the form

$$zv(x)=\lambda_x v(\eta(x))+\beta_x v(x)+\sum_{y\in\eta^{-1}(x)}\lambda_y v(y) \tag{7}$$

Unlike in the classical case, this equation cannot be solved recursively, i.e. setting $v(x_0)$ at a leaf x_0 doesn't allow recursive computation of all other values $v(x)$. Therefore the existence of nonzero solutions of (7) is not obvious. Our aim is to show that such solutions exist for every nonreal z. In Proposition 23 we show that for real values of z the equation may not admit nonzero solutions.

For $x\in\Gamma$ let J_x denote the truncation of the Jacobi matrix J to the subtree Γ_x, i.e. the matrix with the parameters λ_y^x, β_y^x so that

$$\lambda_y^x=\begin{cases}\lambda_y & \text{for } y\in\Gamma_x\setminus\{x\}\\ 0 & \text{for } y\notin\Gamma_x\setminus\{x\}\end{cases}\qquad\qquad \beta_y^x=\begin{cases}\beta_y & \text{for } y\in\Gamma_x\\ 0 & \text{for } y\notin\Gamma_x\end{cases}$$

Lemma 1. *Fix a vertex $x\in\Gamma$. Assume there exists a nonzero function $v\in\mathcal{F}(\Gamma_x\cup\{\eta(x)\})$ and $z\notin\mathbb{R}$ such that $Jv(y)=zv(y)$ for $y\in\Gamma_x$. Then $v(\eta(x))\neq 0$.*

Proof. Assume for a contradiction that $v(\eta(x))=0$. Let w denote the truncation of v to Γ_x. Thus $J_xw=zw$. Moreover $w\neq 0$. Therefore z must be a real number, as J_x is a finite dimensional symmetric linear operator. □

Lemma 2. *Fix a vertex $x\in\Gamma$. Assume there exists $0\neq v\in\mathcal{F}(\Gamma_x)$ and $z\notin\mathbb{R}$ such that $(Jv)(y)=zv(y)$ for $y\in\Gamma_x\setminus\{x\}$. Then*

$$zv(x)\neq\beta_x v(x)+\sum_{y\in\eta^{-1}(x)}\lambda_y v(y).$$

Proof. Assume for a contradiction that

$$zv(x)=\beta_x v(x)+\sum_{y\in\eta^{-1}(x)}\lambda_y v(y).$$

Define the function $u\in\mathcal{F}(\Gamma_x\cup\{\eta(x)\})$ by setting $u(y)=v(y)$ for $y\in\Gamma_x$ and $u(\eta(x))=0$. Then $(Ju)(y)=zu(y)$ for $y\in\Gamma_x$. In view of Lemma 1 we get a contradiction. □

Corollary 3. *Assume there exists a function $v\neq 0$ on Γ and $z\notin\mathbb{R}$ such that $(Jv)(x)=zv(x)$ for $x\in\Gamma$. Then v does not vanish on Γ.*

Proof. Assume for a contradiction that $v(x)=0$ for a vertex x. By Lemma 1 we get that the function v vanishes identically on Γ_x. From the recurrence relation

$$zv(x)=\lambda_x v(\eta(x))+\beta_x v(x)+\sum_{y\in\eta^{-1}(x)}\lambda_y v(y),$$

we get $v(\eta(x)) = 0$. Therefore v vanishes identically on $\Gamma_{\eta(x)}$. Applying the same procedure infinitely many times we achieve that v vanishes at every vertex of Γ. □

Lemma 4. *For any nonreal number z and any $x_0 \in \Gamma$ with $l(x_0) \geq 1$ there exists a nonzero function v defined on Γ_{x_0} satisfying*

$$(Jv)(x) = zv(x), \quad x \in \Gamma_{x_0} \setminus \{x_0\}. \tag{8}$$

Moreover the function v cannot vanish and is unique up to a constant multiple.

Proof. We will use induction on the height $l(x_0)$. Assume $l(x_0) = 1$. Set $v(x_0) = 1$. Let $x \in \eta^{-1}(x_0)$. Then $l(x) = 0$. We want to have

$$zv(x) = \lambda_x v(x_0) + \beta_x v(x).$$

Thus we may set

$$v(x) = \frac{\lambda_x v(x_0)}{z - \beta_x}.$$

In this way (8) is fulfilled.

Assume the conclusion is true for all vertices on height n. Let $l(x_0) = n+1$. Consider vertices $x_1, x_2, \ldots, x_k \in \eta^{-1}(x_0)$. Then $l(x_j) = n$ for $j = 1, 2, \ldots, k$. By induction hypothesis, for every vertex x_j there exists a nonzero function v_j defined on Γ_{x_j} satisfying

$$(Jv_j)(x) = zv_j(x), \quad x \in \Gamma_{x_j} \setminus \{x_j\}.$$

We have

$$\Gamma_{x_0} = \bigcup_{j=1}^{k} \Gamma_{x_j} \cup \{x_0\}.$$

We are going to define the function v on Γ_{x_0} in the following way: set

$$v(x) = c_j v_j(x), \quad \text{for } x \in \Gamma_{x_j},$$

with $c_1 = 1$. In this way we get

$$(Jv)(x) = zv(x), \quad x \in \Gamma_{x_j} \setminus \{x_j\}, \quad j = 1, 2, \ldots, k.$$

In order to conclude the proof we must show that

$$(Jv)(x_j) = zv(x_j), \quad j = 1, 2, \ldots, k.$$

Thus we want to have

$$zc_j v_j(x_j) = \lambda_{x_j} v(x_0) + \beta_{x_j} c_j v_j(x_j) + \sum_{y \in \eta^{-1}(x_j)} \lambda_y c_j v_j(y),$$

i.e.

$$\lambda_{x_j} v(x_0) = c_j \left(zv_j(x_j) - \beta_{x_j} v_j(x_j) - \sum_{y \in \eta^{-1}(x_j)} \lambda_y v_j(y) \right). \tag{9}$$

The expression in the brackets on the right hand side is nonzero for every $j = 1, 2, \dots, k$ by Lemma 2. Therefore (9) is satisfied for an appropriate choice of the value $v(x_0)$ and nonzero constants $c_2, c_3, \dots, c_k$.

By Lemma 1 the function v cannot vanish at any vertex. Moreover if there was another function $\tilde{v}$ satisfying the conclusion of Lemma 3, then $v - c\tilde{v}$ would also satisfy the conclusion and would vanish for an appropriate choice of the constant c. Thus $v = c\tilde{v}$. □

Corollary 5. *For any nonreal number z there exists a nonzero function v so that*

$$(Jv)(x) = zv(x), \quad x \in \Gamma.$$

The function v cannot vanish and is unique up to a constant multiple.

Proof. Fix a leaf x_0. By Lemma 4, for any subtree $\Gamma_{\eta^k(x_0)}$ there exists a unique function v_k defined on $\Gamma_{\eta^k(x_0)}$ so that

$$v_k(x_0) = 1, \qquad (Jv_k)(x) = zv_k(x), \quad \text{for } x \in \Gamma_{\eta^k(x_0)} \setminus \{\eta^k(x_0)\}.$$

By unicity we have

$$v_{k+1}(x) = v_k(x), \quad \text{for } x \in \Gamma_{\eta^k(x_0)}.$$

Define

$$v(x) = v_k(x), \quad \text{for } x \in \Gamma_{\eta^k(x_0)}.$$

Since

$$\Gamma = \bigcup_{k=1}^{\infty} \Gamma_{\eta^k(x_0)},$$

the function v is defined at every vertex of Γ, and the conclusion follows. □

Remark. In Chapter 5 we are going to show that the conclusion of Corollary 2 may not be true for real numbers z, namely the eigenvalue equation $Jv = 0$ may not admit nonzero solutions. Observe that for classical Jacobi matrices (when $\Gamma = \mathbb{N}_0$) the recurrence relation

$$zv(n) = \lambda_n v(n+1) + \beta_n v(n) + \lambda_{n-1} v(n-1) \tag{10}$$

($\lambda_{-1} = 0$) admits a unique, up to constant multiple, nonzero solution for any $z \in \mathbb{C}$. In case of a tree it may easily happen that the equation $Jv = 0$ admits infinitely many linearly inedependent nonzero solutions. Indeed, assume $\lambda_x \equiv 1$ and $\beta_x \equiv 0$. Consider the tree Γ so that $\#\eta^{-1}(x) \geq 2$ for all vertices x on height 1. For any such vertex x choose $x_1, x_2 \in \eta^{-1}(x)$. Then the function $v_x = \delta_{x_1} - \delta_{x_2}$ satisfies $Jv_x = 0$.

