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Orthogonal polynomials and Banach algebras

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Summary: One of the major problem in the theory of orthogonal polynomials is the determination of those orthogonal polynomial systems which have the nonnegative linearization property. Orthogonal polynomials sharing this property establish a convolution structure and therefore give rise to a Banach algebra. In these lecture notes we will show how the nonnegative linearization property is related to some maximum principle of certain boundary value problem. From this relation, we derive sufficient conditions on the orthogonal polynomial system in order to satisfy the nonnegative linearization. Finally, we will turn our attention to the Banach algebra structure generated by the orthogonal polynomials.

Contents

1. Introduction

One of the central problems in the theory of orthogonal polynomials is to determine which orthogonal systems have the property that the coefficients in the expansion of the product

$$p_n(x)p_m(x) = \sum_k c(n,m,k)p_k(x)$$

are nonnegative for any n, m and k. This property called nonnegative linearization has important consequences in a number of topics like

- pointwise estimates of polynomials,
- convolution structures associated with orthogonal polynomials,
- positive definitness of orthogonal polynomials,
- limit theorems for random walks associated with recurrence relations.

Usually in these applications the explicit formulas for the coefficients $c(n,m,k)$ are not necessary. The positivity is the only property that counts.

Actually for many orthogonal systems these coefficients have been computed explicitly in the past. Nonetheless, still there are many systems, including the nonsymmetric Jacobi polynomials, for which such explicit expressions are not available. That's why any criteria which imply nonnegative linearization are of great importance.

Some orthogonal polynomial systems show up as matrix coefficients of irreducible representations of classical matrix groups. Then the product formula can be interpreted in terms of the decomposition of tensor product of two such representations into irreducible components. In such case the coefficients $c(n,m,k)$ are products of multiplicities of representations and lengths of certain cyclic vectors. In this way they are always nonnegative.

The main part of these notes is devoted to study the nonnegative linearization problem and the convolution structure it induces. We are going to state and prove certain criteria for nonnegative linearization. The interesting part of this property is the connection to a maximum principle of certain discrete boundary value problem.

We do not expect the reader to know much about orthogonal polynomials. For this we start with a concise introduction to orthogonal polynomials which contains basic facts we are going to use in other sections. The proofs are complete (at least in our opinion), although certain points are left to the reader. These places are indicated as `Exercises.` There is a number of examples illustrating the theoretical results.

The notes contain three open problems that we were trying to solve in the past but so far we didn't succeed. It would be great if one of the readers could solve any of these problems.

2. Introduction to orthogonal polynomials

2.1. **Preliminaries.** The notion of orthogonality comes from elementary geometry. Usually the first time we encounter this notion in the context of function spaces is while studying Fourier series. For example, most of the students know the following formulas.

$$\int_0^\pi \cos m\theta \cos n\theta \, d\theta = 0 \qquad \text{for } n \neq m,$$
$$\int_0^\pi \sin m\theta \sin n\theta \, d\theta = 0 \qquad \text{for } n \neq m.$$

This means the trigonometric polynomials $\cos n\theta$ are orthogonal to each other and $\sin n\theta$ are orthogonal to each other as well, relative to the inner product

$$(f,g) = \int_0^\pi f(\theta)\overline{g(\theta)}\, d\theta.$$

What about the algebraic polynomials ? Can we produce orthogonal algebraic polynomials ? One can show by induction and by trigonometry (`Exercise`) that there are algebraic polynomials T_n and U_n of exact degree n, such that

$$T_n(\cos\theta) = \cos n\theta,$$
$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

The polynomials T_n are called the Chebyshev polynomials, while U_n are called the Chebyshev polynomials of the second kind. We have

$$T_0 = 1, \qquad T_1(x) = x, \qquad T_2(x) = 2x^2 - 1, \qquad T_3(x) = 4x^3 - 3x,$$
$$U_0 = 1, \qquad U_1(x) = 2x, \qquad U_2(x) = 4x^2 - 1, \qquad U_3(x) = 8x^3 - 4x.$$

Also orthogonality relations can be carried over to the polynomials T_n and U_n. Indeed, by substitution $x = \cos\theta$ we obtain for $n \neq m$

$$0 = \int_0^\pi \cos n\theta \cos m\theta \, d\theta = \int_{-1}^1 T_n(x)T_m(x)\frac{dx}{\sqrt{1-x^2}},$$

$$0 = \int_0^\pi \sin n\theta \sin m\theta \, d\theta = \int_0^\pi \frac{\sin n\theta}{\sin\theta}\frac{\sin m\theta}{\sin\theta}\sin^2\theta \, d\theta$$
$$= \int_{-1}^1 U_n(x)U_m(x)\sqrt{1-x^2}\, dx.$$

Thus T_n are orthogonal to each other in the Hilbert space $L^2((-1,1), w(x)dx)$, with the weight $w(x) = (1-x^2)^{-1/2}$, while U_n are orthogonal with the weight $w(x) = (1-x^2)^{1/2}$.

The trigonometric polynomials $\cos n\theta$ and $\sin n\theta$ satisfy the simple differential equations

$$\frac{d^2}{d\theta^2}(\cos n\theta) = -n^2\cos n\theta,$$
$$\frac{d^2}{d\theta^2}(\sin n\theta) = -n^2\sin n\theta.$$

By changing the variables $x = \cos\theta$ we get (`Exercise`)

$$(1-x^2)\frac{d^2}{dx^2}T_n - x\frac{d}{dx}T_n = -n^2T_n,$$
$$(1-x^2)\frac{d^2}{dx^2}U_n - 3x\frac{d}{dx}U_n = -n(n+2)U_n.$$

Moreover, by using the trigonometric identities

$$2\cos\theta\cos n\theta = \cos(n+1)\theta + \cos(n-1)\theta,$$
$$2\cos\theta\sin n\theta = \sin(n+1)\theta + \sin(n-1)\theta,$$

we obtain the recurrence relations

$$\begin{aligned} xT_0 &= T_1, \\ 2xT_n &= T_{n+1} + T_{n-1}, && n \geq 1, \\ 2xU_0 &= U_1, \\ 2xU_n &= U_{n+1} + U_{n-1}, && n \geq 1. \end{aligned}$$

These formulas allow to compute the polynomials recursively, assuming that $T_0 = U_0 = 1$. Observe that the formulas for T_n and U_n are different only at $n = 0$.

The fact that the polynomials are eigenfunctions of second order differential equation imply immediately the orthogonality of the polynomials with respect to a certain weight. This is a point at which we can consider more general situation.

2.2. **Differential equations.** Assume we are given a sequence of functions $\varphi_n(x)$ which are $C^\infty([-1,1])$ and satisfy second order differential equation of the form

$$L\varphi_n = \lambda_n \varphi_n,$$

where

$$L = \frac{1}{w(x)} \frac{d}{dx} \left(A(x) \frac{d}{dx} \right),$$

and $A(x), w(x)$ are positive and smooth in $(-1,1)$. Assume also that $A(x)$ is continuous on $[-1,1]$ and $A(-1) = A(1) = 0$. Let $\int_{-1}^{1} w(x)dx < +\infty$. It turns out that these assumptions imply the functions φ_n are orthogonal for different values of λ_n in the Hilbert space $L^2((-1,1), w(x)dx)$. Indeed, let $\lambda_n \neq \lambda_m$. Then

$$\begin{aligned} \lambda_n \int_{-1}^{1} \varphi_n \varphi_m w(x)\, dx &= \int_{-1}^{1} \frac{d}{dx} \left(A(x) \frac{d\varphi_n}{dx} \right) \varphi_m \, dx \\ &= -\int_{-1}^{1} A(x) \frac{d\varphi_n}{dx} \frac{d\varphi_m}{dx}\, dx = \lambda_m \int_{-1}^{1} \varphi_n \varphi_m w(x)\, dx. \end{aligned}$$

For the polynomials T_n we have

$$w(x) = \frac{1}{\sqrt{1-x^2}}, \qquad A(x) = \sqrt{1-x^2},$$

while for U_n

$$w(x) = \sqrt{1-x^2}, \qquad A(x) = (1-x^2)^{3/2}.$$

In general, if the differential operator L is of the form

$$L = C(x)\frac{d^2}{dx^2} + D(x)\frac{d}{dx},$$

we have (`Exercise`)

$$w(x) = \frac{1}{C(x)} \exp \int \frac{D(x)}{C(x)}\,dx,$$
$$A(x) = C(x)w(x).$$

Bochner considered differential equation of the form

$$Ly = a_2(x)y'' + a_1(x)y' + a_0(x)y = \lambda y,$$

where $a_i(x)$ are polynomials. He wanted to determine all the cases for which for any $n \geq 0$ there is a number λ_n and a polynomial of degree exactly n, which is a solution to this equation. Assume this property holds. Then considering $n = 0$ implies that $a_0(x)$ is a constant polynomial. We can actually take $a_0 = 0$. Next, considering $n = 1$ yields that a_1 is a linear polynomial. In similar way we can derive that a_2 is a quadratic polynomial. Bochner showed that there are actually 5 different cases, so that we cannot reduce one case to another by affine change of variables. These cases are

$$Ly = (1-x^2)y'' + [a - bx]y',$$
$$Ly = xy'' + (a-x)y',$$
$$Ly = y'' - 2xy',$$
$$Ly = x^2y'' + xy',$$
$$Ly = x^2y'' + 2(x+1)y'.$$

It can be checked directly that the polynomial solutions of the last two equations do not lead to orthogonality.

Consider the first differential operator. By using the method described before, the polynomial solutions to that equation are orthogonal with respect to the weight

$$\begin{aligned} w(x) &= \frac{1}{1-x^2} \exp \int \frac{a-bx}{1-x^2}dx \\ &= (1-x)^{(a+b-2)/2}(1+x)^{(b-a-2)/2}. \end{aligned}$$

Let $a = \alpha - \beta$ and $b = \alpha + \beta + 2$. Then $w(x) = (1-x)^\alpha(1+x)^\beta$. This weight is integrable on $(-1, 1)$ only when $\alpha, \beta > -1$. The corresponding orthogonal polynomials are called the Jacobi polynomials. We can recognize the Chebyshev polynomials case for $\alpha = \beta = -1/2$ and the Chebyshev polynomials of the second kind case for $\alpha = \beta = 1/2$.

