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Hypergroups of compact type

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Abstract

We consider commutative hypergroups with translation operators which are compact on L^2 resp. L^1. It will be shown that such hypergroups are necessarily discrete and that in the case of compact translations on L^1 the support of the Plancherel measure coincides with the set of all characters and the hypergroup must be symmetric. Furthermore we will show that a certain type of Reiter's condition is fulfilled.
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1. Introduction

In this paper we are going to study commutative hypergroups X for which the translation operator $T_x - I$ is compact on the space $L^2(X, m)$ (resp. $L^1(X, m)$) for each $x \in X$, where m is the Haar measure on X. These hypergroups will be called compact type (resp. strong compact type). The most prominent examples of such hypergroups are the polynomial hypergroups generated by the little q-Legendre polynomials (see [4]). We will present a full description of such hypergroups.

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The motivation for the investigations in this paper arose from our work on the so-called Reiter's condition of type $\mathscr{P}_1$ for locally compact commutative hypergroups (see [2,3]). A precise definition of Reiter's condition will be given in Section 3. This condition is of particular interest in spectral analysis of the Banach algebra $L^1(X,m)$. An important problem is to investigate whether the maximal ideals of $L^1(X,m)$ have a bounded approximate identity.

Hypergroups on $\mathbb{N}_0$ generated by orthogonal polynomials are of special interest. In [3] we studied polynomial hypergroups in some detail with respect to the $\mathscr{P}_1$-condition. We were able to show that there is a bounded approximate identity in the maximal ideal which is generated by a character, if and only if, the generalized Reiter condition is fulfilled at this character (see [3, Theorem 3.2].) In [3] it has been shown that the $\mathscr{P}_1$-condition is satisfied at every character which is absolutely summable. Using this result we observed that the $\mathscr{P}_1$-condition is fulfilled at every nontrivial character if the Jacobi operator is compact on ℓ^2 resp. ℓ^1. This observation led us to the more general question whether we have the same situation for hypergroups of compact type (resp. strong compact type). This question will be answered completely in this paper.

The paper is organized as follows. In Section 2 we present the basic facts on hypergroups as far as they are necessary to understand the paper. Section 3 contains the main results as well as some examples.

2. Preliminaries

Throughout this paper we will denote by X a locally compact commutative hypergroup. For the convolution of two elements $x, y \in X$ we write $x * y$ which is defined as $\delta_x * \delta_y$, where δ_x is the point measure at the point x. The involution of an element $x \in K$ will be denoted by $\tilde{x}$. Let $C_c(X)$ denote the space of all continuous functions on X with compact support. For a given $x \in X$ and a function $f \in C_c(X)$ the translation $T_x f$ of f is given by

$$T_x f(y) = f(x * y) = \delta_x * \delta_y(f).$$

The commutativity of X ensures the existence of a Haar measure m on X, i.e., a regular positive Borel measure $m \neq 0$ such that

$$\int_X f(y)\,\mathrm{d}m(y) = \int_X T_x f(y)\,\mathrm{d}m(y), \quad \text{for all } f \in C_c(K),\ x \in X.$$

The spaces $L^p(X,m)$, $1 \leqslant p \leqslant \infty$, are defined as usual. It is well known that the translation operator T_x can be extended to the spaces $L^p(X,m)$, $1 \leqslant p \leqslant \infty$, and to the space of bounded continuous functions $C_b(X)$ on X and moreover, that this operator is bounded on each of this spaces.

Let $M(X)$ be the set of all regular bounded Borel measures. For a measure $\mu \in M(X)$ we have a bounded operator

$$T_\mu : B \to B, \quad T_\mu f = \mu * f = \int_X T_x f\,\mathrm{d}\mu(x),$$

where B can be one of the spaces $C_b(X)$ or $L^p(X,m)$, $1 \leqslant p \leqslant \infty$. In the case $\mathrm{d}\mu = f\,\mathrm{d}m$ we will write T_f.

Let $\mathscr{C}$ denote the C^*-algebra generated by the operators T_f, $f \in C_c(X)$, acting on $L^2(X,m)$.