3. Polynomials and zeros

The classical Jacobi matrices are related to orthogonal polynomials. Namely setting $v_0 = 1$ in (10) gives that $v(n) = p_n(z)$, where p_n is a polynomial of order n, with real coefficients. The question arises if Jacobi matrices on trees are connected to polynomials, as well. In general we cannot expect that the solution of $Jv = zv$ will satisfy that $v(t) = P_t(z)$, where P_t is a polynomial for every $t \in \Gamma$. But we may expect that $P_t(z)$ is a polynomial for t in a subtree Γ_x for some $x \in \Gamma$.

Proposition 6. *Let $x \in \Gamma$. There exists a nonzero solution v_x of $Jv_x = zv_x$, so that for any $t \in \Gamma'_x$ the function $v_x(t) = P_{x,t}(z)$ is a polynomial with real coefficients and positive leading coefficient. Moreover if $t \in \Gamma'_y \subset \Gamma'_x$ then the polynomial $P_{x,t}$ is divisible by $P_{y,t}$.*

Proof. We will use induction on the height $l(x)$. Let $l(x) = 0$. By Corollary 5 there is a nonzero solution v of $Jv = zv$. Then $v(x) \neq 0$. Let

$$v_x = v(x)^{-1}v.$$

Hence $v_x(x) = 1$. By $Jv_x = zv_x$ evaluated at x we get

$$v_x(\eta(x)) = \frac{z - \beta_x}{\lambda_x}.$$

Assume now the conclusion is valid for vertices on height n. Let $l(x) = n+1$. By induction hypothesis, for any $y \in \eta^{-1}(x)$ there is a nonzero solution v_y so that $P_{y,t}(z)$ is a polynomial with real coefficients for $t \in \Gamma'_y$. In particular the polynomial $v_y(x) = P_{y,x}(z)$ has real coefficients. Moreover by Lemma 1 the polynomial $P_{y,x}(z)$ cannot vanish for $z \notin \mathbb{R}$. Fix $y_1 \in \eta^{-1}(x)$ and let

$$v_x = \frac{\mathrm{LCM}\{P_{y,x}(z) \,:\, y \in \eta^{-1}(x)\}}{P_{y_1,x}(z)}\, v_{y_1}. \tag{11}$$

Since $Jv_{y_1} = zv_{y_1}$ we get $Jv_x = zv_x$. Moreover v_x does not vanish for $z \notin \mathbb{R}$. Since $v_{y_1}(x) = P_{y_1,x}(z)$ we obtain

$$P_{x,x}(z) = v_x(x) = \mathrm{LCM}\{P_{\eta(y),x}(z) \,:\, \eta(y) \in \eta^{-1}(x)\}. \tag{12}$$

Since the value $v_x(x)$ determines the solution, the function v_x does not depend on the choice of $y_1 \in \eta^{-1}(x)$. Thus the formula (11) and the above reasoning is valid for any choice of $y \in \eta^{-1}(x)$. Hence

$$v_x = \frac{\mathrm{LCM}\{P_{\tilde{y},x}(z) \,:\, \tilde{y} \in \eta^{-1}(x)\}}{P_{y,x}(z)}\, v_y, \qquad y \in \eta^{-1}(x). \tag{13}$$

By (13) and by induction hypothesis the value $v_x(t)$ is a polynomial in z for any $t \in \bigcup_{y\in\eta^{-1}(x)} \Gamma_y \cup \{x\} = \Gamma_x$. By the recurrence relation also the value $v_x(\eta(x))$ is a polynomial. Moreover by (13) the polynomial $v_x(t)$ is divisible by $v_y(t)$ for any $y \in \eta^{-1}(x)$ and $t \in \Gamma'_y$. This implies the last part of the conclusion. $\square$

Remarks The formulas (12) and (13) imply that for $y \in \eta^{-1}(x)$ and $t \in \Gamma_\eta(y)$ we have

$$P_{x,t}(z) = P_{x,x}(z)\frac{P_{y,t}(z)}{P_{y,x}(z)}. \tag{14}$$

Let $y \in \Gamma_x$. Then y and x are connected in Γ_x by a path $y = y_0, y_1, \ldots, y_n = x$. By iterating (14) we get

$$P_{x,y}(z) = \frac{P_{y_n,y_n}(z)}{P_{y_{n-1},y_n}(z)} \cdot \frac{P_{y_{n-1},y_{n-1}}(z)}{P_{y_{n-2},y_{n-1}}(z)} \cdot \ldots \cdot \frac{P_{y_1,y_1}(z)}{P_{y_0,y_1}(z)} P_{y_0,y_0}(z).$$

Let $y \in \Gamma_{\tilde{x}} \subset \Gamma_x$. Then $\tilde{x} = y_k$ for some k, $0 \le k \le n$. Hence

$$P_{x,y}(z) = \frac{P_{y_n,y_n}(z)}{P_{y_{n-1},y_n}(z)} \cdot \frac{P_{y_{n-1},y_{n-1}}(z)}{P_{y_{n-2},y_{n-1}}(z)} \cdot \ldots \cdot \frac{P_{y_{k+1}}(z)}{P_{y_k,y_{k+1}}(z)} P_{\tilde{x},y}(z).$$

These formulas and (12) imply that the polynomial $P_{x,y}(z)$ can be described in terms of the polynomials of the form $P_{t,t'}(z)$ for $t \in \Gamma_x$.

Corollary 7. *Let $z \notin \mathbb{R}$. Let $\{x_n\}_{n=0}^\infty$ be an infinite path in Γ so that $l(x_n) = n$. Let v be a nonzero solution of $(Jv)(x) = zv(x)$ so that $v(x_0) = 1$. Then for any vertex $x \in \Gamma_{x_n}$ we have*

$$v(x) = \frac{a_{n,x}(z)}{b_n(z)},$$

where $a_{n,x}(z)$ and $b_n(z)$ are polynomials with real coefficients. Moreover the polynomial b_{n+1} is divisible by b_n.

Proof. Consider the subtree Γ_{x_n}. Let $x \in \Gamma_{x_n}$. By Proposition 6 there is a solution v_n so that $v_n(x)$ and $v_n(x_0)$ are polynomials with real coefficients. Then

$$v(x) = \frac{v_n(x)}{v_n(x_0)}$$

satisfies $v(x_0) = 1$. By the last part of Proposition 1 the polynomial $v_{n+1}(x_0)$ is divisible by $v_n(x_0)$. $\square$

Theorem 8. *The polynomials $P_{x,y}(z)$, $y \in \Gamma_\eta(x)$, have only real zeros. Moreover for any $x \in \Gamma$ the zeros of $P_{x,x}$ and $P_{x,\eta(x)}$ are single, and the zeros of $P_{x,x}$ interlace with the zeros of $P_{x,\eta(x)}$, i.e. if $x_1 < x_2 < \ldots < x_n$ denote the zeros of $P_{x,\eta(x)}$, then $P_{x,x}$ has $n-1$ zeros $y_1 < y_2 < \ldots < y_{n-1}$ and*

$$x_1 < y_1 < x_2 < y_2 < \ldots < y_{n-1} < x_n.$$

Proof. We will use induction on $l(x)$. Let $l(x)=0$. Then $P_{x,x}=1$ and $P_{x,\eta(x)} = (z-\beta_x)/\lambda_x$. Assume the conclusion is valid for $l(x)=n-1$. Let $l(x)=n$. By the recurrence relation we have

$$\lambda_x P_{x,\eta(x)}(z) = (z-\beta_x)P_{x,x}(z) - \sum_{j=1}^{k} \lambda_{y_j} P_{x,y_j}(z), \tag{15}$$