The second differential equation leads to the so called Laguerre polynomials and the third equations leads to the Hermite polynomials. One can easily find the weight $w(x)$ in these cases (`Exercise`).

The three families of polynomials are called the classical orthogonal polynomials.

2.3. **General orthogonal polynomials and recurrence relations.** As we have seen before, the Chebyshev polynomials of both kinds satisfy simple recurrence relations. This is another point at which we can generalize our considerations.

Let μ be a finite positive measure on the real line. We can always assume that the total mass is 1, i.e. that μ is a probability measure. This measure can be absolutely continuous like $d\mu(x) = w(x)dx$ or not. Let us assume for simplicity that the measure μ has bounded support. In other words, we require that $\mu(\mathbb{R} \setminus [-a, a]) = 0$. To avoid technical problems we assume that our measure μ cannot be concentrated on finitely many points. However we admit measures which are concentrated on countably many points (actually there are concrete interesting measures of this kind). The measure μ gives rise to the Hilbert space $L^2(\mathbb{R}, \mu)$ of square integrable functions with respect to μ. A complex valued function f belongs to that space if and only if $\int |f(x)|^2 d\mu(x) < +\infty$. The inner product of two functions is given by

$$(f, g) = \int f(x)\overline{g(x)}\, d\mu(x).$$

For example, all continuous functions are in $L^2(\mathbb{R}, \mu)$, because they are bounded in $[-a, a]$. Once we have a Hilbert space, we immediately seek for a nice orthogonal basis in it. If no basis pops up in natural way, we pick up any natural linearly independent sequence of functions and try to make it orthogonal using the Gram–Schmidt procedure. The most natural independent set in our case is the sequence of monomials

$$1, x, x^2, \dots, x^n, \dots.$$

The monomials are linearly independent in $L^2(\mu)$, because we have assumed that μ is not concentrated on finitely many points[1]. Now, by applying the Gram-Schmidt procedure, we obtain a sequence of polynomials (linear combinations of monomials) p_n with the following properties.

(i) $p_0 = 1$.

(ii) $p_n(x) = k_n x^n + \ldots + k_0$, $k_n > 0$.

(iii) $p_n \perp \{1, x, \ldots, x^{n-1}\}$.

(iv) $\int p_n^2(x) d\mu(x) = 1$.

[1] Actually all what follows can be carried over to the case of the measures concentrated on finitely many points (`Exercise`).

Thus $\{p_n\}_{n=0}^{\infty}$ is an orthonormal system (basis) in $L^2(\mathbb{R},\mu)$. We have span$\{p_0,p_1,\ldots,p_n\} = \operatorname{span}\{1,x,\ldots,x^n\}$. The polynomial xp_n has degree $n+1$, hence it can be expressed in terms of the first $n+2$ polynomials.

$$xp_n(x) = \gamma_n p_{n+1}(x) + \beta_n p_n(x) + \alpha_n p_{n-1}(x) + \ldots.$$

Lower order term in the expansion of $xp_n(x)$ vanish, because $\int (xp_n)p_k d\mu = \int p_n(xp_k)d\mu = 0$ for $k \le n-2$. Thus we get the recurrence relation

$$xp_n(x) = \gamma_n p_{n+1}(x) + \beta_n p_n(x) + \alpha_n p_{n-1}(x).$$

Let's investigate the coefficients in this expansion. We have $\gamma_n > 0$ as the ratio of the leading coefficients:

$$\gamma_n = \frac{k_n}{k_{n+1}}.$$

By orthogonality, we have the following.

$$\gamma_n = \int xp_n(x)p_{n+1}(x)d\mu(x),$$
$$\alpha_n = \int xp_{n-1}(x)p_n(x)d\mu(x),$$
$$\beta_n = \int xp_n^2(x)d\mu(x).$$

Hence

$$\alpha_{n+1} = \gamma_n > 0, \qquad \beta_n \in \mathbb{R}.$$

Let $\lambda_n = \gamma_n$. Then

$$\boxed{xp_n = \lambda_n p_{n+1} + \beta_n p_n + \lambda_{n-1}p_{n-1}.} \tag{2.1}$$

For $n = 0$ the formula reduces to

$$xp_0 = \lambda_0 p_1 + \beta_0 p_0.$$

It is customary to set $p_{-1} = \lambda_{-1} = 0$. Once the numbers λ_n and β_n are known, we can compute the polynomials p_n one by one. Indeed,

$$p_{n+1} = \frac{1}{\lambda_n}\left[(x-\beta_n)p_n - \lambda_{n-1}p_{n-1})\right].$$

By Pythagoras theorem, we get

$$\lambda_n^2 + \beta_n^2 + \lambda_{n-1}^2 = \int_{-a}^{a} x^2 p_n^2\, d\mu(x)$$
$$\le a^2 \int_{-a}^{a} p_n^2\, d\mu = a^2.$$

Hence the sequences λ_n and β_n are bounded.

As we have seen before, polynomials, which are eigenfunctions of certain differential operators, are automatically orthogonal with respect to a weight simply associated with the coefficients of the equations. The question arises if the similar statement is true for polynomials satisfying a recurrence relation (2.1), with $\lambda_n > 0$ and $\beta_n \in \mathbb{R}$. We assume that the sequence of polynomials satisfies (2.1). Are the polynomials orthogonal with respect to some measure on $\mathbb{R}$? The answer is yes, but the measure cannot be easily computed in terms of the coefficients. This theorem is due to Favard (1935).

We will give a sketch of the proof assuming, for simplicity, that the coefficients in (2.1) are bounded. In the space of all polynomials we introduce an inner product such that

$$(p_n, p_m) = \delta_n^m.$$

This is possible, because the polynomials p_n are linearly independent. We complete the space of polynomials with respect to this inner product thus obtaining a Hilbert space $\mathcal{H}$. The elements of this space are series of the form

$$\sum_{n=0}^{\infty} a_n p_n, \qquad \sum_{n=0}^{\infty} |a_n|^2 < +\infty.$$

Let M be the linear operator acting on the polynomials by multiplication with the variable x. It can be shown (`Exercise`), by using (2.1), that

(i) M extends to a bounded operator on $\mathcal{H}$,
(ii) M is selfadjoint.

By spectral theorem, there is a resolution of the identity $E(x)$ such that

$$p(M) = \int p(x)\, dE(x)$$

for any polynomial p. Let $d\mu(x) = d(E(x)1, 1)$. Then μ is a probability measure and

$$\int p_n(x)p_m(x)d\mu(x) = (p_n(M)p_m(M)1, 1) = (p_m(M)1, p_n(M)1) = (p_n, p_m) = \delta_n^m.$$

The similar reasoning works for unbounded coefficients, but the operator M is symmetric and unbounded. In order to use spectral theorem, first one has to find a selfadjoint extension of this operator. It can happen that this extension is not unique, and also the orthogonality measure is not unique. We are not going to discuss this topic here. It belongs to the problem of moments.

2.4. **Zeros of orthogonal polynomials.** Assume p_n are polynomials orthogonal with respect to a probability measure μ as constructed in the previous subsection.

Theorem 2.1. *The polynomial p_n has n distinct real roots. If* $\operatorname{supp}\mu \subseteq (-\infty, b]$, *then $p_n(x) > 0$ for any $x \geq b$. If* $\operatorname{supp}\mu \subseteq [a, +\infty)$, *then $(-1)^n p_n(x) > 0$ for any $x \leq a$. In particular, if* $\operatorname{supp}\mu \subseteq [a, b]$ *all the roots of p_n lie in (a, b).*

Proof. Let $x_1, x_2, \ldots, x_m$ be distinct real roots of p_n which have odd multiplicity. Clearly $m \leq n$. Observe that the polynomial

$$p_n(x)(x - x_1)(x - x_2)\ldots(x - x_m)$$

is nonnegative on $\mathbb{R}$. Therefore

$$\int p_n(x)(x - x_1)(x - x_2)\ldots(x - x_m)\, d\mu(x) > 0.$$

The integral cannot vanish, because it would imply that μ is concentrated on finitely many points. By orthogonality, the integral vanishes if $m < n$. Therefore, to avoid contradiction, we must have $m = n$. In order to prove the second part of the theorem, it suffices to take only those roots which lie in $(-\infty, b)$ or $(a, +\infty)$, and integrate over $(-\infty, b)$ or $(a, +\infty)$, respectively. ■

3. Nonnegative linearization

3.1. **Preliminaries.** By the well known trigonometric identity, we have

$$\cos m\theta \cos n\theta = \frac{1}{2}\cos(n - m)\theta + \frac{1}{2}\cos(n + m)\theta. \tag{3.1}$$

Another trigonometric identity is

$$\frac{\sin(m+1)\theta}{\sin\theta}\,\frac{\sin(n+1)\theta}{\sin\theta} = \frac{\sin(n-m+1)\theta}{\sin\theta} + \frac{\sin(n-m+3)\theta}{\sin\theta} + \ldots + \frac{\sin(n+m+1)\theta}{\sin\theta}. \tag{3.2}$$

Remark. The identity (3.2) has a natural group theoretic interpretation. Let $G = SU(2)$ denote the special unitary group of 2×2 unitary matrices with complex coefficients and determinant 1. This group acts on finite dimensional space $\mathcal{H}_n$ of polynomials in two variables z_1 and z_2, homogeneous of degree n, by the rule

$$p(z) \mapsto p(g^{-1}z),$$

where $g \in G$, $z = (z_1, z_2)^T$ and p is a polynomial. This space has dimension $n + 1$, as it is spanned by $z_1^n, z_1^{n-1}z_2, \ldots, z_1 z_2^{n-1}, z_2^n$. The group action gives rise to a group representation π_n, according to

$$\pi_n(g)p(z) = p(g^{-1}z).$$

The representations π_n are irreducible, i.e. the space of $\mathcal{H}_n$ has no nontrivial subspace invariant for all the operators $\pi_n(g)$. It can be shown that every irreducible representation is of the form π_n for some n. The character of the representation π_n is, by definition, the function χ_n on G defined as

$$\chi_n(g) = \mathrm{Tr}\pi_n(g).$$

The characters of different (inequivalent) representations are orthogonal in $L^2(G, m)$, where m is the Haar measure on G. Let $e^{i\theta}$ and $e^{-i\theta}$ be the eigenvalues of $g \in G$. It turns out (`Exercise` *) that χ_n depends only on θ and

$$\chi_n(g) = \frac{\sin(n+1)\theta}{\sin\theta}. \tag{3.3}$$

Consider the tensor product $\pi_m \otimes \pi_n$ of representations. This new representation is no longer irreducible hence it decomposes into irreducible components. It can be shown that for $n \geq m$ we have

$$\pi_m \otimes \pi_n = \pi_{n-m} \oplus \pi_{n-m+2} \oplus \ldots \oplus \pi_{n+m}.$$