The convolution of two functions $f, g \in L^1(X, m)$ is defined by

$$f * g\,(x) = \int_X f(y)\, T_{\tilde{y}} g(x)\, \mathrm{d}m(y).$$

With this product and the *-operation $f^*(x) = \overline{f(\tilde{x})}$ the Banach space $L^1(X, m)$ becomes a Banach *-algebra.

The set of all characters of X will be denoted by $\mathscr{X}_{\mathrm{b}}(X)$, i.e., the set

$$\mathscr{X}_{\mathrm{b}}(X) = \{\gamma \in C_{\mathrm{b}}(X) : \gamma \neq 0 \text{ and } T_x\gamma(y) = \gamma(x)\gamma(y) \text{ for all } x, y \in X\}.$$

Let $\hat{X}$ be the set of all hermitian characters, i.e.,

$$\hat{X} = \{\gamma \in \mathscr{X}_{\mathrm{b}}(X) : \gamma^* = \gamma\}.$$

The Fourier transform of a function $f \in L^1(X, m)$ will be denoted by $\mathscr{F} f(\gamma)$ resp. $\hat{f}(\gamma)$ depending on whether γ is in $\mathscr{X}_{\mathrm{b}}(X)$ or in $\hat{X}$. There is a uniquely determined positive regular Borel measure π on $\hat{X}$ with

$$\int_X |f(x)|^2\, \mathrm{d}m(x) = \int_{\hat{X}} |\hat{f}(\gamma)|^2\, \mathrm{d}\pi(\gamma)$$

for all $f \in L^2(X, m)$. The measure π is called the Plancherel measure and its support will be denoted by $\mathscr{S}$. It is well known that

$$\mathscr{S} = \{\gamma \in \hat{K} : |\hat{f}(\gamma)| \leqslant \|T_f\| \text{ for all } f \in L^1(X, m)\}.$$

For more details on hypergroups we refer to monograph [1].

Let a_j be a family of numbers indexed by elements j from a set J. By $\lim_{j\to\infty} a_j = a$ we mean that for any $\varepsilon > 0$ we have $|a_j - a| < \varepsilon$ for all but finitely many j.

3. The results

The first proposition is known (see for example [7, Theorem 3.4]). We state it here, with a different proof, for the sake of completeness.

Proposition 1. *Let X be a commutative hypergroup with Haar measure m. If the support $\mathscr{S}$ of the Plancherel measure π is compact then the hypergroup X is discrete.*

Proof. By assumption we have $\pi(\mathscr{S}) < +\infty$. Let f_i be a net of functions with compact support $K_i \subset X$ such that $\bigcap_i K_i = \{e\}$ and $\|f_i\|_2 = 1$. Then $\hat{f_i} \to 1$ pointwise and by the dominated convergence theorem also in $L^2(\mathscr{S}, \pi)$. Thus by the Plancherel formula the sequence f_i is convergent in $L^2(X, m)$ and its limit cannot be anything else but $m(\{e\})^{-1/2}\delta_e$. We may assume that $m(\{e\}) = 1$.

We will show now that no x in X can be an accumulation point. Assume for a contradiction that there is a net x_i such that $x_i \neq x$ and $x_i \to x$ in X. Since for every $f \in L^2(X, m)$ the mapping

$$X \ni x \to T_x f \in L^2(X, m)$$

is continuous and $\delta_e \in L^2(X, m)$ we have

$$m(\{x_i\})^{-1}\delta_{x_i} = T_{\tilde{x}_i}\delta_e \to T_{\tilde{x}}\delta_e = m(\{x\})^{-1}\delta_x,$$

where the convergence is with respect to the norm of $L^2(X, m)$. On the other hand,

$$\|m(\{x_i\})^{-1}\delta_{x_i} - m(\{x\})^{-1}\delta_x\|_2^2 = m(\{x_i\})^{-1} + m(\{x\})^{-1} > m(\{x\})^{-1},$$

which gives a contradiction. □

We now introduce a special type of hypergroups which we are going to study in detail.