where $\eta^{-1}(x)=\{y_1,y_2,\ldots,y_k\}$. By (14), with $t=y_j$, we get

$$P_{x,y_j}(z) = P_{x,x}(z)\,\frac{P_{y_j,y_j}(z)}{P_{y_j,x}(z)}. \tag{16}$$

By induction hypothesis the zeros of $P_{y_j,y_j}(z)$ are real and single and interlace with the zeros of $P_{y_j,x}(z)$ for any j. This implies

$$\deg P_{y_j,x} = \deg P_{y_j,y_j} + 1.$$

In view of (15) and (16) we get

$$\deg P_{x,\eta(x)} = \deg P_{x,x} + 1.$$

Let r be a root of $P_{x,x}(z)$. We are going to study the sign of $P_{x,\eta(x)}(r)$ making use of (15). If $P_{y_j,x}(r)\neq 0$, then (16) implies $P_{x,y_j}(r)=0$. But since $P_{x,x}(r)=0$ then $P_{y_{j_0},x}(r_1)=0$ for some j_0, by (12). Consider the quantity

$$P_{x,y_{j_0}}(r+\varepsilon) = P_{x,x}(r+\varepsilon)\,\frac{P_{y_{j_0},y_{j_0}}(r+\varepsilon)}{P_{y_{j_0},x}(r+\varepsilon)},$$

where $\varepsilon>0$ is infinitesimally small. We have

$$\frac{P_{y_{j_0},y_j}(r+\varepsilon)}{P_{y_{j_0},x}(r+\varepsilon)} > 0,$$

as the polynomials $P_{y_{j_0},y_{j_0}}(z)$ and $P_{y_{j_0},x}(z)$ have the same number of roots to the right of $r+\varepsilon$, by induction hypothesis and by the fact that the leading coefficients are positive. Consider the limit

$$\begin{aligned} P_{x,y_{j_0}}(r) &= \lim_{\varepsilon\to 0^+} P_{x,x}(r+\varepsilon)\,\frac{P_{y_{j_0},y_{j_0}}(r+\varepsilon)}{P_{y_{j_0},x}(r+\varepsilon)} \\ &= P_{y_{j_0},y_{j_0}}(r)\lim_{\varepsilon\to 0^+}\frac{P_{x,x}(r+\varepsilon)}{P_{y_{j_0},x}(r+\varepsilon)}. \end{aligned} \tag{17}$$

The polynomials $P_{y,x}$, for $y\in\eta^{-1}(x)$, have single roots by induction hypothesis. Thus the limit in the right hand side of (17) is nonzero in view of (12). Since $P_{y_{j_0},y_{j_0}}(r)\neq 0$ (by induction hypothesis) we get that $P_{x,j_0}(r)\neq 0$. Hence the sign of the limit is determined by the sign of $P_{x,x}(r+\varepsilon)$. By plugging $z=r$ into (15) we get that $P_{x,\eta(x)}(r)$ and $P_{x,x}(r+\varepsilon)$ have opposite signs.

Consider now two consecutive roots $r_1 < r_2$ of $P_{x,x}(z)$. The signs of $P_{x,x}(r_1+\varepsilon)$ and $P_{x,x}(r_2+\varepsilon)$ are opposite. Therefore the signs of $P_{x,\eta(x)}(r_1)$ and $P_{x,\eta(x)}(r_2)$ are also opposite. Thus $P_{x,\eta(x)}(z)$ must vanish in the interval (r_1, r_2).

Assume now that r is the largest root of $P_{x,x}(z)$. Then $P_{x,x}(r+\varepsilon) > 0$ for small positive ε. By the above reasoning we have $P_{x,\eta(x)}(r) < 0$, which means that $P_{x,\eta(x)}$ must vanish somewhere to the right of r, as the leading coefficient is positive. Similarly if r is the smallest root of $P_{x,x}(z)$ then the signs of $P_{x,\eta(x)}(r)$ and $P_{x,x}(r+\varepsilon)$ are opposite. But since the degree of $P_{x,\eta(x)}$ is by one greater than the degree of $P_{x,x}(z)$ and the leading coefficients are positive, we get that $P_{x,\eta(x)}$ must vanish below r. □

Theorem 9. *Let $x \in \Gamma$. Let r belong to the spectrum of J_x. Then r satisfies at least one of the two conditions*

(a) $P_{x,\eta(x)}(r) = 0$.
(b) *There exist $y \in \Gamma_x$ and $y_1, y_2 \in \eta^{-1}(y)$ so that $P_{y_1,y}(r) = P_{y_2,y}(r) = 0$.*

Proof. First we will show that the numbers described in the theorem belong to the spectrum of J_x. Assume $P_{x,\eta(x)}(r) = 0$. By Theorem 8 we have $P_{x,x}(r) \neq 0$. By Lemma 4 for any nonreal z there is a solution v_x of the equation $Jv_x = zv_x$ so that $v_x(y) = P_{x,y}(z)$ for $y \in \Gamma_{\eta(x)}$. Let

$$u(y) = \lim_{\varepsilon \to 0} P_{x,y}(r + i\varepsilon), \quad y \in \Gamma_x.$$

Then u satisfies $J_x u = ru$. Moreover u is nonzero as $u(x) = P_{x,x}(r) \neq 0$.

Assume now that there exist $y \in \Gamma_x$ and $y_1, y_2 \in \eta^{-1}(y)$ so that $P_{y_1,y}(r) = P_{y_2,y}(r) = 0$. By the above reasoning there are two nonzero solutions u_1, u_2, defined on Γ_{y_1}, Γ_{y_2}, respectively, of the equations $J_{y_1} u_1 = ru_1$ and $J_{y_2} u_2 = ru_2$ and $u_1(y_1) \neq 0$, $u_2(y_2) \neq 0$. Consider the function u_{y_1,y_2} defined on Γ_y as follows

$$u_{y_1,y_2}(t) = \begin{cases} \lambda_{y_2} u_2(y_2) u_1(t) & t \in \Gamma_{y_1}, \\ -\lambda_{y_1} u_1(y_1) u_2(t) & t \in \Gamma_{y_2}, \\ 0 & t \notin \Gamma_{y_1} \cup \Gamma_{y_2}. \end{cases}$$

Then $u_{y_1,y_2} \neq 0$ and $J_x u_{y_1,y_2} = J u_{y_1,y_2} = r u_{y_1,y_2}$.

Assume that for $y \in \Gamma_x$ and $y_1, y_2, \ldots, y_n \in \eta^{-1}(y)$ we have

$$P_{y_1,y}(r) = P_{y_2,y}(r) = \ldots = P_{y_n,y}(r) = 0.$$

Then the eigenvectors

$$u_{y_1,y_2}, \ldots, u_{y_1,y_n}$$

are linearly independent, as the support of u_{y_1,y_i} coincides with $\Gamma_{y_1} \cup \Gamma_{y_i}$. Hence the dimension of the space spanned by these eigenvectors is at least $n-1$.

In the previous part of the proof we have constructed eigenvectors corresponding to the set of numbers described in the theorem. We will calculate the dimension of the space spanned by these eigenvectors. The proof will be complete if the dimension coincides with the dimension of the space $\ell^2(\Gamma_x)$, i.e. with $\#\Gamma_x$. We will use induction with respect to the height $l(x)$. Assume the conclusion is valid for $l(x) = n$. Let $l(x) = n+1$. Denote $\eta^{-1}(x) = \{y_1, y_2, \ldots, y_k\}$. Let $n_j = \mathrm{deg} P_{y_j,x}$. Every eigenvector of J_{y_j} corresponding to the case (b) is an eigenvector of J_x as well. Therefore, by induction hypothesis, the dimension of the linear span of all eigenvectors of J_{y_i} corresponding to the case (b) is equal

$$\#\Gamma_{y_i} - \deg P_{y_i,x}.$$

Such eigenvectors corresponding to J_{y_i} and J_{y_j}, for $i \neq j$ have disjont supports, hence the total dimension of the eigenvectors corresponding to the case (b) for $J_{y_1}, \ldots, J_{y_k}$ is equal

$$\sum_{j=1}^{k} \#\Gamma_{y_j} - \sum_{j=1}^{k} \mathrm{deg} P_{y_j,x}$$

Consider the product

$$P_{y_1,x}(z) \ldots P_{y_k,x}(z).$$

We know that every polynomial $P_{y_j,x}$ has single roots. We have

$$P_{y_1,x}(z) \ldots P_{y_k,x}(z) = c \prod_{l=1}^{L} (z - r_l)^{n_l}.$$

By the reasoning performed in the first part of the proof, the root r_l gives rise to $n_l - 1$ linearly independent eigenvectors of J_x. Moreover the degree of the polynomial $P_{x,\eta(x)}$ is equal to $L+1$ as

$$\deg P_{x,\eta(x)} = \deg P_{x,x} + 1,$$

and $\deg P_{x,x} = L$ (cf. (12)). The roots of $P_{x,\eta(x)}$ lead to $L+1$ linearly independent eigenvectors of J_x, which are linearly independent from the ones constructed in (b), as they do not vanish at x. Summarizing the number of linearly independent eigenvectors of J_x is not less then

$$\sum_{j=1}^{k} \#\Gamma_{y_j} - \sum_{j=1}^{k} \mathrm{deg} P_{y_j,x} + \sum_{l=1}^{L} (n_l - 1) + L + 1 = \sum_{j=1}^{k} \#\Gamma_{y_j} + 1 = \#\Gamma_x.$$