Computing characters of both sides of the formula gives

$$\chi_m\chi_n = \chi_{n-m} + \chi_{n-m+2} + \ldots + \chi_{n+m}.$$

We can rewrite formulas (3.1) and (3.2) in terms of the Chebyshev polynomials using the relationship described in Section 2.1.

$$\begin{aligned} T_mT_n =& \tfrac{1}{2}T_{n-m} + \tfrac{1}{2}T_{n+m}, \\ U_mU_n =& U_{n-m} + U_{n-m+2} + \ldots + U_{n+m}. \end{aligned}$$

Thus we obtained that the product of two Chebyshev polynomials can be expressed as a sum of these polynomials with nonnegative coefficients. We are interested if this property holds also for other orthogonal polynomials, especially the classical orthogonal polynomials. Let $\{p_n\}_{n=0}^\infty$ be a system of orthonormal polynomials, as constructed in Section 2.3. The product p_np_m is a polynomial of degree $n+m$ and it can be expressed with respect to the polynomial basis $\{p_k\}_{k=0}^\infty$. We obtain

$$\boxed{p_n(x)p_m(x) = \sum_{k=|n-m|}^{n+m} c(n,m,k)p_k(x),}$$

where

$$c(n,m,k) = \int p_np_mp_k\,d\mu.$$

The sum ranges from $|n-m|$, because if $k < |n-m|$ then $k+m < n$ or $k+n < m$. Hence $\deg(p_kp_m) < \deg(p_n)$ or $\deg(p_kp_n) < \deg p_m$. In both cases the integral of $p_np_mp_k$ vanishes. We are interested in determining when

$$c(n,m,k) \geq 0 \quad \text{for all } n,m,k.$$

If this is satisfied, we say that the system $\{p_n\}_{n=0}^{\infty}$ admits nonnegative linearization or for short we will say it has the property (P). This property has been studied by many authors since 1894.

The numbers $c(n, m, k)$ are called the linearization coefficients. In the last part we will show the consequences of nonnegative linearization to convolution structures associated with orthogonal polynomials. Here we will state and prove one important estimate that follows from the property (P). Assume that $\operatorname{supp}\mu \subset (-\infty, \xi]$. For example, $\operatorname{supp}\mu \subset [-1, 1]$ in the case of the Chebyshev polynomials. Then $p_n(\xi) > 0$, because the leading coefficient is positive and $p_n(x)$ cannot vanish in $[\xi, \infty)$. We have

$$p_n^{2N}(x) = \sum_{k=0} d_k p_k(x), \quad \text{where } d_k \geq 0.$$

Then

$$\int p_n^{2N}(x)\, d\mu(x) = d_0 \leq \sum_{k=0} d_k p_k(\xi) = p_n^{2N}(\xi),$$

$$\left(\int p_n^{2N}(x) d\mu(x) \right)^{1/2N} \leq p_n(\xi).$$

Taking the limit with $N \to \infty$ gives (`Exercise`)

$$\boxed{|p_n(x)| \leq p_n(\xi) \quad \text{for } x \in \operatorname{supp}\mu.}$$

3.2. **Renormalization.** We will be dealing with systems of polynomials orthogonal with respect to a measure μ, but we do not insist that they are necessarily orthonormal. How can one obtain such polynomials ?

Let $\{p_n\}_{n=0}^{\infty}$ be polynomials orthonormal with respect to a measure μ. Let

$$P_n(x) = \sigma_n^{-1} p_n(x) \tag{3.4}$$

for a sequence of positive coefficients σ_n, with $\sigma_0 = 1$. The polynomials P_n are orthogonal relative to the measure μ. By (2.1), we obtain

$$xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}, \tag{3.5}$$

with (`Exercise`)

$$\gamma_n = \frac{\sigma_{n+1}}{\sigma_n} \lambda_n, , \tag{3.6}$$

$$\alpha_n = \frac{\sigma_{n-1}}{\sigma_n} \lambda_{n-1}. \tag{3.7}$$

Therefore $\alpha_n > 0$ for $n \geq 1$ and $\gamma_n > 0$ for $n \geq 0$. It can be shown easily (`Exercise`) that the equations (3.6) and (3.7) are equivalent to

$$\alpha_{n+1}\gamma_n = \lambda_n^2. \tag{3.8}$$

In other words, if polynomials P_n satisfy (3.5) and (3.8), they are related to the orthonormal polynomials p_n by (3.4), for some positive sequence σ_n, or equivalently they are orthogonal with respect to μ and have positive leading coefficients. The polynomials P_n are simply renormalized versions of orthonormal polynomials p_n.

What are the possible choices of renormalized polynomials, once we have orthonormal polynomials satisfying (2.1) ? Assume that we have positive sequences α_n and γ_n such that (3.8) is satisfied. Then we may have

$$xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1},$$
$$xP_n = \alpha_{n+1} P_{n+1} + \beta_n P_n + \gamma_{n-1} P_{n-1}, \tag{3.9}$$
$$xP_n = P_{n+1} + \beta_n P_n + \lambda_{n-1}^2 P_{n-1}. \tag{3.10}$$

All these systems are renormalized versions of p_n, because in each case the formula (3.8) is satisfied.We should have used different symbols to denote these polynomials according to the choice of normalization, because the polynomials are not equal for the same value of n. They are equal only modulo the coefficient depending on n.

We will deal with polynomials P_n satisfying (3.5) and (3.8). Does it affect the property (P) ? Fortunately, the property (P) for the polynomials p_n is satisfied if and only if it is satisfied for P_n. This follows from the fact that

$$p_n(x)p_m(x) = \sum_{k=|n-m|}^{n+m} c(n,m,k)p_k(x)$$

immediately implies

$$P_n(x)P_m(x) = \sum_{k=|n-m|}^{n+m} g(n,m,k)P_k(x),$$

where

$$g(n,m,k) = \frac{\sigma_k}{\sigma_n \sigma_n} c(n,m,k).$$

3.3. **History.** For $\alpha, \beta > -1$ the Jacobi polynomials $J_n^{\alpha,\beta}$ are orthogonal with respect to the weight

$$d\mu_{\alpha,\beta}(x) = c_{\alpha,\beta}(1-x)^\alpha(1+x)^\beta dx, \ -1 < x < 1.$$

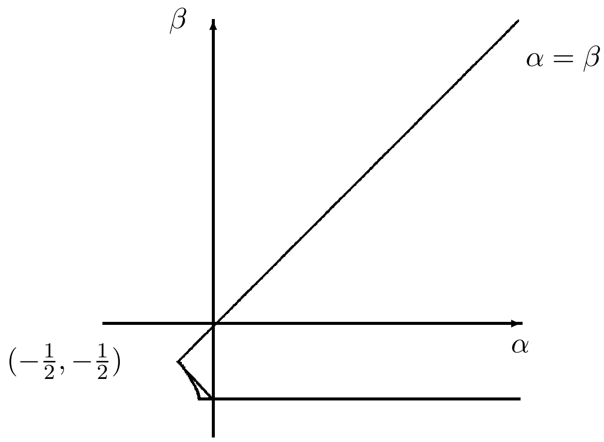
As special cases, we have

Chebyshev	$\alpha=\beta=-\frac{1}{2}$	$T_n(\cos\theta)=\cos n\theta$
Legendre	$\alpha=\beta=0$	Lebesgue measure
Chebyshev II	$\alpha=\beta=\frac{1}{2}$	$U_n(\cos\theta)=\dfrac{\sin(n+1)\theta}{\sin\theta}$
Gegenbauer	$\alpha=\beta$	

In the next table we listed the basic achievements for the Jacobi polynomials. The third column shows the range of the parameters for which nonnegative linearization have been shown.

1919	Dougall	$\alpha=\beta>-\frac{1}{2}$
1962	Hylleraas	$\alpha=\beta>-\frac{1}{2}$ $\alpha=\beta+1$
1970	Gasper	$\alpha\ge\beta>-1$ $\alpha+\beta+1\ge 0$
1970	Gasper	$\alpha\ge\beta>-1$ $c(2,2,2)\ge 0$

The last result gives necessary and sufficient conditions for the property (P) for the Jacobi polynomials. Graphically, the region given by Gasper's results is depicted below. The difference between these results is a tiny region enclosed by the curve that starts at $(-\frac{1}{2},-\frac{1}{2})$ and the lines $\beta=-1$ and $\alpha+\beta+1=0$.



The equation of the curve is

$$a(a+3)^2(a+5)=b^2(a^2-7a-24),$$

where $a = \alpha+\beta+1$ and $b = \alpha-\beta$. The problem has been studied also for other polynomials.

1894	Rogers	q-ultraspherical
1981	Bressoud	q-ultraspherical
1981	Rahman	continuous q-Jacobi $0 < q < 1$, $\alpha \ge \beta > -1$ $\alpha+\beta+1 \ge 0$
1983	Gasper	continuous q-Jacobi $0 < q < 1$, $\alpha \ge \beta > -1$ $c(2,2,2) \ge 0$,
1983	Lasser	Associated Legendre

Actually, Dougall's and Hylleraas' theorems follow from the paper of Rogers [2], by taking limits with $q \to 1$ or $q \to -1$, but Rogers' paper have been rediscovered only around 1970. Definitions and basic properties of these polynomials can be found in [14].

The methods used by the authors didn't allow them to generalize their results to other orthogonal polynomials. The first general theorem is due to Askey [2].

Theorem 3.1 (Askey 1970)**.** *If the sequences λ_n and β_n are nondecreasing, then the system $\{p_n\}$ satisfies the property (P).*

Example. The Gegenbauer polynomials P_n satisfy

$$xP_n = \frac{n+2\alpha+1}{2n+2\alpha+1}P_{n+1} + \frac{n}{2n+2\alpha+1}P_{n-1}.$$

We have $\beta_n \equiv 0$ and

$$\lambda_n^2 = \alpha_{n+1}\gamma_n = \frac{1}{4}\left[1 + \frac{1-4\alpha^2}{4(n+\alpha+1)^2-1}\right].$$

λ_n is nondecreasing if and only if $\alpha \ge \frac{1}{2}$. The case $-\frac{1}{2} \le \alpha < \frac{1}{2}$ is left open, although the property (P) follows from explicit formulas.

