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Polynomial bases for continuous function spaces

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Abstract

Let $S \subset \mathbb{R}$ denote a compact set with infinite cardinality and $C(S)$ the set of real continuous functions on S. We investigate the problem of polynomial and orthogonal polynomial bases of $C(S)$. In case of $S = \{s_0, s_1, s_2, \ldots\} \cup \{\sigma\}$, where $(s_k)_{k=0}^{\infty}$ is a monotone sequence with $\sigma = \lim_{k\to\infty} s_k$, we give a sufficient and necessary condition for the existence of a so-called Lagrange basis. Furthermore, we show that little q-Jacobi polynomials which fulfill a certain boundedness property constitute a basis in case of $S_q = \{1, q, q^2, \ldots\} \cup \{0\}$, $0 < q < 1$.

1 Introduction

One important goal in approximation theory is the representation of functions with respect to a set of simple functions. Here, we focus on the Banach space $C(S)$ of real continuous functions on a compact set $S \subset \mathbb{R}$ with infinite cardinality. Among the continuous functions polynomials are the most simple to deal with. Hence, further on we discuss the representation of $f \in C(S)$ with respect to a sequence of polynomials $(P_k)_{k=0}^{\infty}$. Moreover, it is profitable to look for a sequence with

$$\deg P_k = k \quad \text{for all } k \in \mathbb{N}_0, \tag{1}$$

which guarantees that every polynomial has a finite representation.

Of special interest are orthogonal polynomial sequences with respect to a probability measure π on S, where a representation is based on the Fourier coefficients

$$\hat{f}(k) = \int_S f(x) P_k(x)\, d\pi(x), \quad k \in \mathbb{N}_0, \tag{2}$$

of $f \in C(S)$.

Let us recall some important facts about orthogonal polynomials, see [3]. An orthogonal polynomial sequence $(P_k)_{k=0}^{\infty}$ with compact support S and property (1) satisfies a three term recurrence relation

$$P_1(x)P_k(x) = a_k P_{k+1}(x) + b_k P_k(x) + c_k P_{k-1}(x), \quad k \in \mathbb{N}, \tag{3}$$

starting with

$$P_0(x) = a_0 \text{ and } P_1(x) = (x-b)/a, \tag{4}$$

where the coefficients are real numbers with $c_k a_{k-1} > 0,\ k \in \mathbb{N}$, and $(c_k a_{k-1})_{k=1}^{\infty}$, $(b_k)_{k=1}^{\infty}$ are bounded sequences. The other way around, if we construct $(P_k)_{k=0}^{\infty}$ by (3) with coefficients satisfying the conditions above, then we get an orthogonal polynomial sequence with compact support S.
The sequence of kernels $(K_n)_{n=0}^{\infty}$ is defined by

$$K_n(x,y) = \sum_{k=0}^{n} P_k(x)P_k(y)h(k) = \sum_{k=0}^{n} p_k(x)p_k(y), \tag{5}$$

where

$$h(k) = (\int_S P_k^2(x)\, d\pi(x))^{-1} = \frac{1}{a_0^2} \frac{\prod_{i=0}^{k-1} a_i}{\prod_{i=1}^{k} c_i}, \quad k \in \mathbb{N}_0, \tag{6}$$

and $(p_k)_{k=0}^{\infty}$ is the orthonormal polynomial sequence with respect to π defined by

$$p_k = \sqrt{h(k)} P_k. \tag{7}$$

For $z \in S$ it holds

$$(K_n(z,z))^{-1} = \min_{Q \in \mathcal{P}_{(n)}, Q(z)=1} \int_S (Q(x))^2\, d\pi(x), \tag{8}$$

where $\mathcal{P}_{(n)}$ denotes the set of polynomials with degree less or equal n. One of the most important tools is the Christoffel-Darboux formula

$$\begin{aligned} K_n(x,y) &= a_n h(n) \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{P_1(x) - P_1(y)} \\ &= a\sqrt{c_{n+1}a_n} \frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}. \end{aligned} \tag{9}$$

The linearization coefficients $g(i,j,k)$ are defined in terms of

$$P_i P_j = \sum_{k=0}^{\infty} g(i,j,k) P_k = \sum_{k=|i-j|}^{i+j} g(i,j,k) P_k, \quad i,j \in \mathbb{N}_0, \tag{10}$$

where $g(i,j,|i-j|), g(i,j,i+j) \neq 0$. The nonnegativity of the linearization coefficients is sufficient for a special boundedness property, which we will introduce in Section 4.

2 Polynomial bases for $C(S)$

Let us first refer to the concept of a basis.