□

Remark. By analyzing the proof we may observe that if r satisfies the assumption (a) only, then r is a single eigenvalue of J_x. The same occurs if r satisfies (b), but not (a), for a single $y \in \Gamma_x$ and just one pair of $y_1, y_2 \in \eta^{-1}(y)$. Otherwise any number r satisfying either (a) or (b) is a multiple eigenvalue of J_x.

4. Essential self-adjointness and defect indices

Let $z \notin \mathbb{C}$. The function $v \in \ell^2(\Gamma)$ belongs to the defect space N_z if v is orthogonal to $\mathrm{Im}\,(zI - J) = (zI - J)(\mathcal{F}(\Gamma))$. In particular v is orthogonal to $(zI - J)\delta_x$ for any $x \in \Gamma$. This implies $Jv = \overline{z}v$. The dimension of the defect space N_z is called the defect index. It is known that the defect index is constant on the upper-half plane and on the lower-half plane. In our case the defect index is constant on $\mathbb{C} \setminus \mathbb{R}$ as $Jv = \overline{z}v$ is equivalent to $J\overline{v} = z\overline{v}$. We refer to [4, 6] for the theory of symmetric operators in Hilbert space and its self-adjoint extensions.

Proposition 10. *The defect indices of the operator J cannot be greater than 1.*

Proof. Fix a nonreal number z. Let $v \in \ell^2(\Gamma)$ satisfy $v \neq 0$ and $Jv = zv$. By Corollary 3 the function v is unique up to a constant multiple. □

Proposition 10 implies

Corollary 11. *Let J be a Jacobi matrix on Γ. Fix a nonreal number z and let v denote the unique, up to a constant multiple, nonzero solution of the equation $Jv = zv$. Then J is essentially self-adjoint if and only if $v \notin \ell^2(\Gamma)$.*

Theorem 12. *There exist Jacobi matrices on Γ which are not essentially self-adjoint.*

Proof. We set $\beta_x \equiv 0$. Fix a nonreal number z. Choose an infinite path x_n in Γ so that $l(x_n) = n$. We will construct a matrix J by induction on n. Assume we have constructed a matrix J on $\Gamma_{x_{n-1}} \setminus \{x_{n-1}\}$ and a nonvanishing function v on $\Gamma_{x_{n-1}}$ so that

$$\|v\,|_{\Gamma_{x_{n-1}}}\|_2^2 \leq 1 - 2^{-(n-1)}$$

and

$$(Jv)(x) = zv(x), \quad x \in \Gamma_{x_{n-1}} \setminus \{x_{n-1}\}.$$

We want to extend the definition of J and v so that the conclusion remains valid when $n-1$ is replaced by n.

Our first task is to define $\lambda_{x_{n-1}}$ and $v(x_n)$ so that

$$zv(x_{n-1}) = \lambda_{x_{n-1}} v(x_n) + \sum_{y \in \eta^{-1}(x_{n-1})} \lambda_y v(y),$$

i.e.

$$\lambda_{x_{n-1}} v(x_n) = zv(x_{n-1}) - \sum_{y \in \eta^{-1}(x_{n-1})} \lambda_y v(y). \tag{18}$$

The right hand side of (18) cannot vanish by Lemma 2. We will define $\lambda_{x_{n-1}}$ and $v(x_n)$ so as to satisfy (18). By specifying $\lambda_{x_{n-1}}$ large enough we may assume that

$$|v(x_n)|^2 \leq 2^{-n-1}.$$

For any $y \in \eta^{-1}(x_n)$ and $y \neq x_{n-1}$ consider the subtree $\Gamma_y \setminus \{y\}$. Set $\lambda_x = 1$ for any $x \in \Gamma_y \setminus \{y\}$. By Lemma 4 there is a nonzero solution v_y defined on Γ_y satisfying

$$(Jv_y)(x) = zv_y(x), \quad x \in \Gamma_y \setminus \{y\}.$$

We may assume that

$$\sum_{y \in \eta^{-1}(x_n) \setminus \{x_{n-1}\}} \|v_y \,|_{\Gamma_y}\|_2^2 \leq 2^{-n-1}.$$

We want to define the numbers λ_y for $y \in \eta^{-1}(x_n)$ and $y \neq x_{n-1}$ so that

$$zv_y(y) = (Jv_y)(y) = \lambda_y v_y(x_n) + \sum_{x \in \eta^{-1}(y)} \lambda_x v_y(x).$$

Hence we want to have

$$\lambda_y = \frac{zv_y(y) - \displaystyle\sum_{x \in \eta^{-1}(y)} \lambda_x v_y(x)}{v(x_n)}. \tag{19}$$

By Lemma 2 the numerator (19) cannot vanish. We may multiply v_y by a constant of absolute value 1 so that the expression on the right hand side of (19) becomes positive. In this way the values λ_y for $y \in \eta^{-1}(x_n)$ and $y \neq x_{n-1}$ are defined. We extend the definition of v to Γ_{x_n} by setting

$$v(x) = v_y(x), \; x \in \Gamma_y, \; y \neq x_{n-1}.$$

On the way we have also extended the definition of J so that

$$(Jv)(x) = zv(x), \quad x \in \Gamma_{x_n} \setminus \{x_n\}.$$

Moreover by construction we have

$$\begin{aligned} \|v \,|_{\Gamma_{x_n}}\|_2^2 = \|v \,|_{\Gamma_{x_{n-1}}}\|_2^2 + \sum_{y \in \eta^{-1}(x_n), y \neq x_{n-1}} \|v \,|_{\Gamma_y}\|_2^2 + |v(x_n)|^2 \\ \leq 1 - 2^{-(n-1)} + 2^{-n-1} + 2^{-n-1} = 1 - 2^{-n}. \end{aligned}$$

□

Remark 1. The Jacobi matrix J constructed in the proof satisfies $\beta_x \equiv 0$ and $\lambda_x = 1$ for vertices x whose distance from the path $\{x_n\}$ is greater than 2.

Remark 2. Another way of proving Theorem 12 is as follows. Fix any Jacobi matrix J_0 so that the operator J_0 is bounded on $\ell^2(\Gamma)$. For example we may set $\beta_x \equiv 0$ and $\lambda_x = (\#\eta^{-1}(y))^{-1/2}$, whenever $x \in \eta^{-1}(y)$. Let S denote the operator acting according to the rule

$$Sv(x) = \lambda_x v(\eta(x)).$$

By Γ_0 we denote the set of leaves, i.e. vertices of height 0. Then

$$\|Sv\|_2^2 = \sum_{x\in\Gamma} |Sv(x)|^2 = \sum_{x\in\Gamma} \lambda_x^2 |v(\eta(x))|^2$$
$$= \sum_{y\in\Gamma\setminus\Gamma_0} |v(y)|^2 \sum_{x\in\eta^{-1}(y)} \lambda_x^2 = \sum_{y\in\Gamma\setminus\Gamma_0} |v(y)|^2 \le \|v\|_2^2.$$

The operator S is thus bounded. The adjoint operator S^* acts by the rule

$$S^*v(x) = \sum_{y\in\eta^{-1}(x)} \lambda_y v(y), \qquad x \notin \Gamma_0,$$
$$S^*v(x) = 0, \qquad x \in \Gamma_0.$$

Then $J_0 = S + S^*$ is the Jacobi matrix such that $\|J_0\|_{2\to 2} \le 2$. Fix an infinite path $\{x_n\}$ and a sequence of positive numbers $\{\lambda_n\}$. Let J_1 be the degenerate Jacobi matrix defined by $\beta_x \equiv 0$ and $\lambda_{x_n} = \lambda_n$, $\lambda_x = 0$ for $x \notin \{x_n\}$. Choose the coefficients λ_n so that the classical Jacobi matrix associated with the coefficients λ_n and $\beta_n \equiv 0$ is not essentially self-adjoint. For example set $\lambda_n = 2^n$. Let $J = J_0 + J_1$. The matrix J is nondegenerate. Moreover J is not essentially self-adjoint as a bounded perturbation of non essentially self-adjoint operator ([6], cf. Prop. 8.6 [4]).