Askey called for some more general criteria that would be able to capture at least the Legendre polynomials case and he stated the problem of finding them, if possible. In order to formulate these criteria, we will pass to renormalized polynomials. Assume that there is renormalization of the system p_n such that the polynomials P_n satisfy $xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1}$.

Theorem 3.2 (Sz. 1992)**.** *If α_n, β_n, $\alpha_n+\gamma_n$ are nondecreasing and $\alpha_n \le \gamma_n$ for all n, then the system $\{p_n\}_{n=0}^{\infty}$ has the property (P).*

[2] Richard Askey pointed it out to me.

In the case when $\beta_n \equiv 0$, which is equivalent to the fact that the orthogonality measure μ is symmetric about the origin, we can split assumptions according to the parity of the index n.

Theorem 3.3 (Sz. 1992)**.** *If $\beta_n \equiv 0$ and α_{2n}, α_{2n+1}, $\alpha_{2n}+\gamma_{2n}$, $\alpha_{2n+1}+\gamma_{2n+1}$ are nondecreasing and $\alpha_n \le \gamma_n$ for all n, then the system $\{p_n\}_{n=0}^{\infty}$ has the property (P).*

All the mentioned results, with exception of Gasper's second paper on the Jacobi polynomials and on the q-Jacobi polynomials, follow from Theorems 3.2 and 3.3. Moreover, the associated polynomials are covered as well, because the assumptions are invariant for the shift of the index. The associated polynomials of order k are defined by the recurrence relation

$$xP_n^{(k)} = \gamma_{n+k}P_{n+1}^{(k)} + \beta_{n+k}P_n^{(k)} + \alpha_{n+k}P_{n-1}^{(k)}.$$

Example. For the Gegenbauer polynomials we have $\beta_n \equiv 0$, $\alpha_n + \gamma_n \equiv 1$ and

$$\alpha_n = \frac{n}{2n+2\alpha+1} \nearrow \frac{1}{2} \qquad \text{iff} \qquad \alpha \ge -\frac{1}{2},$$

$$\alpha_n \le \gamma_n \qquad \text{iff} \qquad \alpha \ge -\frac{1}{2}.$$

Example. The generalized Chebyshev polynomials are orthogonal with respect to $d\mu(x) = |x|^{2\beta+1}(1-x^2)^{\alpha}dx$ on the interval $(-1,1)$, where $\alpha, \beta > -1$. They satisfy

$$\begin{aligned} xP_{2n} &= \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1}\,P_{2n+1} + \frac{n}{2n+\alpha+\beta+1}\,P_{2n-1}, \\ xP_{2n-1} &= \frac{n+\alpha}{2n+\alpha+\beta}\,P_{2n} + \frac{n+\beta}{2n+\alpha+\beta}\,P_{2n-2}. \end{aligned}$$

We have $\beta_n \equiv 0$ and $\alpha_n + \gamma_n \equiv 1$. The assumptions of Theorem 3.3 are satisfied if and only if $\alpha \ge \beta$ and $\alpha+\beta+1 \ge 0$. Moreover, we have

$$J_n^{\alpha,\beta}(2x^2-1) = P_{2n}(x),$$

where $J_n^{\alpha,\beta}$ denote the Jacobi polynomials (`Exercise`).

This trick, to relate the orthogonal polynomials to other polynomials orthogonal with respect to a symmetric measure, is very useful in the context of nonnegative linearization.

3.4. **Discrete boundary value problem.** We are going to prove Theorems 3.2 and 3.3 by using a discrete version of second order hyperbolic boundary value problem in two variables.

In application of these theorems we have used the normalization in which the polynomials P_n satisfied (3.5). In the proof we will switch normalization to the one given by (3.9). As we know, it is irrelevant in which normalization we show property (P). Hence, let

$$xP_n = \alpha_{n+1}P_{n+1} + \beta_n P_n + \gamma_{n-1}P_{n-1}.$$

We are interested in proving the nonnegativity of the coefficients in the expansion

$$P_n P_m = \sum_{k=|n-m|}^{n+m} g(n,m,k)P_k.$$

By orthogonality, we have that

$$\int_{\mathbb{R}} P_n P_m P_k d\mu = \left(\int_{\mathbb{R}} P_k^2 d\mu\right) g(n,m,k). \tag{3.11}$$

Let $u(n,m)$ be a matrix indexed by $n,m \geq 0$. Define two operators L_1 and L_2 acting on the matrices by the rule

$$\begin{aligned}(L_1 u)(n,m) &= \alpha_{n+1}u(n+1,m) + \beta_n u(n,m) + \gamma_{n-1}u(n-1,m),\\ (L_2 u)(n,m) &= \alpha_{m+1}u(n,m+1) + \beta_m u(n,m) + \gamma_{m-1}u(n,m-1).\end{aligned}$$

Observe that, by the recurrence relation, if we take $u(n,m) = P_n(x)P_m(x)$ for some x, then

$$L_1 u = xu, \qquad L_2 u = xu.$$

Thus, for such matrices u we have

$$Hu := (L_1 - L_2)u = 0.$$

The last equality holds also for the special matrix

$$u(n,m) = g(n,m,k) = \left(\int_{\mathbb{R}} P_k^2 d\mu\right)^{-1} \int_{\mathbb{R}} P_n(x)P_m(x)P_k(x)d\mu(x). \tag{3.12}$$

Additionally, we have

$$u(n,0) = g(n,0,k) = \begin{cases} 1 & k = n, \\ 0 & k \neq n. \end{cases}$$

Hence the matrix u satisfies $Hu = 0$ and has nonnegative boundary values.

Proposition 3.1. *The polynomials P_n have the property (P) if and only if every matrix $u = \{u(n,m)\}$ such that*

$$\begin{cases} Hu &= 0, \\ u(n,0) &\geq 0, \end{cases} \tag{3.13}$$

satisfies $u(n,m) \geq 0$ for $n \geq m \geq 0$.

Proof. The "if" direction is clear, because we can always assume $n \geq m$ and, as we have seen above, the matrix $u(n, m) = g(n, m, k)$ satisfies (3.13). Assume P_n have the property (P). Let $u = u(n, m)$ be any solution to (3.13). Set

$$\tilde{u}(n, m) = \sum_{k=0}^{\infty} u(k, 0) g(n, m, k).$$

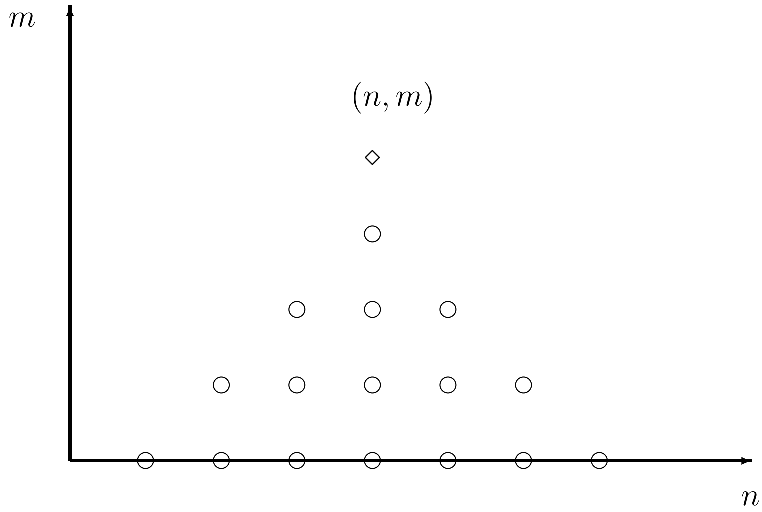
Then $\tilde{u}$ is also a solution. Moreover, $\tilde{u}(n, m) \geq 0$ and $\tilde{u}(n, 0) = u(n, 0)$. By the uniqueness of the solution, we have $u(n, m) = \tilde{u}(n, m) \geq 0$. ■

Now we are reduced down to showing that, under certain assumptions on the coefficients α_n, β_n and γ_n, the boundary problem (3.13) has nonnegative solutions. To this end, we will give an equivalent condition which is much more convenient for direct applications.

For each point (n, m), with $n \geq m \geq 0$, let $\Delta_{n,m}$ denote the set of lattice points in the plane defined by

$$\Delta_{n,m} = \{(i, j) \,:\, 0 \leq j \leq i, |n - i| < m - j\}.$$

The set $\Delta_{n,m}$ is depicted below (the points are marked with empty circles).



Let H^* be the adjoint operator to H with respect to the inner product of matrices

$$\langle u, v \rangle = \sum_{n,m=0}^{\infty} u(n, m) \overline{v(n, m)}.$$

This operator acts according to

$$\begin{aligned}(H^* v)(n, m) =& \gamma_n v(n + 1, m) + \beta_n v(n, m) + \alpha_n v(n - 1, m) \\ & - \gamma_m v(n, m + 1) - \beta_m v(n, m) - \alpha_m v(n, m - 1).\end{aligned}$$

The following lemma is needed for proving the next theorem.

Lemma 3.1. *There exists a matrix $v_{n,m}(i,j)$ such that*

(i) $\operatorname{supp} v_{n,m} \subset \Delta_{n,m}$.
(ii) $(H^* v_{n,m})(n,m) = -1$.
(iii) $(H^* v_{n,m})(i,j) = 0$ *for* $1 \le j < m$.

Proof. The conditions (ii) and (iii) provide m^2 linear equations with m^2 unknowns $v_{n,m}(i,j)$ for $(i,j) \in \Delta_{n,m}$. These equations are independent, because the coefficients α_i and γ_i do not vanish for all i. Hence, the system can be solved. ■

Theorem 3.4 (Sz. 2001)**.** *The boundary problem (3.13) admits nonnegative solutions if and only if for every (n,m), with $n \ge m \ge 0$, there exists a matrix $v_{n,m}(i,j)$ such that*

(i) $\operatorname{supp} v_{n,m} \subset \Delta_{n,m}$.
(ii) $(H^* v_{n,m})(n,m) < 0$.
(iii) $(H^* v_{n,m})(i,j) \ge 0$ *for* $(i,j) \ne (n,m)$.