Definition 2.1 *A sequence $(\Phi_k)_{k=0}^{\infty}$ in $C(S)$ is called basis if for every $f \in C(S)$ there exists a unique sequence of $(\varphi_k)_{k=0}^{\infty}$ of real numbers such that*

$$f = \sum_{k=0}^{\infty} \varphi_k \Phi_k, \tag{11}$$

where $\lim_{n\to\infty} \sum_{k=0}^{n} \varphi_k \Phi_k$ is with respect to the sup-norm. A basis $(\Phi_k)_{k=0}^{\infty}$ of polynomials is called polynomial basis. A polynomial basis with (1) is called Faber basis.

There is a famous result of Faber [4] in 1914 that in case of S being an interval $[c, d]$ there doesn't exist a polynomial basis $(P_k)_{k=0}^{\infty}$ of $C([c, d])$ with property (1). Concerning $C([c, d])$ great efforts have been made in constructing polynomial bases and to minimize the degrees as far as possible. In 1977 Temlyakov [20] has investigated a method of construction, where the growth of the degrees fulfills $\deg P_k \le C\, k \log\log(k)$. Later on, in 1985 Bochkarev [2] has used the Fejér kernel to construct a basis with linear bounds, that is $\deg P_k \le 4k$. In 1987 Privalov [14] published a somehow negative result, which implies the result of Faber. Namely, if there is a polynomial basis $(P_k)_{k=0}^{\infty}$ of $C([c, d])$, then there exists a $\delta > 0$ such that $\deg P_k \ge (1 + \delta)\ k$ for all $k \ge k_0$, where k_0 is a proper integer. Also, in 1991 Privalov gave a positive result, see [15]. He proved that for any $\epsilon > 0$ there exists a polynomial basis of $C([c, d])$ with $\deg P_k \le (1 + \epsilon)\ k$. Such a basis is called *polynomial basis of optimal degree* (with respect to ϵ).

If we are searching for Banach spaces $C(S)$ equipped with a Faber basis, then we have to choose S different from an interval. In this setting spaces $C(S)$ with a so-called Lagrange basis are discussed in [16]. In Section 3 we investigate a basic class of compact sets S and give a sufficient and necessary condition for the existence of a Lagrange basis.

Note, that the results mentioned above are not based upon the fact of orthogonality. In case of orthogonality the question 'Does $(P_k)_{k=0}^{\infty}$ constitute an orthogonal polynomial basis of $C(S)$' is equivalent to the question if any function $f \in C(S)$ is represented by its Fourier series

$$\sum_{k=0}^{\infty} \hat{f}(k) P_k h(k). \tag{12}$$

In this particular branch of study there also are some positive results. In 1996 Kilgore, Prestin and Selig [7] constructed an orthogonal polynomial basis of optimal degree with respect to the Chebyshev weight of first kind ($\alpha = \beta = -\frac{1}{2}$) using wavelet methods. Later on, in 1998 Girgensohn [6] gave optimal polynomial

bases for all of the four Chebyshev weights ($\alpha = \pm\frac{1}{2},\ \beta = \pm\frac{1}{2}$) and in 2001 Skopina [17] succeeded for Legendre weights ($\alpha = \beta = 0$). The general problem for Jacobi weights $(1-x)^\alpha(1+x)^\beta dx$, $\alpha, \beta > -1$, seems still to be open.
In order to check if an orthogonal polynomial sequence $(P_k)_{n=0}^\infty$ constitutes a basis of $C(S)$ we have to show

$$\sup_{x\in S}\int_S |K_n(x,y)|\, d\pi(y) \le C \quad \text{for all } n \in \mathbb{N}_0. \tag{13}$$

We should mention that the sequence $(P_k)_{n=0}^\infty$ is a basis of $C(S)$ if and only if it is a basis of $L^1(S,\pi)$, see [11]. For the discussion of an example based on little q-Jacobi polynomials see Section 5.

3 Lagrange bases

In [16] we have introduced the concept of a Lagrange basis. Let $S \subset \mathbb{R}$ be a compact set and $(s_k)_{k=0}^\infty$ a sequence of distinct points in S. Define as usual the Lagrange basic functions L_n^k as

$$L_n^k(x) = \frac{\prod_{i=0,i\neq k}^n (x - s_i)}{\prod_{i=0,i\neq k}^n (s_k - s_i)} \quad \text{for all} \quad n \in \mathbb{N}_0,\ k = 0, 1, \ldots, n. \tag{14}$$

and

$$l_k(x) = L_k^k(x) \quad \text{for all} \quad k \in \mathbb{N}_0. \tag{15}$$

Definition 3.1 *The sequence $(l_k)_{k=0}^\infty$ is called sequence of Lagrange polynomials with respect to $(s_k)_{k=0}^\infty$. If $(l_k)_{k=0}^\infty$ is a basis of $C(S)$, then we call $(l_k)_{k=0}^\infty$ a Lagrange basis of $C(S)$ with respect to $(s_k)_{k=0}^\infty$.*