The next theorem provides a relation between Jacobi matrices on the tree Γ and classical Jacobi matrices associated with the infinite paths of Γ.

Theorem 13. *Assume a Jacobi matrix J on Γ is not essentially self-adjoint and $\beta_x \equiv 0$. Choose an infinite path $\{x_n\}$ with $l(x_n) = n$. Then the classical Jacobi matrix J_0 with $\lambda_n = \lambda_{x_n}$ and $\beta_n \equiv 0$ is not essentialy self-adjoint.*

Before proving Theorem 13 we will need the following lemma.

Lemma 14. *Let J be a Jacobi matrix on Γ with $\beta_x \equiv 0$. Let $Jv = iv$ and $v(x_0) = 1$ for a vertex x_0 on height 0. Then the function $\tilde v(x) = i^{-l(x)} v(x)$ is positive.*

Proof. By assumptions we have

$$iv(x) = \lambda_x v(\eta(x)) + \sum_{y\in\eta^{-1}(x)} \lambda_y v(y).$$

Thus

$$\tilde v(x) = \lambda_x \tilde v(\eta(x)) - \sum_{y\in\eta^{-1}(x)} \lambda_y \tilde v(y). \tag{20}$$

We know that $\tilde v$ cannot vanish and is unique up to a constant multiple. The function $\operatorname{Re}\tilde v$ satisfies (20) and takes the value 1 at x_0. Thus $\tilde v = \operatorname{Re}\tilde v$, i.e. $\tilde v$ is real valued. We will show that $\tilde v(x)$ is positive by induction. Observe that if $\tilde v(x)$ is positive for any vertex on height zero then by (20) $\tilde v$ is positive.

Assume the opposite, i.e. $\tilde{v}$ is negative at some vertices on height zero. Since $\tilde{v}(x_0) = 1$ there are two vertices y_1, y_2 on height zero so that $y_1' = y_2'$ and $\tilde{v}(y_1) > 0$, $\tilde{v}(y_2) < 0$. By (20) evaluated at $x = y_1$ and $x = y_2$ we get that $\tilde{v}(y_1') > 0$ and $\tilde{v}(y_2') < 0$, which gives a contradiction.

□

Proof of Theorem 13. By (20) evaluated at $x = x_n$ we obtain

$$\tilde{v}(x_n) = \lambda_{x_n}\tilde{v}(x_{n+1}) - \sum_{y\in\eta^{-1}(x_n)} \lambda_y \tilde{v}(y).$$

Hence

$$\lambda_{x_n}\tilde{v}(x_{n+1}) - \lambda_{x_{n-1}}\tilde{v}(x_{n-1}) = \tilde{v}(x_n) + \sum_{\substack{y\in\eta^{-1}(x_n)\\ y\neq x_{n-1}}} \lambda_y \tilde{v}(y) > 0.$$

The last inequality follows from Lemma 14. Therefore

$$\begin{aligned} \tilde{v}(x_{2n}) &\geq \frac{\lambda_{x_0}\lambda_{x_2}\ldots\lambda_{x_{2n-2}}}{\lambda_{x_1}\lambda_{x_3}\ldots\lambda_{x_{2n-1}}}\tilde{v}(x_0), \\ \tilde{v}(x_{2n+1}) &\geq \frac{\lambda_{x_1}\lambda_{x_3}\ldots\lambda_{x_{2n-1}}}{\lambda_{x_2}\lambda_{x_4}\ldots\lambda_{x_{2n}}}\tilde{v}(x_1). \end{aligned}$$

By assumptions the seqeunce $\tilde{v}(x_n)$ is square summable. Thus

$$\sum_{n=1}^{\infty}\left(\frac{\lambda_{x_0}\lambda_{x_2}\ldots\lambda_{x_{2n-2}}}{\lambda_{x_1}\lambda_{x_3}\ldots\lambda_{x_{2n-1}}}\right)^2 + \left(\frac{\lambda_{x_1}\lambda_{x_3}\ldots\lambda_{x_{2n-1}}}{\lambda_{x_2}\lambda_{x_4}\ldots\lambda_{x_{2n}}}\right)^2 < \infty. \tag{21}$$

The last inequality is equivalent to not essential self-adjointness of the classical Jacobi matrix J_0 with $\lambda_n = \lambda_{x_n}$ and $\beta_n \equiv 0$. Indeed, let p_n and q_n denote the polynomials of the first and the second kind associated with J_0, i.e.

$$\begin{aligned} xp_n(x) &= \lambda_n p_{n+1}(x) + \lambda_{n-1}p_{n-1}(x), \quad n \geq 0, \\ xq_n(x) &= \lambda_n q_{n+1}(x) + \lambda_{n-1}q_{n-1}(x), \quad n \geq 1, \end{aligned}$$

where $p_{-1} = 0$ $p_1 = 1$ and $q_0 = 0$, $q_1 = 1/\lambda_0$. Then (21) reduces to

$$\sum_{n=1}^{\infty}[p_n^2(0) + q_n^2(0)] < \infty.$$

□

Remark. The assumption $\beta_x \equiv 0$ in Thm. 13 is essential. Indeed, there exists a Jacobi matrix J on Γ, which is not essentially self-adjoint, but the classical Jacobi matrix J_0 associated with the path $\{x_n\}$ is essentially self-adjoint. Indeed, for every vertex x_n, $n \geq 1$, fix a vertex $y_{n-1} \neq x_{n-1}$ in $\eta^{-1}(x_n)$. Let P denote the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(\{x_n, y_n\}_{n=0}^{\infty})$. We will consider Jacobi matrices J so that $\beta_x = 0$ for $x \notin \{y_n\}_{n=0}^{\infty}$. Let $J_1 = PJP$.

First, we are going to study the essential self-adjointness of the operator J_1. To this end consider the equation $(J_1)v(x) = zv(x)$. This is equivalent to

$$\begin{aligned} zv(x_n) &= \lambda_{x_n} v(x_{n+1}) + \lambda_{x_{n-1}} v(x_{n-1}) + \lambda_{y_{n-1}} v(y_{n-1}), \\ zv(y_{n-1}) &= \lambda_{y_{n-1}} v(x_n) + \beta_{y_{n-1}} v(y_{n-1}). \end{aligned}$$

We have

$$v(y_{n-1}) = \frac{\lambda_{y_{n-1}}}{z - \beta_{y_{n-1}}} v(x_n). \tag{22}$$

Hence

$$zv(x_n) = \lambda_{x_n} v(x_{n+1}) + \lambda_{x_{n-1}} v(x_{n-1}) + \frac{\lambda_{y_{n-1}}^2}{z - \beta_{y_{n-1}}} v(x_n).$$

Set $z = i$, $v_n := v_{x_n}$, $\mu_n := \lambda_{y_{n-1}}$, $\lambda_n := \lambda_{x_n}$ and $\beta_n := \beta_{y_{n-1}}$. Then

$$\left[\frac{\beta_n \mu_n^2}{1+\beta_n^2} + \left(1 + \frac{\mu_n^2}{1+\beta_n^2} \right) i \right] v_n = \lambda_n v_{n+1} + \lambda_{n-1} v_{n-1}.$$

Set $\mu_n^2 = 1 + \beta_n^2$. Then we obtain

$$2iv_n = \lambda_n v_{n+1} - \beta_n v_n + \lambda_{n-1} v_{n-1}.$$

Assume the classical Jacobi matrix with coefficients λ_n and $-\beta_n$ is not essentially self-adjoint. Then the sequence v_n is square summable. Moreover (22) implies

$$|v(y_{n-1})|^2 = \frac{\mu_n^2}{1+\beta_n^2} |v_n|^2 = |v_n|^2.$$

Hence

$$\|v\|^2 = \sum_{n=0}^{\infty} |v(x_n)|^2 + |v(y_n)|^2 < \infty,$$

i.e. the operator J_1 is not essentialy self-adjoint. Let J_2 be any bounded Jacobi matrix on Γ. Then the Jacobi matrix $J = J_1 + J_2$ is not essentially self-adjoint.