Proof. ($\Leftarrow$) Let $u(n,m)$ satisfy the boundary value problem (3.13). It suffices to show that $u(n,m) \ge 0$. We will use induction in m. Assume $u(i,j) \ge 0$ for $j < m$. Then

$$\begin{aligned} 0 &= \langle Hu, v_{n,m} \rangle = \langle u, H^* v_{n,m} \rangle \\ &= (H^* v_{n,m})(n,m)\, u(n,m) + \sum_{j<m} (H^* v_{n,m})(i,j)\, u(i,j). \end{aligned}$$

Hence

$$-(H^* v_{n,m})(n,m) u(n,m) = \sum_{j<m} (H^* v_{n,m})(i,j) u(i,j) \ge 0.$$

Thus $u(n,m) \ge 0$.
($\Rightarrow$) In order to complete the proof, it suffices to show that for $v_{n,m}$ as in Lemma 3.1 we have

$$(H^* v_{n,m})(i,0) \ge 0 \quad \text{for } i = |n-m|, |n-m|+1, \ldots, n+m-1, n+m.$$

Let u be any solution of (3.13). By Lemma 3.1, we have

$$\begin{aligned} 0 &= \langle Hu, v_{n,m} \rangle = \langle u, H^* v_{n,m} \rangle \\ &= -u(n,m) + \sum_{i=|n-m|}^{n+m} (H^* v_{n,m})(i,0) u(i,0). \end{aligned}$$

Thus

$$u(n,m) = \sum_{i=|n-m|}^{n+m} (H^* v_{n,m})(i,0) u(i,0).$$

Assume $(H^* v_{n,m})(i_0, 0) < 0$. Take the boundary condition $u(i, 0) = 0$ for $i \neq i_0$ and $u(i_0, 0) = 1$. Then $u(n, m) < 0$, which contradicts the assumptions. ■

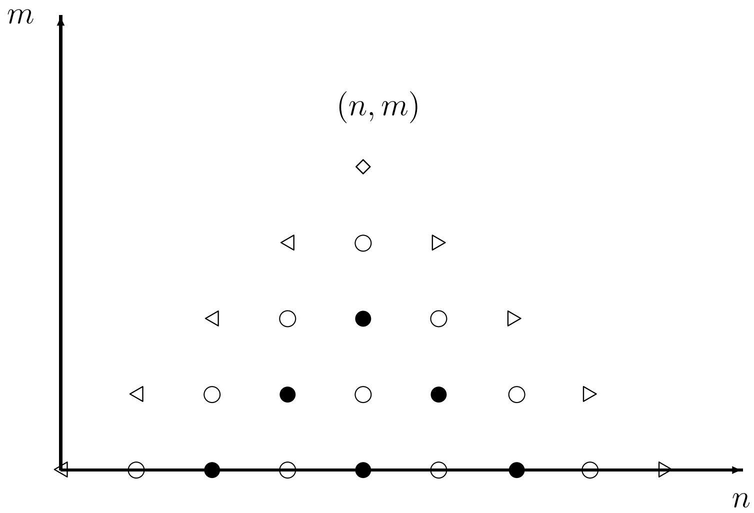
In view of the preceding theorem, it suffices now to come up with a suitable choice of matrices $v_{n,m}$ in order to get some conditions for nonnegative linearization. The funny thing is that the obvious choice

$$v_{n,m}(i, j) = 1 \qquad \text{for } (i, j) \in \Delta_{n,m}$$

is a complete failure (`Exercise`). This may be a reason for which Askey's problem remained unsolved for such a long time. But, if we assign the value 1 to every other point in $\Delta_{n,m}$, interesting outcome follows. Let

$$v_{n,m}(i, j) = \begin{cases} 1 & (i, j) \in \Delta_{n,m},\ (n + m) - (i + j) \text{ odd}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.14}$$

The points where $v_{n,m}$ does not vanish are marked by empty circles.



Then $\operatorname{supp} v$ consists of $\circ$, while $\operatorname{supp} H^* v$ consists of $\circ, \bullet, \triangleleft, \triangleright$ and $\diamond$. Moreover, we have

$$(H^* v_{n,m})(i, j) = \begin{cases} -\alpha_m & (i, j) \ - \ \diamond, \\ \beta_i - \beta_j & (i, j) \ - \ \circ, \\ \alpha_i + \gamma_i - \alpha_j - \gamma_j & (i, j) \ - \ \bullet, \\ \alpha_i - \alpha_j & (i, j) \ - \ \triangleright, \\ \gamma_i - \alpha_j & (i, j) \ - \ \triangleleft. \end{cases}$$

If the assumptions of Theorem 3.2 are satisfied, then

$$(H^*v)(i,j) \geq 0 \text{ for } j < m,$$
$$(H^*v)(n,m) < 0.$$

In this way the conclusion of Theorem 3.2 holds.

Let's turn to Theorem 3.3, i.e. to the case when $\beta_n \equiv 0$. First observe that, by the recurrence relation,

$$xP_n = \alpha_{n+1}P_{n+1} + \gamma_{n-1}P_{n-1}$$

the polynomials with even indices involve only even powers of x, while the polynomials with odd indices involve only odd powers. In order to show nonnegative linearization, we have to show nonnegativity of the integral of the triple product $\int P_nP_mP_kd\mu$. By symmetry of the measure μ, the condition

$$\int_{\mathbb{R}} P_nP_mP_kd\mu \neq 0$$

implies that one of the indices n, m or k is an even number. Since it is possible to interchange the roles of n, m and k, we can always assume that k is an even number. Observe that if $u(n,m) = g(n,m,2k)$ then $u(2n+1,0) = 0$ for any n. Therefore, instead of studying (3.13), we can study

$$\begin{cases} Hu &= 0, \\ u(2n,0) &\geq 0, \\ u(2n+1,0) &= 0. \end{cases} \tag{3.15}$$

We would like to show that this boundary problem has nonnegative solutions. It can be checked directly that $u(n,m) = 0$ if $n+m$ is an odd number. Thus, it suffices to consider those (n,m) for which both n, m are even or odd. For such (n,m) the matrix $H^*v_{n,m}$, where $v_{n,m}$ is defined by (3.14), satisfies the assumptions of Theorem 3.4 exactly when the assumptions of Theorem 3.3 are fulfilled.

Remark. Let

$$\begin{aligned} (L_1u)(n,m) &= \alpha_{n+1}u(n+1,m) + \beta_nu(n,m) + \gamma_{n-1}u(n-1,m), \\ (L_2u)(n,m) &= \alpha'_{m+1}u(n,m+1) + \beta'_mu(n,m) + \gamma'_{m-1}u(n,m-1). \end{aligned}$$

Let $H = L_1 - L_2$. Consider the boundary value problem for $u = \{u(n,m)\}_{n,m=0}^{\infty}$

$$\begin{cases} Hu &= 0, \\ u(n,0) &\geq 0. \end{cases}$$

Assume that

$$\begin{aligned}\beta_n \geq & \beta'_m,\\ \alpha_n \geq & \alpha'_m,\\ \alpha_n + \gamma_n \geq & \alpha'_m + \gamma'_m,\\ \gamma_n \geq & \alpha'_m\end{aligned}$$

for $n \geq m \geq 0$. Following the lines of the proof of Theorem 3.2, we can show that $u(n,m) \geq 0$ for $n \geq m$.

One can use this to study the integrals of the form

$$\int P_n \widetilde{P}_m P_k d\mu(x)$$

where P_n and $\widetilde{P}_n$ are two systems of orthogonal polynomials and μ is the orthogonality measure for P_n.

3.5. **Quadratic transformation.** Theorems 3.2 and 3.3 are more efficient when we deal with polynomials orthogonal with respect to a symmetric measure, i.e. when $\beta_n \equiv 0$. There is a way of reducing nonsymmetric case to the symmetric one by a special transformation. Assume we are dealing with a nonsymmetric measure μ with support contained in the interval $[-1,1]$. Let $\nu(y)$ be a new symmetric measure such that

$$d\nu(y) = \frac{1}{2} d\mu(2y^2-1) \quad \text{for } y > 0$$

and

$$\nu(\{0\}) = \mu(\{-1\}).$$

This transformation consists in shifting the measure μ by 1 to the right, scaling to the interval $[0,1]$, substituting $x = y^2$ and symmetrizing about the origin. The resulting measure is symmetric and supported in the interval $[-1,1]$. The point is that the polynomials orthogonal with respect to ν and μ are closely related. Indeed, denote by $Q_n(y)$ the polynomials orthogonal relative to $d\nu(y)$.Let $n \neq m$. By substituting $x = 2y^2 - 1$, we obtain

$$\begin{aligned}0 &= \int_{-1}^{1} Q_{2n}(y) Q_{2m}(y) d\nu(y)\\ &= \int_{-1}^{1} Q_{2n}\left(\sqrt{\tfrac{x+1}{2}}\right) Q_{2m}\left(\sqrt{\tfrac{x+1}{2}}\right) d\mu(x).\end{aligned}$$

The polynomial Q_{2n} involves only even powers of the variable, which follows from symmetry of the measure. Thus $P_n(x) = Q_{2n}\left(\sqrt{\frac{x+1}{2}}\right)$ is a polynomial of degree n in x. For different indices these polynomials are orthogonal to each other relative to $d\mu(x)$. Therefore, if Q_n admit nonnegative linearization so do P_n. Moreover, the recurrence relations for P_n and Q_n are related, as well. We have

$$Q_{2n}(y) = P_n(2y^2 - 1).$$

Let

$$yQ_n(y) = \gamma_n Q_{n+1}(y) + \alpha_n Q_{n-1}(y).$$

Then, in view of $x = 2y^2 - 1$, we get

$$\begin{aligned} xP_n(x) &= (2y^2-1)P_n(2y^2-1) = (2y^2-1)Q_{2n}(y) \\ &= 2\gamma_{2n}\gamma_{2n+1}P_{n+1}(x) + [2\alpha_{2n+1}\gamma_{2n} + 2\alpha_{2n}\gamma_{2n-1} - 1]P_n(x) \\ &\quad +2\alpha_{2n}\alpha_{2n-1}P_{n-1}(x). \end{aligned}$$

Remark. If we normalize the polynomials Q_n in such a way that $Q_n(1) = 1$, then the coefficients α_{2n} and α_{2n+1} are the canonical moments of the measure μ, as defined in the Holger Dette lectures.