Definition 1. We say that a hypergroup X is of compact type if for every $x \in X$ the operator $T_x - I$ is compact on the space $L^2(X, m)$.

Now we will show that hypergroups of compact type are necessarily discrete and moreover, have a special dual structure.

Theorem 1. *Every commutative hypergroup of compact type is discrete. The support of the Plancherel measure is given by* $\mathscr{S} = \{\gamma_i | i \in I\} \cup \{1\}$, *where* $\gamma_i \to 1$. *In particular, every nontrivial character in* $\mathscr{S}$ *is square summable.*

Proof. Let $f \in C_c(X)$. The operators $T_x - I$ are compact on $L^2(X, m)$ and so is the convolution operator

$$T_f - \hat{f}(1)I = \int_X f(x)[T_{\tilde{x}} - I]\,\mathrm{d}m(x).$$

Since the limit of compact operators is compact the operator $T_f - \hat{f}(1)I$ is compact for every $f \in L^1(X, m)$.

Via the Gelfand transform, $T_f - \hat{f}(1)I$ is mapped into the multiplication operator $M_{\hat{f}-\hat{f}(1)}$ acting on $L^2(\mathscr{S}, \pi)$. For $\varepsilon > 0$ consider the closed set

$$\mathscr{S}_\varepsilon = \{\gamma \in \mathscr{S} : |\hat{f}(\gamma) - \hat{f}(1)| \geqslant \varepsilon\}.$$

The closed subspace

$$V_\varepsilon = \{g \in L^2(\mathscr{S}, \pi) : \operatorname{supp} g \subset \mathscr{S}_\varepsilon\}$$

of $L^2(\mathscr{S}, \pi)$ is invariant for the operator $M_{\hat{f}-\hat{f}(1)}$. Moreover, this operator restricted to V_ε is invertible. By compactness of this operator the space V_ε must be finite dimensional. Therefore the set $\mathscr{S}_\varepsilon$ is finite for any $\varepsilon > 0$ and for any $f \in L^1(X, m)$. This, by definition of the topology on $\mathscr{S}$, implies that $\mathscr{S}$ is of the form as claimed in the statement of the theorem, i.e., the trivial character is the only accumulation point in $\mathscr{S}$ and $\mathscr{S}$ is compact. □

Definition 2. We say that a hypergroup X is of strong compact type if for every x in X the operator $T_x - I$ is compact on the space $L^1(X, m)$.

Strong compact type implies compact type as the following results state.

Proposition 2. *Let X be a commutative hypergroups of strong compact type. Then every nontrivial bounded character is absolutely integrable.*

Proof. According to the Theorem of Riesz on compact operators on Banach spaces, for each $x \in X$ the spectrum of the operator T_x consists of 1 and countably many eigenvalues $\lambda_{x,1}, \lambda_{x,2}, \ldots$ different from 1. For an eigenvalue λ let $N_{x,\lambda}$ denote the corresponding eigenspace. Since the operators T_x commute with each other the space $N_{x,\lambda}$ is invariant for every translation T_y.

Let M be the family of all finite dimensional invariant subspaces of $L^1(X,m)$ for $\{T_x\}_{x\in X}$. A subspace V in M will be called minimal if it does not admit a proper subspace in M.

Let V be a minimal subspace. Then for every x and λ if $N_{x,\lambda} \cap V \neq \{0\}$ then $N_{x,\lambda} \cap V = V$. Fix $x \in X$. Hence there exists λ such that $N_{x,\lambda} \cap V \neq \{0\}$ or $N_{x,\lambda} \cap V = \{0\}$ for each λ. In the first case we have $V \subseteq N_{x,\lambda}$. Otherwise, $T_x V = \{0\}$. Thus for each $x \in X$ we have either $T_x\gamma = 0$ for each $\gamma \in V$ or $T_x\gamma = \lambda\gamma$, for each $\gamma \in V$. In both cases every function $\gamma \in V$ is a common eigenfunction for all the operators T_x and the eigenvalue depends only on x and V. Therefore V must be a one-dimensional space spanned by a continuous and self-adjoint character γ. Let a function $0 \neq \alpha \in V$. Then $T_x\alpha = \gamma(x)\alpha$, where $\gamma(x)$ depends only on x and V. We may assume that α is continuous by replacing it with $T_f\alpha$, where $f \in C_c(X)$, if necessary. Of course there exists f such that $T_f\alpha \neq 0$. We have