The matrix J_0 is associated with the coefficients $\lambda_n = \lambda_{x_n}$ and $\beta_n \equiv 0$. Thus, in order to conclude the reasoning, it suffices to prove the following.

Lemma 15. *There exists a classical Jacobi matrix J*

$$Jx_n = \lambda_n x_{n+1} - \beta_n x_n + \lambda_{n-1} x_{n-1}$$

which is not essentially self-adjoint, so that the Jacobi matrix

$$J' x_n = \lambda_n x_{n+1} + \lambda_{n-1} x_{n-1}$$

is essentially self-adjoint.

Proof. We will assume that $\beta_n \neq 0$. Not essential self-adjointness of J is equivalent to the fact that every solution of the recurrence relation

$$0 = \lambda_n x_{n+1} - \beta_n x_n + \lambda_{n-1} x_{n-1}, \qquad n \geq 1,$$

is square summable. Assume the sequence x_n satisfies this recurrence relation. Then

$$\begin{aligned} \beta_{2n} x_{2n} &= \lambda_{2n} x_{2n+1} + \lambda_{2n-1} x_{2n-1}, \\ \beta_{2n+1} x_{2n+1} &= \lambda_{2n+1} x_{2n+2} + \lambda_{2n} x_{2n}. \end{aligned} \tag{23}$$

Thus

$$\begin{aligned} x_{2n+1} &= \frac{\lambda_{2n+1}}{\beta_{2n+1}} x_{2n+2} + \frac{\lambda_{2n}}{\beta_{2n+1}} x_{2n}, \\ x_{2n-1} &= \frac{\lambda_{2n-1}}{\beta_{2n-1}} x_{2n} + \frac{\lambda_{2n-2}}{\beta_{2n-1}} x_{2n-2} \end{aligned} \tag{24}$$

Plugging in the last two equations into (23) results in

$$\left(\beta_{2n} - \frac{\lambda_{2n}^2}{\beta_{2n+1}} - \frac{\lambda_{2n-1}^2}{\beta_{2n-1}} \right) x_{2n} = \frac{\lambda_{2n}\lambda_{2n+1}}{\beta_{2n+1}} x_{2n+2} + \frac{\lambda_{2n-2}\lambda_{2n-1}}{\beta_{2n-1}} x_{2n-2}.$$

Let $\beta_{2n-1} = a\lambda_{2n-1}$ and

$$\beta_{2n} = \frac{\lambda_{2n}^2}{\beta_{2n+1}} + \frac{\lambda_{2n-1}^2}{\beta_{2n-1}}.$$

Then

$$0 = \lambda_{2n} x_{2n+2} + \lambda_{2n-2} x_{2n-2}.$$

Choose an increasing sequence λ_{2n} so that every solution u_{2n} of the last equation is square summable. Assume also that $\lambda_{2n} = \lambda_{2n+1}$. Then by (24) we get

$$|x_{2n+1}| \leq |a||x_{2n+2}| + \frac{\lambda_{2n}}{\lambda_{2n+1}} |x_{2n}| = |a||x_{2n+2}| + |x_{2n}|.$$

Thus the sequence x_n is square summable, i. e. the Jacobi matrix J is not essentially self-adjoint.

On the other hand the Jacobi matrix J', under assumption $\lambda_{2n} = \lambda_{2n+1}$, is essentially self-adjoint. Indeed, the sequence $x_{2n-1} = 0$ and $x_{2n} = (-1)^n$ satisfies $J'x = 0$ and it is not square summable. □

Remark. Following the proof it is possible to construct the coefficients λ_n and β_n explicitly. Let $\lambda_{2n+1} = \lambda_{2n} = q^n$ for $q > 1$. Then $\beta_{2n+1} = aq^n$ and $\beta_{2n} = a^{-1}[q^n + q^{n-1}]$.

The following lemma is straightforward but useful.

Lemma 16. *Consider a symmetric operator A on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}_0$ be a finite dimensional subspace of $D(A) \subset \mathcal{H}$ and let $P_{\mathcal{H}_0}$ denote the orthogonal projection onto $\mathcal{H}_0$. Define the operator $\tilde{A} : \mathcal{H}_0^\perp \to \mathcal{H}_0^\perp$ by*

$$\tilde{A} = (I - P_{\mathcal{H}_0})A(I - P_{\mathcal{H}_0}).$$

The operator $\tilde{A}$ is essentialy self-adjoint if and only if A is essentially self-adjoint.

Theorem 17. *Assume J is not essentially self-adjoint. Fix a leaf x_0 and a nonreal number z. Let a function $u_z(x)$ satisfy $u_z \neq 0$, $u_z(x_0) = 0$ and*

$$(Ju_z)(x) = zu_z(x), \quad x \neq x_0.$$

Then u_z is square summable on Γ.

Proof. Let $\mathcal{H}_0 = \mathbb{C}\delta_{x_0}$. The operator $\tilde{J}$ acts on $\ell^2(\Gamma \setminus \{x_0\})$ and is not essentially self-adjoint by Lemma 16. Moreover if $\tilde{u}_z$ denotes the truncation of u_z to $\tilde{\Gamma} = \Gamma \setminus \{x_0\}$ we have

$$(\tilde{J}\tilde{u}_z)(x) = z\tilde{u}_z(x), \quad x \in \tilde{\Gamma}.$$

By Corollary 3, applied to $\tilde{\Gamma}$, we know that $\tilde{u}_z$ cannot vanish. Since $\tilde{J}$ is not essentially self-adjoint there exists a function $0 \neq \tilde{v} \in \ell^2(\tilde{\Gamma})$ so that

$$(\tilde{J}\tilde{v})(x) = z\tilde{v}(x), \quad x \in \tilde{\Gamma}.$$

By Lemma 4, applied to $\tilde{\Gamma}$, we get that $\tilde{u}_z(x) = c\tilde{v}(x)$ for $x \in \tilde{\Gamma}$. □

Fix a leaf x_0, i.e. $l(x_0) = 0$ and let $x_n = \eta^n(x_0)$. By Corollary 3, for a nonreal number z, there exist two nonzero solutions v_z and u_z on Γ such that

$$v_z(x_0) = 1, \; v_z(x_1) = \frac{z - \beta_{x_0}}{\lambda_{x_0}}, \quad u_z(x_0) = 0, \; u_z(x_1) = \frac{1}{\lambda_{x_0}} \tag{25}$$

$$(Jv_z)(x) = zv_z(x), \; (Ju_z)(x) = zu_z(x), \quad x \in \Gamma \setminus \{x_0\}. \tag{26}$$

Observe that we have

$$(Jv_z)(x) = zv_z(x), \quad \text{for } x \in \Gamma.$$

The functions v_z and u_z satisfying (25) and (26) will be called *the solution* and *the associated solution* of the equation

$$(Jf)(x) = zf(x), \quad x \in \Gamma \setminus \{x_0\}.$$

Summarizing we get

Proposition 18. *Assume a Jacobi matrix J on Γ is not essentially self-adjoint. Then for any nonreal number z every solution of the equation*

$$zv(x) = \lambda_x v(\eta(x)) + \beta_x v(x) + \sum_{y \in \eta^{-1}(x)} \lambda_y v(y), \qquad x \neq x_0$$

is square summable.

Fix a leaf x_0 and remove from Γ the links from the infinite path $\{x_n\}_{n=0}^\infty$, where $x_n = \eta^n(x_0)$. In this way the tree Γ splits into infinite number of finite subtrees of the form $\Gamma_n := \Gamma_{x_n} \setminus \Gamma_{x_{n-1}}$. In other words Γ_n consists of x_n and all its descendants with exception of x_{n-1} and its descendants.