Obviously the recurrence relation for P_n is much more complicated than the one for Q_n. That is why it is convenient to test Theorem 3.3 on the symmetric polynomials Q_n, instead of applying Theorem 3.2 to the nonsymmetric polynomials P_n. We have already experienced this when we considered the nonsymmetric Jacobi polynomials.

Example. The Askey–Wilson polynomials satisfy the recurrence relation

$$2xP_n(x) = A_nP_{n+1}(x) + [a + a^{-1} - (A_n + C_n)]P_n(x) + C_nP_{n-1}(x),$$

$$\begin{cases} A_n = \dfrac{(1-abcdq^{n-1})(1-abq^n)(1-acq^n)(1-adq^n)}{a(1-abcdq^{2n-1})(1-abcdq^{2n})}, \\[2ex] C_n = \dfrac{a(1-q^n)(1-bcq^n)(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \end{cases}$$

Let Q_n be the polynomials defined by the recurrence relation

$$2xQ_n(x) = \gamma_n Q_{n+1}(x) + \alpha_n Q_{n-1}(x),$$

where

$$\begin{cases} \alpha_{2n} = -ab\dfrac{(1-q^n)(1-cdq^{n-1})}{1-abcdq^{2n-1}}, \\[2ex] \gamma_{2n} = \dfrac{(1-abcdq^{n-1})(1-abq^n)}{1-abcdq^{2n-1}}, \\[2ex] \alpha_{2n+1} = -\dfrac{a(1-bcq^n)(1-bdq^n)}{b(1-abcdq^{2n})}, \\[2ex] \gamma_{2n+1} = \dfrac{(1-acq^n)(1-adq^n)}{(1-abcdq^{2n})}. \end{cases}$$

It can be checked, by using recurrence relations, that

$$Q_{2n}(x) = P_n(2a^{-1}x^2 + \tfrac{1}{2}(b+b^{-1})).$$

We have

$$\alpha_{2n} + \gamma_{2n} = 1 - ab \quad \text{and} \quad \alpha_{2n+1} + \gamma_{2n+1} = 1 - ab^{-1}.$$

We notice that the recurrence relation for the polynomials Q_n makes it possible to apply Theorem 3.3 and get reasonable conditions on the parameters. At the same time, working directly with the recurrence relation for P_n seems hopeless.

Theorem 3.5 (Sz 1996)**.** *Let a, b, c, d and q satisfy*

(i) $0 \le q < 1$,
(ii) $ac < 1$, $ad < 1$, $bc < 1$, $bd < 1$,
(iii) $a > 0$, $b < 0$ *and* $cd < 0$,
(iv) $a + b \le 0$ *and* $c + d \le 0$,
(v) $ab + 1 \ge 0$ *and* $cd + q \ge 0$.

Then the polynomials Q_n, as well as the Askey-Wilson polynomials P_n, have non-negative product linearization.

By specifying the parameters $\alpha = q^{\alpha+\frac{1}{2}}$, $b = -q^{\beta+\frac{1}{2}}$ and $c = -d = q^{\frac{1}{2}}$, we get the Rahman theorem on the continuous q-Jacobi polynomials.

4. Commutative Banach algebras

4.1. **Preliminaries.** We are going to construct and study certain Banach algebras associated with systems of orthogonal polynomials. Here we will recall basic facts which we are going to use.

A Banach algebra B is a Banach space in which we can also multiply elements in such a way that B becomes an algebra and

$$\|ab\| \le \|a\|\|b\|$$

for any $a, b \in B$. The algebra is called commutative if $ab = ba$ for all $a, b \in B$. The algebra is unital if there exists an element e, called the unit, such that $ea = ae = a$ for any $a \in B$. An element a is called invertible if there exists b such that $ab = ba = e$. The set of all complex numbers z such that $ze - a$ is not invertible is called the spectrum of a and denoted by $\sigma(a)$. The algebra is called $*$-algebra if there is a conjugation such that

(i) $(xy)^* = y^* x^*$.
(ii) $\|x^*\| = \|x\|$.

The principal example, that we are going to generalize later on to the orthogonal polynomials setting, is

$$\ell^1(\mathbb{Z}) = \left\{ a = \{a(n)\}_{-\infty}^{\infty} \,:\, \|a\| = \sum_{-\infty}^{\infty} |a(n)| < +\infty \right\}.$$

The multiplication is given by convolution of sequences

$$a * b(n) = \sum_{-\infty}^{\infty} a(m) b(n-m).$$

The algebra $\ell^1(\mathbb{Z})$ is commutative and unital, because the element

$$\delta_0(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

is the unit. The conjugation is given by complex conjugation of the terms.

Other examples of Banach algebras (not necessarily commutative or unital) are:

(i) $C[0,1]$ with sup-norm and pointwise multiplication (more generally $C(X)$, where X is a compact Hausdorff space).
(ii) $B(\mathcal{H})$ - the algebra of all bounded operators on a Hilbert space with operator norm, or any norm closed subalgebra of $B(\mathcal{H})$.
(iii) $L^1(G, m)$ where G is a locally compact group and m is left invariant Haar measure with multiplication given by convolution

$$f * g(x) = \int_G f(y) g(y^{-1}x) \, dm(y).$$

Assume we are dealing with a commutative unital Banach algebra B. We may think of $\ell^1(\mathbb{Z})$ or $C(X)$. Take an element in the algebra. If we are dealing with $C(X)$ it is easy to determine if our element is invertible or not, while for elements in the algebra $\ell^1(\mathbb{Z})$ it is not so obvious. It was Gelfand who invented a way of mapping B into certain $C(X)$ space such that an element in B is invertible if and only if its image in $C(X)$ is invertible.

Let X be the set of all multiplicative linear functionals on B, i.e. those linear functionals $\varphi : B \to \mathbb{C}$ which satisfy

$$\varphi(ab) = \varphi(a)\varphi(b).$$

It can be shown that such functionals are necessarily continuous with norm bounded by 1:

$$|\varphi(a)| \le \|a\|, \qquad a \in B.$$

Thus $X \subseteq (B^*)_1$. In B^* we have various topologies to choose from like norm topology, weak topology and weak-star topology. By definition, the weak-star topology is the weakest topology such that the mappings $B^* \ni \psi \mapsto \psi(a)$ are continuous for $a \in B$. We endow $(B^*)_1$ with the weak-star topology. Then, by the Banach-Alaoglu Theorem, $(B^*)_1$ is a compact set and hence X is compact as a closed (`Exercise`) subset of $(B^*)_1$.

For each element $a \in B$ we define the Fourier-Gelfand transform $\widehat{a}$ as a function on X according to

$$\widehat{a}(\varphi) = \varphi(a), \qquad \varphi \in X.$$

By definition, $\widehat{a}$ is a continuous function on X.

Theorem 4.1 (Gelfand)**.** *The mapping $a \mapsto \widehat{a}$ is a homomorphism of norm 1 from B to $C(X)$. Moreover, the element a is invertible if and only if $\widehat{a}$ does not vanish on X. More generally, the spectrum of a is equal to the range of $\widehat{a}$.*

If we have an element a in B we can raise it to any power and add these powers with complex coefficients. In other words, we can apply any polynomial to a. Also since B is a Banach algebra we can operate on a with any entire function by using its Taylor series. Can we go beyond that ? We can operate on a with functions $F(z)$ which are holomorphic in the neighborhood of the spectrum of a or the range of $\widehat{a}$ with a help of the Cauchy integral formula

$$F(a) := \frac{1}{2\pi i}\int_\gamma \frac{F(z)}{ze - a}\,dz,$$

where γ is a closed Jordan curve enclosing the range of $\widehat{a}$. In this way, we defined an element $F(a)$ in B such that $\widehat{F(a)} = F \circ \widehat{a}$ (`Exercise`). Let's turn to our principal example $\ell^1(\mathbb{Z})$. The multiplicative functionals are given by (`Exercise` *)

$$\{a(n)\}_{n=-\infty}^{\infty} \mapsto \widehat{a}(\theta) = \sum_{-\infty}^{\infty} a(n)e^{in\theta}.$$

In other words, $X \cong \mathbb{T}$. Hence, a sequence a is invertible in ℓ^1, i.e. $a^{-1} \in \ell^1$ if and only if its transform does not vanish on $\mathbb{T}$. This theorem is due to Wiener.

Also, if a function $F(z)$ is holomorphic in the neighborhood of the set $\{\sum_{-\infty}^{\infty} a(n)\, e^{in\theta} : \theta \in \mathbb{R}\}$ then $F(a)$ belongs to ℓ^1. This result is due to Lévy.

4.2. **Convolution associated with orthogonal polynomials.** The systematic study of this topic have been undertaken by Lasser [16]. Let an orthonormal polynomial system $\{p_n\}_{n=0}^{\infty}$ satisfy property (P). Assume that $\operatorname{supp}\mu \subset (-\infty, \xi]$. For example, $\operatorname{supp}\mu \subset [-1,1]$ in the case of the Jacobi polynomials. Then $p_n(\xi) > 0$ and

$$\boxed{|p_n(x)| \le p_n(\xi) \quad \text{for } x \in \operatorname{supp}\mu.}$$

It is more convenient to use renormalized polynomials $R_n(x)$, where

$$R_n(x) = \frac{p_n(x)}{p_n(\xi)},$$

because

$$\boxed{|R_n(x)| \le 1 \quad \text{for } x \in \operatorname{supp}\mu.}$$

The new polynomials also have property (P). Let

$$R_n(x)R_m(x) = \sum_{k=|n-m|}^{n+m} g(n,m,k)R_k(x). \tag{4.1}$$

Hence $g(n,m,k) \ge 0$. Plugging in $x = \xi$ gives

$$\sum_{k=|n-m|}^{n+m} g(n,m,k) = 1. \tag{4.2}$$

We would like to use the coefficients $g(n,m,k)$ to define convolution of sequences by mimicking standard convolution on $\ell^1(\mathbb{Z})$. Since the polynomials R_n are not orthonormal, we will make use of the numbers

$$h(n) = \left(\int R_n^2(x)d\mu(x)\right)^{-1} = p_n(\xi)^2.$$

There is an important symmetry relation involving $g(n,m,k)$ and $h(n)$. Indeed, multiplying both sides of (4.1) by $R_k(x)$ and integrating with $d\mu(x)$ gives

$$\int R_n R_m R_k d\mu = g(n,m,k)h(k)^{-1}.$$

Therefore,

$$g(n,m,k)h(k)^{-1} = g(n,k,m)h(m)^{-1} = g(k,m,n)h(n)^{-1}. \tag{4.3}$$

First of all, we will have to work with the weighted $\ell^1(\mathbb{N}_0, h)$ space consisting of sequences $\{a(n)\}_{n=0}^{\infty}$ such that

$$\sum_{n=0}^{\infty} |a(n)|h(n) < +\infty.$$

Let $\{a(n)\} \in \ell^1(\mathbb{N}_0, h)$. The series $\sum a(n)R_n(x)h(n)$ is uniformly convergent on $\operatorname{supp}\mu$, because $R_n(x)$ are bounded by 1. We introduce the convolution operation in $\ell^1(\mathbb{N}_0, h)$ by setting

$$\{a(n)\} * \{b(n)\} = \{c(n)\}$$

if and only if

$$\left(\sum_{n=0}^{\infty} a(n)R_n(x)h(n)\right)\left(\sum_{n=0}^{\infty} b(n)R_n(x)h(n)\right) = \sum_{n=0}^{\infty} c(n)R_n(x)h(n). \tag{4.4}$$