$$\gamma(x)\alpha(y) = T_x\alpha(y) = T_y\alpha(x) = \gamma(y)\alpha(x).$$

Hence there exists a constant c such that $\alpha(x) = c\gamma(x)$ for any $x \in X$. In particular γ is continuous. Thus $T_f V$ is a one-dimensional space spanned by the continuous function $\gamma(x)$. By taking any approximate unit f_n in $L^1(X,m)$ we can actually show that the space V is spanned by γ. We then have

$$(T_x\gamma)(y) = \gamma(x)\gamma(y). \tag{1}$$

Observe that γ is bounded by 1. Indeed, since the operators T_x are contractions on $L^1(X,m)$ we have

$$|\gamma(x)|\,\|\gamma\|_{L^1(X,m)} = \|T_x\gamma\|_{L^1(X,m)} \leqslant \|\gamma\|_{L^1(X,m)}.$$

Hence $|\gamma(x)| \leqslant 1$. Moreover, we have that $\gamma \in L^1(X,m) \cap L^\infty(X,m)$, hence by the Schwarz inequality $\gamma \in L^2(X,m)$. Therefore γ is a self-adjoint character because

$$\gamma(\tilde{x})\|\gamma\|_2 = \langle T_{\tilde{x}}\gamma, \gamma\rangle = \langle \gamma, T_x\gamma\rangle = \overline{\gamma(x)}\|\gamma\|_2.$$

We claim that every finite dimensional subspace V invariant for $\{T_x\}_{x\in X}$ is a direct sum of the minimal subspaces. Indeed, V contains a minimal subspace W. This subspace is spanned by a character γ. Hence $\gamma \in L^1(X,m) \cap L^\infty(X,m)$. Let V' be the space of all functions in V which are orthogonal to γ with respect to the standard inner product. Then V' is also invariant. Indeed, let $\eta \in V'$ and $x \in X$. Then $T_x\eta \in V$ and

$$\langle T_x\eta, \gamma\rangle = \langle \eta, T_{\tilde{x}}\gamma\rangle = \gamma(x)\langle \eta, \gamma\rangle = 0.$$

Thus $T_x\eta \in V'$. Now we can repeat the same procedure for V'.

Thus we have proved that every finite dimensional subspace V invariant for $\{T_x\}_{x\in X}$ is spanned by self-adjoint characters. In particular every subspace $N_{x,\lambda}$ is spanned by such characters. In particular we have $N_{x,\lambda} \subset L^1(X,m) \cap L^\infty(X,m)$.

Let $M_{x,\lambda}$ denote the eigenspace for the operator T_x on the space $L^\infty(X, m)$ corresponding to the eigenvalue $\lambda \neq 1$. We have

$$[(T_x)_{L^1 \to L^1}]^* = (T_{\tilde{x}})_{L^\infty \to L^\infty}.$$

The Fredholm alternative yields that

$$\begin{aligned} &\dim N_{x,\lambda} = \dim M_{x^*,\lambda}, \\ &\dim N_{\tilde{x},\lambda} = \dim M_{x,\lambda}. \end{aligned}$$

Moreover, since $N_{x,\lambda} \subset L^\infty$, we have

$$\begin{aligned} &N_{x,\lambda} \subset M_{x,\lambda}, \\ &N_{\tilde{x},\lambda} \subset M_{\tilde{x},\lambda}. \end{aligned}$$

This implies

$$\begin{aligned} &N_{x,\lambda} = M_{x,\lambda}, \\ &N_{\tilde{x},\lambda} = M_{\tilde{x},\lambda}. \end{aligned} \tag{2}$$