Lemma 19. *Let $x \in \Gamma_n$, for some $n \geq 1$. Then $v_z(x_n)u_z(x) = u_z(x_n)v_z(x)$.*

Proof. By Lemma 1 we know that v_z and u_z cannot vanish. Both functions satisfy $(Ju_z)(x) = zu_z(x)$, $(Jv_z)(x) = zv_z(x)$ for $x \in \Gamma_n \setminus \{x_n\}$. By Lemma 4 we get $v_z(x) = cu_z(x)$ for $x \in \Gamma_n$. Plugging in $x = x_n$ gives the conclusion. □

Proposition 20. *For the solution v_z and the associated solution u_z we have*

$$\begin{vmatrix} v_z(x_n) & u_z(x_n) \\ v_z(x_{n+1}) & u_z(x_{n+1}) \end{vmatrix} = \frac{1}{\lambda_{x_n}}.$$

Proof. By (26) we get for $n \geq 1$

$$\begin{aligned} \lambda_{x_n} v_z(x_{n+1}) &= zv_z(x_n) - \beta_{x_n} v_z(x_n) - \lambda_{x_{n-1}} v_z(x_{n-1}) - \sum_{y \in \eta^{-1}(x_n) \setminus \{x_{n-1}\}} \lambda_y v_z(y), \\ \lambda_{x_n} u_z(x_{n+1}) &= zu_z(x_n) - \beta_{x_n} u_z(x_n) - \lambda_{x_{n-1}} u_z(x_{n-1}) - \sum_{y \in \eta^{-1}(x_n) \setminus \{x_{n-1}\}} \lambda_y u_z(y). \end{aligned}$$

Observe that $\eta^{-1}(x_n) \setminus \{x_{n-1}\} \subset \Gamma_n$. Hence Lemma 19 implies

$$v_z(x_n)u_z(y) = u_z(x_n)v_z(y).$$

Now on multiplying the equations by $u_z(x_n)$ and $v_z(x_n)$, respectively, and subtracting sidewise gives

$$\lambda_{x_n} \begin{vmatrix} v_z(x_n) & u_z(x_n) \\ v_z(x_{n+1}) & u_z(x_{n+1}) \end{vmatrix} = \lambda_{x_{n-1}} \begin{vmatrix} v_z(x_{n-1}) & u_z(x_{n-1}) \\ v_z(x_n) & u_z(x_n) \end{vmatrix}$$

The conclusion follows as

$$\lambda_{x_0} \begin{vmatrix} v_z(x_0) & u_z(x_0) \\ v_z(x_1) & u_z(x_1) \end{vmatrix} = 1.$$

□

The following theorem provides a natural analog of Carleman criterion for essential self-adjointness.

Theorem 21. *Let J be a Jacobi matrix associated with the coefficients λ_x and β_x. Let x_n denote any infinite path so that $l(x_n) = n$. Assume*

$$\sum_{n=1} \frac{1}{\lambda_{x_n}} = \infty.$$

Then the operator J is essentially self-adjoint.

Proof. The result follows by the standard argument from Proposition 20. If J was not essentially self-adjoint then the functions v and u would be square summable, thus the series $\sum \lambda_{x_n}^{-1}$ would be summable. □

Remark. The assumption does not depend on the choice of the infinite path, as any two such paths will meet at a certain vertex.

5. Nonnegative Jacobi matrices on trees

We say that a matrix J is positive definite if

$$(Jv, v) \geq 0, \quad v \in \mathcal{F}(\Gamma).$$

The next theorem gives characterization of positive definite Jacobi matrices on Γ.

Theorem 22. (i) *Assume there exists a positive function $m(x)$ on Γ such that*

$$\beta_x m(x) \geq \lambda_x m(\eta(x)) + \sum_{y \in \eta^{-1}(x)} \lambda_y m(y), \qquad x \in \Gamma. \tag{27}$$

Then the matrix J is positive definite

(ii) *If the matrix J is positive definite there exists a positive function $m(x)$ on Γ such that*

$$\beta_x m(x) = \lambda_x m(\eta(x)) + \sum_{y \in \eta^{-1}(x)} \lambda_y m(y), \qquad x \in \Gamma. \tag{28}$$

Proof. (i). For $x \in \Gamma$ let

$$\alpha_x = \lambda_x \frac{m(x)}{m(\eta(x))}, \quad \gamma_x = \lambda_x \frac{m(\eta(x))}{m(x)}.$$

Thus, on dividing by $m(x)$, the formula (27) takes the form

$$\beta_x \geq \gamma_x + \sum_{y \in \eta^{-1}(x)} \alpha_y, \qquad x \in \Gamma. \tag{29}$$

We have (see (5))

$$
\begin{aligned}
(Jv, v) &= ((S + S^* + M)v, v) \\
&= \sum_{x\in\Gamma} \beta_x |v(x)|^2 + 2\mathrm{Re} \sum_{x\in\Gamma} \lambda_x \overline{v(x)} v(\eta(x)) \\
&\geq \sum_{x\in\Gamma} \beta_x |v(x)|^2 - 2 \sum_{x\in\Gamma} \lambda_x |v(x)|\,|v(\eta(x))| \\
&= \sum_{x\in\Gamma} \beta_x |v(x)|^2 - 2 \sum_{x\in\Gamma} \sqrt{\alpha_x\gamma_x} |v(x)|\,|v(\eta(x))| \\
&\geq \sum_{x\in\Gamma} \beta_x |v(x)|^2 - \sum_{x\in\Gamma} \gamma_x |v(x)|^2 - \sum_{x\in\Gamma} \alpha_x |v(\eta(x))|^2 \\
&= \sum_{x\in\Gamma} \beta_x |v(x)|^2 - \sum_{x\in\Gamma} \gamma_x |v(x)|^2 - \sum_{x\in\Gamma} |v(x)|^2 \sum_{y\in\eta^{-1}(x)} \alpha_y \\
&= \sum_{x\in\Gamma} \left(\beta_x - \gamma_x - \sum_{y\in\eta^{-1}(x)} \alpha_y \right) |v(x)|^2 \geq 0.
\end{aligned}
$$

(ii) Consider the operator U acting by the rule

$$Uv(x) = (-1)^{l(x)} v(x).$$

Clearly U is a unitary operator. Let

$$\tilde{J} = -U^* J U.$$

Then $\tilde{J}$ is a nonpositive definite operator and

$$\tilde{J}v(x) = \lambda_x v(\eta(x)) - \beta_x v(x) + \sum_{y\in\eta^{-1}(x)} \lambda_y v(y).$$

Fix an infinite path x_n so that $l(x_n) = n$. Thus $\Gamma = \bigcup_{n=0}^{\infty} \Gamma_{x_n}$. Let P_n denote the orthogonal projection from $\ell^2(\Gamma)$ onto $\ell^2(\Gamma_{x_n})$ and $\tilde{J}_n = P_n \tilde{J} P_n$. Then $\tilde{J}_n$ is a bounded nonpositive linear operator. Therefore

$$-a_n I < \tilde{J}_n \leq 0 < \tfrac{1}{n} I,$$

for a positive constant a_n. Hence

$$0 < \tilde{J}_n + a_n I < \left(a_n + \tfrac{1}{n}\right) I.$$

We have

$$0 < ((\tilde{J}_n + a_n I)\delta_x, \delta_x) = a_n - \beta_x, \qquad x \in \Gamma_{x_n}. \tag{30}$$

Observe that

$$(\tilde{J}_n + a_n I)\delta_x = \lambda_x \delta_{\eta(x)} + (a_n - \beta_x)\delta_x + \sum_{y\in\eta^{-1}(x)} \lambda_y \delta_y,\ x \in \Gamma_{x_n} \setminus \{x_n\}. \tag{31}$$

Let

$$f_n := (\tfrac{1}{n}I - \tilde{J}_n)^{-1}\delta_{x_0} = [(a_n + \tfrac{1}{n})I - (\tilde{J}_n + a_n I)]^{-1}\delta_{x_0} = \sum_{k=0}^{\infty} \frac{1}{(a_n + \frac{1}{n})^{k+1}} (\tilde{J}_n + a_n I)^k \delta_{x_0}.$$