The convolution can be computed explicitly in terms of the coefficients $g(n, m, k)$. Indeed, we have

$$(a * b)(k) = \sum_{n,m} a(n)b(m)h(n)h(m)h(k)^{-1}g(n, m, k).$$

Let's check if the operation is well defined.

$$\begin{aligned}\|a * b\|_1 &= \sum_k \left| \sum_{n,m} a(n)b(m)h(n)h(m)h(k)^{-1}g(n, m, k)\right| h(k) \\ &\le \sum_{n,m} |a(n)||b(m)|h(n)h(m) \sum_k g(n, m, k) = \|a\|_1\|b\|_1.\end{aligned}$$

The space $\ell^1(\mathbb{N}_0, h)$ with the operation $*$ becomes a commutative Banach algebra. Associativity and commutativity immediately follow from the multiplication rule (4.4).

Let's compute the multiplicative functionals of this algebra. By (4.4), the mappings

$$a = \{a(n)\} \mapsto \sum_{n=0}^{\infty} a(n)R_n(z)h(n)$$

for all $z \in \mathbb{C}$ such that $|R_n(z)| \le 1$ for any $n \in \mathbb{N}_0$, give rise to multiplicative functionals. It can be shown that every multiplicative functional is of that form (`Exercise` *) (`Hint:` Let φ be a multiplicative functional. Then there is a complex number z such that $\varphi(\delta_1) = R_1(z)h(1)$. Show that $\varphi(\delta_n) = R_n(z)h(n)$.).

The space of multiplicative functionals $\mathcal{M}$ of the algebra $\ell^1(\mathbb{N}, h)$ is thus given by

$$\mathcal{M} = \{z \in \mathbb{C} : |R_n(z)| \le 1, n \in \mathbb{N}\}.$$

We showed that $\operatorname{supp}\mu \subset \mathcal{M}$.

The set $\operatorname{supp}\mu$ is usually known explicitly, while $\mathcal{M}$ can be sometimes not easy to determine. Thus, it would be convenient to have $\operatorname{supp}\mu = \mathcal{M}$. Under additional assumptions, like

$$\liminf h(n)^{1/n} \le 1,$$

one can show (Sz 1995) that

$$\boxed{\{x \in \mathbb{C} : |R_n(x)| \le 1, n \in \mathbb{N}_0\} = \operatorname{supp}\mu.}$$

In particular, for the Jacobi polynomials we have $h(n) = \mathrm{O}(n^{2\alpha+1})$. More generally, by Nevai, Totik and Zhang (1991), if the coefficients in the recurrence relation for the orthonormal polynomials p_n are convergent and $\xi \in \operatorname{supp}\mu$, then

$$\limsup h(n)^{1/n} \le 1.$$

Now we are in a position to apply analogs of Wiener and Lévy theorems in the context of convolution algebras associated with the polynomials R_n. Let $\mathcal{A}$ denote the space of functions $f(x)$ on $\operatorname{supp}\mu$ such that

$$f(x) = \sum_{n=0}^{\infty} a_n R_n(x) h(n),$$

where $\{a_n\} \in \ell^1(\mathbb{N}_0, h)$. The space $\mathcal{A}$ is an analog of the space of absolutely convergent Fourier series.

Theorem 4.2 (Wiener). *Let $f \in \mathcal{A}$ and $f(x) \neq 0$ for $x \in \operatorname{supp}\mu$. Then $1/f$ also belongs to $\mathcal{A}$.*

Theorem 4.3 (Lévy). *Let $f \in \mathcal{A}$ and let G be a function holomorphic in an open set containing $\{f(x) : x \in \operatorname{supp}\mu\}$. Then $G(f)$ belongs to $\mathcal{A}$.*

5. Open problems

1. Determine the range of (α, β) for which the generalized Chebyshev polynomials have property (P). We know that this holds for $\alpha \ge \beta > -1$ and $\alpha + \beta + 1 \ge 0$. The property can hold only in the region, where it is valid for the Jacobi polynomials.

2. Find any criteria for the property (P) for finite systems of orthogonal polynomials, like the Krawtchouk polynomials. The Krawtchouk polynomials are orthogonal relative to the measure

$$\mu = \sum_{n=0}^{N} \binom{N}{n} p^{N-n}(1-p)^n \delta_n.$$

They satisfy

$$xK_n = (1-p)(N-n)K_{n+1} + [p(N-n) + (1-p)n]K_n + pnK_{n-1}.$$

Eagleson showed that they satisfy the property (P) if and only if $p \le \frac{1}{2}$. Let Q_n satisfy

$$\begin{aligned} yQ_{2n} &= (N-n)Q_{2n+1} + nQ_{2n-1} \\ yQ_{2n-1} &= (1-p)Q_{2n} + pQ_{2n-2}. \end{aligned}$$

Then $K_n(y^2) = Q_{2n}(y)$. It can be shown that the polynomials Q_n admit nonnegative linearization for $p \le \frac{1}{2}$. The problem is to find a general criterion that would handle this case.

3. Assume orthonormal polynomials satisfy property (P) and

$$\xi \in \operatorname{supp} \mu \subseteq (-\infty, \xi]. \tag{5.1}$$

Does it imply

$$\limsup h(n)^{1/n} \le 1 \ ?$$

Recall that $h(n) = p_n(\xi)^2$.

Does (5.1) imply that

$$\mathcal{M} = \operatorname{supp} \mu \ ? \tag{5.2}$$

The evaluation of the Gelfand transform at the point ξ corresponds to the trivial character of $\ell^1(\mathbb{N}_0, h)$.

It is worthwhile comparing (5.1) and (5.2) with the group setting. The condition (5.1) corresponds to the fact that the trivial representation is weakly contained in the left regular representation, while (5.2) corresponds to that every irreducible unitary representation is weakly contained in the left regular representation. In the group setting these conditions are equivalent to each other and amount to the fact that the group is amenable.

6. Special topics

6.1. **Discrete measures.** Let

$$xP_n = \gamma_n P_{n+1} + \beta_n P_n + \alpha_n P_{n-1},$$

where

$$\lambda_n^2 = \alpha_{n+1}\gamma_n \to 0$$

and

$$\beta_n \to \beta.$$

By changing variables $x \mapsto x + \beta$, we can reduce to the case

$$\beta_n \to 0.$$

The orthonormal version satisfy

$$xp_n = \lambda_n p_{n+1} + \beta_n p_n + \lambda_{n-1} p_{n-1}. \tag{6.1}$$

Since $\lambda_n \to 0$ and $\beta_n \to 0$, the operator M acting on $L^2(\mu)$ by multiplication by x is compact and selfadjoint. Hence, the spectrum of M consists of a countable sequence of real numbers convergent to 0. Therefore, the orthogonality measure μ, which is supported on the spectrum of M, is discrete. Assume the polynomials

p_n admit nonnegative linearization. It can be shown that the right most end ξ of the support of μ cannot be a mass point. Indeed, by

$$p_n p_m = \sum_{k=|n-m|}^{n+m} c(n,m,k) p_k$$

we have

$$p_n^2(\xi) = \sum_{k=0}^{2n} c(n,n,k) p_k(\xi) \geq c(n,n,0) p_0(\xi) = 1,$$

because $p_0(\xi) = 1$ and $c(n,n,0) = \int p_n^2(x) d\mu(x) = 1$. Hence,

$$p_n^2(\xi) \geq 1.$$

But

$$\mu(\{\xi\}) = \left(\sum_{n=0}^{\infty} p_n^2(\xi)) \right)^{-1} = 0.$$

Therefore, ξ must be an accumulation point of $\operatorname{supp}\mu$, i.e. $\xi = 0$. Summarizing, $\operatorname{supp}\mu$ consists of an increasing sequence of points convergent to 0. The measure μ cannot be thus symmetric. Can we apply Theorem 3.2 in this case ? The answer is no. Theorem 3.2 requires that

$$\alpha_n \leq \alpha_{n+1}, \qquad \alpha_n \leq \gamma_n.$$

Thus,

$$\alpha_n^2 \leq \alpha_{n+1}\gamma_n = \lambda_n^2 \to 0.$$

Therefore, $\alpha_n \to 0$, and α_n cannot be nondecreasing.

We need some other criteria in this case. In order to formulate, them we need to introduce the notion of chain sequences.

A sequence $(a_0, a_1, \ldots, a_n)$ is called a chain sequence if there exist numbers $(g_0, g_1, \ldots, g_{n+1})$, $0 \leq g_i \leq 1$, satisfying $a_i = (1 - g_i) g_{i+1}$ for $0 \leq i \leq n$.

Theorem 6.1 (Młotkowski, Sz. 2001)**.** *Let the polynomials P_n satisfy (6.1). Assume that β_n is increasing and that for every n the sequence*

$$\frac{\lambda_m^2}{(\beta_n - \beta_m)(\beta_n - \beta_{m+1})}, \qquad m = 0, 1, 2, \ldots, n-2,$$

is a chain sequence. Then the polynomials P_n admit nonnegative product linearization.

Corollary 6.1. *Assume $\lambda_n \leq \beta_{n+2} - \beta_{n+1}$ for every $n \geq 0$. Then the property (P) holds.*

Example. Let

$$x p_n = b q^{n+1} p_{n+1} - a q^n p_n + b q^n p_{n-1}.$$

If $a, b > 0$ and $b \leq a(1-q)$, the property (P) holds.