The dual space $\mathscr{X}_{\mathrm{b}}(X)$ can be identified with the set of all bounded characters. Let γ be a bounded character different from 1. Then γ is a common eigenfunction of the operators T_x with eigenvalues $\gamma(x)$. There exists $x \in X$ such that $\gamma(x) \neq 1$. Then $\gamma \in M_{x,\lambda}$ for $\lambda = \gamma(x)$. By (2) we obtain that $\gamma \in N_{x,\lambda} \subset L^1(X, m)$. Therefore γ is absolutely integrable. $\square$

Proposition 3. *Every commutative hypergroup of strong compact type is of compact type.*

Proof. Let $x \in X$. Since $T_x - I$ is compact the spectrum of T_x consists of 1 and at most countably many nonzero eigenvalues $\lambda_1, \lambda_2, \ldots$, such that $\lambda_n \to 1$. The corresponding eigenspaces N_{x,λ_i} are finite dimensional and by the proof of Proposition 2 are contained in $L^2(X, m)$.

In the proof of Proposition 2 we have shown that every character is in $L^2(X, m)$. It is easy to show that the characters form an orthogonal basis of $L^2(X, m)$. Therefore the space $L^2(X, m)$ is spanned by the absolutely integrable characters.

Let γ be a character. Then $(T_x - I)\gamma = (\lambda_i - 1)\gamma$ for some i, because $\gamma \in L^1(X, m)$. Thus $T_x - I$ has the representation of the form

$$T_x - I = \sum_{i=1}^{\infty} (\lambda_i - 1) P_{x,\lambda_i},$$

where P_{x,λ_i} denotes the orthogonal projection onto N_{x,λ_i}. Hence $T_x - I$ is compact on $L^2(X, m)$. $\square$

Theorem 2. *Let X be a commutative hypergroup of strong compact type. Then $\mathscr{X}_{\mathrm{b}}(X) = \mathscr{S}$. In particular the hypergroup X is symmetric, i.e., every bounded character is self-adjoint.*

Proof. Proposition 3and Theorem 1 imply that X is discrete. Let $\mathscr{C}$ be the C^*-algebra generated by the operators T_x acting on the Hilbert space $\ell^2(X, m)$. Since $\ell^1(X, m) * \ell^2(X, m) \subset \ell^2(X, m)$ we have

$\ell^1(X,m) \subset \mathscr{C}$. Moreover, $\ell^1(X,m) \cap \ell^2(X,m)$ is dense in $\mathscr{C}$. Let $f \in \ell^1(X,m) \cap \ell^2(X,m)$ and $\gamma \in \mathscr{X}_{\mathrm{b}}(X),\ \gamma \neq 1$. By Proposition 2 we have $\gamma \in \ell^1(X,m)$. Hence

$$\begin{aligned}\|\gamma\|_2|\langle f,\gamma\rangle| &= \|\langle f,\gamma\rangle\gamma\|_2 = \|\gamma * f\|_2 \leqslant \|\gamma\|_1\|f\|_2\\ &= \|\gamma\|_1\|T_f(\delta_e)\|_2 \leqslant \|\gamma\|_1\|\delta_e\|_2\|T_f\|_{L^2\to L^2}.\end{aligned}$$

In particular the functional $\varphi(f) = \langle f,\gamma\rangle$, for $f \in \ell^1(X,m)$ gives rise to a continuous linear functional on the C^*-algebra $\mathscr{C}$. Since the structure space of $\mathscr{C}$ can be identified with $\mathscr{S}$ we have $\gamma \in \mathscr{S}$. $\square$

Now we study the so-called Reiter's condition for hypergroups. The following results are an extension of the work which has been done in [3]. First we repeat the precise definition of the Reiter condition.