By (30) and (31) the function

$$(\tilde{J}_n + a_n I)^k \delta_{x_0}$$

is nonnegative, and positive on all vertices of Γ_{x_n} at distance from x_0 less or equal to k. Hence $f_n \geq 0$ and $f_n(x) > 0$ for any $x \in \Gamma_{x_n}$. Moreover

$$\tilde{J}_n f_n = \tilde{J}_n(\tfrac{1}{n}I - \tilde{J})_n^{-1}\delta_{x_0} = (\tilde{J}_n - \tfrac{1}{n}I)(\tfrac{1}{n}I - \tilde{J}_n)^{-1}\delta_{x_0} + \tfrac{1}{n}f_n = \tfrac{1}{n}f_n - \delta_{x_0}.$$

This results in

$$\lambda_x f_n(\eta(x)) - \beta_x f_n(x) + \sum_{y\in\eta^{-1}(x)} \lambda_y f_n(y) = \tfrac{1}{n}f_n(x) - \delta_{x_0}(x), \quad x \in \Gamma_{x_n} \setminus \{x_n\}.$$

Let

$$m_n(x) = \frac{f_n(x)}{f_n(x_0)}.$$

Then $m_n(x_0) = 1$ and

$$\lambda_x m_n(\eta(x)) + \sum_{y\in\eta^{-1}(x)} \lambda_y m_n(y) = (\beta_x + \tfrac{1}{n})m_n(x),\ x \in \Gamma_{x_n} \setminus \{x_0, x_n\}, \tag{32}$$

$$\lambda_{x_0} m_n(x_1) \leq (\beta_{x_0} + \tfrac{1}{n}). \tag{33}$$

Observe that for any fixed $t \in \Gamma$ the sequence $m_n(t)$ is bounded. Indeed, assume the opposite. Let t be the vertex closest to x_0, so that $m_n(t)$ is unbounded. Let s be the vertex adjacent to t, so that

$$d(x_0, t) = d(x_0, s) + 1.$$

Then applying (32) with $x = s$ implies that the sequence $m_n(s)$ is unbounded, which gives a contradiction.

Observe also that for any fixed $t \in \Gamma$ the sequence $m_n(t)$ cannot accumulate at zero. Indeed, assume the opposite. Let t be the vertex closest to x_0 so that $m_n(t)$ accumulates at zero. Again let s be the vertex adjacent to t, so that

$$d(x_0, t) = d(x_0, s) + 1.$$

Then applying (32) with $x = t$ implies that the sequence $m_n(s)$ also accumulates at zero, which gives a contradiction.

Consider the sequence of functions m_n. Let m be any pointwise accumulation point of this sequence. Then $m(x) > 0$ and by (32) and (33) we obtain

$$\lambda_x m(\eta(x)) + \sum_{y\in\eta^{-1}(x)} \lambda_y m(y) = \beta_x m(x), \qquad x \in \Gamma \setminus \{x_0\}, \tag{34}$$

$$\lambda_{x_0} m(x_1) \le \beta_{x_0}. \tag{35}$$

In order to get the conclusion (i.e. to guarantee equality also in (35)) we have to modify slightly the function $m(x)$.

Observe that after removing all the edges from the path $\{x_n\}$ the tree Γ splits into the sequence of disjoint trees Γ_n so that $x_n \in \Gamma_n$. By (34) evaluated at $x = x_n$ we have

$$\lambda_{x_n} m(x_{n+1}) + \lambda_{x_{n-1}} m(x_{n-1}) + \sum_{\substack{y\in\eta^{-1}(x_n)\\ y\neq x_{n-1}}} \lambda_y m(y) = \beta_{x_n} m(x_n), \qquad n \ge 1.$$

Let the coefficients c_n be defined by $c_0 = 0$ and

$$\sum_{\substack{y\in\eta^{-1}(x_n)\\ y\neq x_{n-1}}} \lambda_y m(y) = c_n m(x_n), \qquad n \ge 1. \tag{36}$$

Thus

$$\lambda_{x_n} m(x_{n+1}) + \lambda_{x_{n-1}} m(x_{n-1}) = (\beta_{x_n} - c_n) m(x_n), \qquad n \ge 1.$$

This implies $\beta_{x_n} \ge c_n$. Consider the classical Jacobi matrix defined by

$$J_0 u(n) = \lambda_{x_n} u(n+1) + (\beta_{x_n} - c_n) u(n) + \lambda_{x_{n-1}} u(n-1).$$

By Theorem 22(i) the matrix J_0 is positive definite. Let p_n denote the orthogonal polynomials associated with J_0. By the well known result we have $(-1)^n p_n(0) > 0$. Set $\tilde{m}(x_n) = (-1)^n p_n(0)$. Then

$$\lambda_{x_n} \tilde{m}(x_{n+1}) + \lambda_{x_{n-1}} \tilde{m}(x_{n-1}) = (\beta_{x_n} - c_n) \tilde{m}(x_n). \tag{37}$$

Set also

$$\tilde{m}(y) = \frac{\tilde{m}(x_n)}{m(x_n)} m(y), \qquad y \in \Gamma_n. \tag{38}$$

In view of (36), (37) and (38) we get

$$\lambda_{x_n} \tilde{m}(x_{n+1}) + \sum_{y\in\eta^{-1}(x_n)} \lambda_y \tilde{m}(y) = \beta_{x_n} \tilde{m}(x_n), \qquad n \ge 0.$$

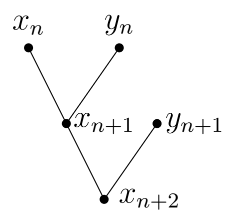
Finally, by (34) and (38) we have

$$\lambda_x m(\eta(x)) + \sum_{y\in\eta^{-1}(x)} \lambda_y m(y) = \beta_x m(x), \quad x \in \Gamma_n \setminus \{x_n\}, n \ge 1.$$

□

Proposition 23. *There exist Jacobi matrices J on trees so that the equation $Jv = tv$ does not admit nonzero solutions for some real values of t.*

Proof. We may admit that $t = 0$. Consider a tree Γ with $\#\eta^{-1}(x) = 2$ for every vertex x, $l(x) \geq 1$. Fix an infinite path x_n, so that $l(x_n) = n$. Then $\eta^{-1}(x_n) = \{x_{n-1}, y_{n-1}\}$ for $n \geq 1$. We will define the coefficients λ_x and β_x on Γ_{y_k} in such a way that the operator J restricted to $\ell^2(\Gamma_{y_k} \setminus \{y_k\})$ is positive. For example we may set $\lambda_x = 1$ and $\beta_x = 4$ for any $x \in \Gamma_{y_k} \setminus \{y_k\}$. In this way if the function v satisfies $(Jv)(y) = 0$ for $y \in \Gamma_{y_k} \setminus \{y_k\}$, then either $v = 0$ or v cannot vanish on Γ_{y_k}. If v does not vanish on Γ_{y_k} its restriction to Γ_{y_k} is unique up to a constant multiple. Let $\lambda_{y_k} = 1$ and set β_{y_k} in such a way that $v(\eta(y_k)) = v(x_{k+1}) = 0$. Set also $\lambda_{x_k} = 1$ and $\beta_{x_k} = 0$ for any k. Thus the matrix J is defined. Assume $Ju = 0$. If u vanishes on every subtree Γ_{y_n} then by the recurrence relation u vanishes at every vertex x_n, with $n \geq 1$, as $\eta(y_n) = x_{n+1}$. Moreover by the recurrence relation evaluated at x_1 we obtain $v(x_0) = 0$, i.e. $v = 0$. If u does not vanish on every subtree Γ_{y_n}, let n be the smallest index for which u does not vanish on Γ_{y_n}.



Then $v(x_k) = 0$ for any $k \leq n$. We must have $v(y_n) \neq 0$. By construction we also get $v(x_{n+1}) = 0$. By the recurrence relation evaluated at x_{n+1} we conclude that $v(x_{n+2}) \neq 0$. This implies that v does not vanish on $\Gamma_{y_{n+1}}$. But by construction $v(x_{n+2}) = 0$, which is a contradiction. □

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