6.2. **Positive definitness.** This part is based on [27]. Let p_n be orthonormal with respect to μ and

$$xp_n = \lambda_n p_{n+1} + \beta_n p_n + \lambda_{n-1} p_{n-1}. \tag{6.2}$$

Following Nevai, we say that a measure μ belongs to the class $M(0,1)$ if

$$\begin{aligned} \lim_{n\to\infty} \lambda_n &= \tfrac{1}{2}, \\ \lim_{n\to\infty} \beta_n &= 0. \end{aligned}$$

By the Theorem of Blumenthal, the support of μ consists of $[-1,1]$ and at most countable set of the real numbers that can accumulate only at ± 1. In 1970 Nevai showed the following.

$$\lim_{r\to\infty} \int_{-\infty}^{\infty} p_{i+r}(x)p_{j+r}(x)p_n(x)\,d\mu(x) = \frac{1}{\pi}\int_{-1}^{1} p_n(x)T_{|i-j|}(x)\left(1-x^2\right)^{-1/2}\,dx,$$

where T_n denotes the nth Chebyshev polynomial. Hence, if p_n satisfy the property (P), then

$$\int_{-1}^{1} p_n(x)T_k(x)\left(1-x^2\right)^{-1/2}\,dx \geq 0.$$

Therefore,

$$p_n(\cos\theta) = \sum_{k=0}^{n} c(n,k)\cos k\theta, \tag{6.3}$$

where $c(n,k) \geq 0$. The last equation implies that the function

$$\theta \mapsto p_n(\cos\theta)$$

is positive definite.

Theorem 6.2. *Let the orthogonal polynomials $P_n(x)$ satisfy (6.2) and $d\mu(x) \in M(0,1)$. Assume that*

(i) the sequences λ_n and β_n are decreasing,

(ii) the connection coefficients $c(n,k)$ in (6.3) are nonnegative.

Then the polynomials p_n satisfy the property (P).

6.3. **Theorem of Karlin and McGregor.** The following result states, in particular, that the kernel $K(x,t) = e^{-tx}$ is positive definite with respect to any system of orthogonal polynomials on the positive half axis. The proof, we are going to present, makes use of the recurrence relation.

Theorem 6.3. *Let $d\mu(x)$ be a probability measure on the half line $[0,\ +\infty)$. Let $P_n(x)$ be the orthogonal polynomials with respect to $d\mu(x)$, normalized so that $P_n(0) > 0$. Then*

$$\int_0^\infty e^{-tx} P_n(x) P_m(x)\, d\mu(x) \ > \ 0 \tag{6.4}$$

for every $t > 0$.

Remark. In particular $\int_0^\infty e^{-tx} P_n(x) d\mu(x) > 0$. Hence,

$$e^{-tx} = \sum_{n=0}^\infty c(t,n) P_n(x),$$

with $c(t,n) > 0$.

Proof. We start by showing the weak inequality in (6.4). First, we consider a measure with bounded support. Let $d\mu(x)$ be supported on the interval $[0,\ a]$. Set $d\mu_a(x) = d\mu(a-x)$. Then we have

$$\begin{aligned}
&\int_0^\infty e^{-tx} P_n(x) P_m(x)\, d\mu(x) = \int_0^a e^{-tx} P_n(x) P_m(x)\, d\mu(x) \\
&= \int_0^a e^{-t(a-x)} P_n(a-x) P_m(a-x)\, d\mu_a(x) \\
&= e^{-ta} \int_0^a e^{tx} P_n(a-x) P_m(a-x)\, d\mu_a(x) \\
&= e^{-ta} \sum_{k=0}^\infty \frac{t^k}{k!} \int_0^a x^k P_n(a-x) P_m(a-x)\, d\mu_a(x).
\end{aligned}$$

We will show that every term of the series is nonnegative. To this end, observe that the polynomials

$$\widetilde{P}_n(x) = P_n(a-x),$$

are orthogonal with respect to the measure $d\mu_a(x)$ and have positive leading coefficients, as they are normalized by $\widetilde{P}_n(a) > 0$ and the corresponding measure is supported on $[0,\ a]$. Hence, by the Favard theorem, they satisfy a recurrence formula

$$x\, \widetilde{P}_n(x) = \gamma_n \widetilde{P}_{n+1}(x) + \beta_n \widetilde{P}_n(x) + \alpha_n \widetilde{P}_{n-1}(x), \tag{6.5}$$

with γ_n and α_n positive. Multiplying (6.5) by $\widetilde{P}_n(x)$ and integrating with $d\mu_a(x)$ gives

$$\beta_n = \left(\int_0^a \widetilde{P}_n^2(x)\, d\mu_a(x) \right)^{-1} \int_0^a x\, \widetilde{P}_n^2(x)\, d\mu_a(x). \tag{6.6}$$

Thus, the coefficients β_n are nonnegative. By applying (6.5) successively k times, we obtain that the integral

$$\int_0^a x^k \widetilde{P}_n(x)\widetilde{P}_m(x)\,d\mu_a(x)$$

vanishes for $k < |n-m|$ and it is strictly positive otherwise. This proves (6.4) in case of compactly supported measures.

If $d\mu(x)$ is an arbitrary measure supported on $[0,\infty)$, then the sequence of measures $d\mu_N(x) = \chi_{[0,N]}(x)d\mu(x)$ converges to $d\mu(x)$ weakly. Let $P_{n,N}(x)$ be the polynomials orthogonal with respect to measure $d\mu_N(x)$, such that $P_{n,N}(0) = 1$. Since the moments of $d\mu_N(x)$ tend to the corresponding moments of $d\mu(x)$ and the coefficients of orthogonal polynomials depend only on the moments of the measure, we have

$$\int_0^\infty e^{-tx}P_n(x)P_m(x)\,d\mu(x) = \lim_{N\to\infty}\int_0^\infty e^{-tx}P_{n,N}(x)P_{m,N}(x)\,d\mu_N(x). \qquad (6.7)$$

This shows that the integral in (6.4) is nonnegative. Now we are going to prove that actually we have strict inequality in (6.4). To this end, observe that the polynomials $P_n(x)$ satisfy the following.

$$x\,P_n(x) = -\gamma_n P_{n+1}(x) + \beta_n P_n(x) - \alpha_n P_{n-1}(x), \qquad (6.8)$$

where γ_n, $\alpha_n > 0$, except for $\alpha_0 = 0$. Let $m < n$. Consider the function

$$f(t) = \int_0^{+\infty} e^{-xt}P_n(x)P_m(x)\,d\mu(x).$$

We have that $f(t) \geq 0$. Assume that $f(t_0) = 0$, at some point t_0. Then $f(t)$ has a minimum at t_0. Thus $f'(t_0) = 0$. We have

$$\begin{aligned} f'(t_0) &= -\int_0^{+\infty} e^{-xt_0}[x\,P_n(x)]P_m(x)\,d\mu(x) \\ &= \gamma_n\int_0^{+\infty} e^{-xt_0}P_{n+1}(x)P_m(x)\,d\mu(x) - \beta_n\int_0^{+\infty} e^{-xt_0}P_n(x)P_m(x)\,d\mu(x) \\ &\quad +\alpha_n\int_0^{+\infty} e^{-xt_0}P_{n-1}(x)P_m(x)\,d\mu(x). \end{aligned}$$

The first and the third integral are nonnegative by the first part of the proof, while the second integral vanishes by our assumption. As $f'(t_0) = 0$, we get that

$$\int_0^{+\infty} e^{-xt_0}P_{n-1}(x)P_m(x)\,d\mu(x) = 0.$$

We can now repeat the argument several times till we get

$$\int_0^{+\infty} e^{-xt_0}P_m(x)P_m(x)\,d\mu(x) = 0,$$

which gives a contradiction. ■

7. Some questions and answers

7.1. Laguerre polynomials – question of K. Runovski, W. Van Assche. Let L_n^α denote the Laguerre polynomials normalized so that $L_n^\alpha(0) = 1$. Let $P_n = (-1)^n L_n^\alpha$. Then

$$xP_n = (n+\alpha+1)P_{n+1} + (2n+\alpha+1)P_n + nP_{n-1}.$$

By Theorem 3.1, the polynomials P_n^α admit nonnegative linearization for any $\alpha > -1$. However, we cannot construct convolution structure the way we did that in Section 4.2, because the polynomials are orthogonal on $[0, +\infty)$, hence there is no point ξ at which the quantities $P_n(\xi)$ have constant positive sign. Consider the Laguerre functions

$$\mathcal{L}_n^\alpha(x) = e^{-x} L_n^\alpha(x).$$

They form an orthogonal basis in $L^2((0, +\infty), e^x x^\alpha dx)$. According to a theorem of Askey and Gasper [4], we have

$$\mathcal{L}_m^\alpha(x)\mathcal{L}_n^\alpha(x) = \sum_{k=0}^{\infty} g(n, m, k)\mathcal{L}_k^\alpha(x),$$

where $g(n, m, k) \geq 0$ for $\alpha \geq \frac{1}{2}(\sqrt{17} - 5)$ and any n, m and k. This gives rise to a convolution structure different from the one constructed in Sec. 4.2.

7.2. Kazhdan property – question of H. Führ. A noncompact group G has the Kazhdan property if the trivial representation is isolated from all the other irreducible unitary representations. For example $Sp(n, 1)$ are Kazhdan groups. In our setting the Kazhdan property would mean that the point ξ is isolated from $\mathcal{M} \cap \mathbb{R}$. Actually, this can never happen. Indeed, we can consider two cases. First, assume that $\xi \in \operatorname{supp} \mu$. As the right most end of $\operatorname{supp} \mu$, the point ξ cannot be a mass point of μ. Thus, ξ must be an accumulation point of $\operatorname{supp} \mu$, i.e. it is not isolated in $\mathcal{M} \cap \mathbb{R}$. Secondly, assume that the right most end ξ_0 of $\operatorname{supp} \mu$ is strictly less than ξ. Then, since the polynomials $R_n(x)$ are increasing on $[\xi_0, +\infty)$, we have

$$0 < R_n(\xi_0) \leq R_n(x) \leq R_n(\xi) = 1.$$

Hence, $[\xi_0, \xi] \subseteq \mathcal{M}$, i.e. ξ is not isolated.

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