Definition 3. Let $\gamma \in \mathscr{X}_{\mathrm{b}}(X)$ be fixed. We say that the $\mathscr{P}_1$-condition with bound M is satisfied in γ ($\mathscr{P}_1(\gamma, M)$ for short) if for each $\varepsilon > 0$ and every compact subset $C \subseteq X$ there exists $g \in L^1(X,m)$ with the following properties:

(i) $\mathscr{F}g(\gamma) = 1$,
(ii) $\|g\|_1 \leqslant M$,
(iii) $\|T_{\tilde{y}}g - \overline{\gamma(y)}g\|_1 < \varepsilon$ for all $y \in C$.

We are now considering hypergroups of strong compact type with respect to the $\mathscr{P}_1$-condition.

Theorem 3. *Let X be a commutative hypergroup of strong compact type. Then the $\mathscr{P}_1(\gamma, M)$ condition is satisfied for each nontrivial character γ.*

Proof. By Proposition 2 and Theorem 1 the hypergroup X is discrete and its dual is compact with the trivial character as the only accumulation point. By the proof of Proposition 2 every nontrivial character belongs to $\ell^1(X,m)$. In the same manner as in [3, proof of Proposition 4.3] we obtain that $\mathscr{P}_1(\gamma, M)$ is satisfied. $\square$

Now we turn to the polynomial hypergroups. Let us recall some basic facts. For a more thorough treatment of this class of hypergroups we refer to [5,6].

Let $\{R_n\}_{n\in\mathbb{N}_0}$ be a polynomial sequence defined by a recurrence relation of the type

$$R_1(x)\,R_n(x) = a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x) \tag{3}$$

for $n \in \mathbb{N}$ with starting polynomials $R_0(x) = 1,\ R_1(x) = 1/a_0(x - b_0)$ and $a_n > 0,\ b_n \geqslant 0$ for all $n \in \mathbb{N}_0$ and $c_n \geqslant 0$ for all $n \in \mathbb{N}$. Let the polynomials be normalized at $x = 1$, i.e.,

$$R_n(1) = 1$$

for all $n \in \mathbb{N}_0$. We also assume that the coefficients in the linearization formula

$$R_n(x)R_m(x) = \sum_{k=|n-m|}^{n+m} g(n,m;k)R_k(x),$$

are nonnegative for all $n,\ m,\ k \in \mathbb{N}_0$. A polynomial sequence with these properties generates a hypergroup structure on $\mathbb{N}_0$.

We can obtain a Banach algebra structure by considering the weighted space $\ell^1(\mathbb{N}_0, h)$ where

$$h(0) = 1, \quad h(1) = \frac{1}{c_1}, \quad h(n) = \frac{a_1 a_2 \ldots\ldots a_{n-1}}{c_1 c_2 \ldots\ldots c_n}\ , \tag{4}$$

with translation operators given by

$$T_n \beta(m) = \sum_{k=|n-m|}^{n+m} g(n, m; k)\beta(k). \tag{5}$$

Proposition 4. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ define a hypergroup on $\mathbb{N}_0$. Then the hypergroup is of compact type, if and only if, $a_n \to 0,\ b_n \to 1$ and $c_n \to 0$ in which case the hypergroup is also of strong compact type.*

Proof. The translations T_n can be defined recursively as follows. We set $T_0 = I$. The operator T_1 is defined on sequences $\{\beta(n)\}_{n=0}^{\infty}$ by the formula

$$T_1 \beta(n) = a_n \beta(n+1) + b_n \beta(n) + c_n \beta(n-1), \quad n \geqslant 0. \tag{6}$$

Using (5) it can be proved that

$$T_1 T_n = a_n T_{n+1} + b_n T_n + c_n T_{n-1}.$$

Plugging in $x = 1$ to the recurrence relation (3) we obtain

$$a_n + b_n + c_n = 1, \quad \text{for } n > 0, \quad \text{and} \quad a_0 + b_0 = 1. \tag{7}$$

Now we get

$$(T_1 - I)T_n = a_n(T_{n+1} - I) + (b_n - 1)(T_n - I) + c_n(T_{n-1} - I).$$

This formula implies that the operator $T_n - I$ can be factored by $T_1 - I$. Therefore the operators $T_n - I$ are compact on $\ell^2(\mathbb{N}, h)$, if and only if, $T_1 - I$ is compact.

Formula (6) implies the following:

$$T_1 \delta_n = c_{n+1}\delta_{n+1} + b_n \delta_n + a_{n-1}\delta_{n-1}, \quad n \geqslant 0. \tag{8}$$

The system $\{\delta_n\}_{n=0}^{\infty}$ forms a basis for either space $\ell^2(\mathbb{N}, h)$ or $\ell^1(\mathbb{N}, h)$. The matrix of the operator T_1 corresponding to this basis is tridiagonal. Therefore the operator $T_1 - I$ is compact on $\ell^2(\mathbb{N}_0, h)$, if and only if,

$$\lim_{n\to\infty} \frac{\langle T_1\delta_n, \delta_{n+1}\rangle}{\|\delta_n\|_2 \|\delta_{n+1}\|_2} = \lim_{n\to\infty} \frac{\langle T_1\delta_{n+1}, \delta_n\rangle}{\|\delta_n\|_2 \|\delta_{n+1}\|_2} = \lim_{n\to\infty} \frac{\langle T_1\delta_n, \delta_n\rangle}{\|\delta_n\|_2^2} = 0.$$

By (4) and (8) the latter holds, if and only if,

$$c_{n+1}a_n \to 0 \quad \text{and} \quad b_n \to 1.$$

By (7) and the fact that a_n and c_n are positive, this is possible only if $a_n \to 0,\ b_n \to 1$ and $c_n \to 0$.

Similarly, the operator $T_1 - I$ is compact on $\ell^1(\mathbb{N}, h)$, if and only if,

$$\lim_{n\to\infty} \frac{\langle T_1\delta_n, \delta_{n+1}\rangle}{\|\delta_n\|_1\|\delta_{n+1}\|_\infty} = \lim_{n\to\infty} \frac{\langle T_1\delta_{n+1}, \delta_n\rangle}{\|\delta_{n+1}\|_1\|\delta_n\|_\infty} = \lim_{n\to\infty} \frac{\langle T_1\delta_n, \delta_n\rangle}{\|\delta_n\|_1\|\delta_n\|_\infty} = 0.$$

By (4) and (8) this holds, if and only if,

$$a_n \to 0, \quad c_n \to 0 \quad \text{and} \quad b_n \to 1. \tag{9}$$

Hence, also $T_n - I$ is compact for every $n \in \mathbb{N}$, provided (9) holds, as we have seen before it can be factored by $T_1 - I$. □

In view of Theorem 3 we get the following.

Corollary 1. *Let $\{R_n\}_{n\in\mathbb{N}_0}$ define a hypergroup on $\mathbb{N}_0$ of compact type. Then the $\mathscr{P}_1(\gamma, M)$ condition is satisfied for every nontrivial character γ.*

Example 1. We consider the little q-Legendre polynomials. They satisfy recurrence relation (3) with

$$a_n = q^n \frac{(1+q)(1-q^{n+1})}{(1-q^{2n+1})(1+q^{n+1})},$$
$$b_n = \frac{(1-q^n)(1-q^{n+1})}{(1+q^n)(1+q^{n+1})},$$
$$c_n = q^n \frac{(1+q)(1-q^n)}{(1-q^{2n+1})(1+q^n)}.$$

By [4] they define the polynomial hypergroup. From the recurrence relation we can read easily that this hypergroup is of strong compact type. Hence the modified Reiter's condition is satisfied for each nontrivial character.

Example 2. Let the orthogonal polynomials R_n satisfy (3) such that b_n is increasing, $b_n \to 1$,

$$c_{n+1}a_n \leqslant (b_{n+2} - b_{n+1})^2,$$

for every $n \geqslant 0$, and $a_n + b_n + c_n = 1$. Then by remarks following [8, Theorem 1] the polynomials R_n give rise to a hypergroup. Moreover $a_n \to 0$ and $c_n \to 0$. Hence the resulting hypergroup is of strong compact type.

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