In case of a Lagrange basis it holds $f = \sum_{k=0}^\infty \varphi_k(f) l_k$ with

$$\varphi_0(f) = f(s_0);\ \ \varphi_k(f) = f(s_k) - \sum_{j=0}^{k-1} \varphi_j(f) l_j(s_k) \quad \text{for all } k \in \mathbb{N}. \tag{16}$$

A sequence $(v_n)_{n=0}^\infty$ of linear operators from $C(S)$ into $C(S)$ is defined by

$$v_n(f) = \sum_{k=0}^n \varphi_k(f) l_k. \tag{17}$$

By simple means we have

$$\sum_{k=0}^n \varphi_k(f) l_k(s_i) = f(s_i) \quad \text{for all } i = 0, 1, \ldots, n, \tag{18}$$

which implies

$$\sum_{k=0}^{n} \varphi_k(f) l_k = \sum_{k=0}^{n} f(s_k) L_n^k. \tag{19}$$

The sequence of Lagrange polynomials $(l_k)_{k=0}^{\infty}$ constitutes a basis of $C(S)$ if and only if

$$\|v_n\| = \max_{x \in S} \sum_{k=0}^{n} |L_n^k(x)| \le C \quad \text{for all } n \in \mathbb{N}_0, \tag{20}$$

see [16].
Further let us assume that $(s_k)_{k=0}^{\infty}$ is strictly increasing or strictly decreasing and

$$S = \{s_0, s_1, s_2, \ldots\} \cup \{\sigma\}, \tag{21}$$

where

$$\sigma = \lim_{k \to \infty} s_k. \tag{22}$$

Using this assumption we derive

$$\|v_n\| = \sum_{k=0}^{n} |L_n^k(\sigma)|, \tag{23}$$

see [16].

Let us now give the main result of this section.

Theorem 3.2 *Assume $(s_k)_{k=0}^{\infty}$ is a strictly increasing or strictly decreasing sequence and $S = \{s_0, s_1, s_2, \ldots\} \cup \{\sigma\}$.
Then $(l_k)_{k=0}^{\infty}$ is a basis of $C(S)$ if and only if there exists $0 < q < 1$ with*

$$|\sigma - s_{k+1}| \le q\, |\sigma - s_k| \quad \textit{for all } k \in \mathbb{N}_0. \tag{24}$$

Proof. Let $(l_k)_{k=0}^{\infty}$ be a basis of $C(S)$. Then there exists $C > 1$ such that $|l_k(\sigma)| < C$ for all $k \in \mathbb{N}_0$, which implies

$$|\sigma - s_{k-1}| < C|s_k - s_{k-1}| \quad \text{for all } k \in \mathbb{N}. \tag{25}$$

If $(s_k)_{k=0}^{\infty}$ is strictly increasing, then $(\sigma - s_{k-1}) < C(s_k - \sigma + \sigma - s_{k-1})$, which is equivalent to

$$(\sigma - s_k) < \frac{C-1}{C}\,(\sigma - s_{k-1}) \quad \text{for all } k \in \mathbb{N}. \tag{26}$$

In case of a strictly decreasing sequence we get the inequality the other way around. Choose $q = (C-1)/C$.
Let us now assume that (24) holds and $(s_k)_{k=0}^{\infty}$ is strictly decreasing. If $k > i$ we get

$$\frac{|\sigma - s_i|}{|s_k - s_i|} = \frac{s_i - \sigma}{s_i - \sigma - (s_k - \sigma)} < \frac{s_i - \sigma}{s_i - \sigma - q^{k-i}(s_i - \sigma)} = \frac{q^i}{q^i - q^k}, \tag{27}$$

and if $k < i$ we get

$$\frac{|\sigma - s_i|}{|s_k - s_i|} < \frac{q^i}{q^k - q^i}. \tag{28}$$

Furthermore, it holds

$$\sum_{k=0}^{n} \prod_{i=0, i\neq k}^{n} \frac{q^i}{|q^k - q^i|} = \sum_{k=0}^{n} \prod_{j=1}^{n-k} \frac{1}{1-q^j} \prod_{i=1}^{k} \frac{1}{1-q^i} \, q^{k(k+1)/2} \tag{29}$$

$$\leq \left(\prod_{j=1}^{\infty} \frac{1}{1-q^j} \right)^2 \sum_{k=0}^{\infty} q^k < \infty. \tag{30}$$

Hence, $\sum_{k=0}^{n} |L_n^k(\sigma)| \leq C$ for all $n \in \mathbb{N}_0$, which implies that $(l_k)_{k=0}^{\infty}$ constitutes a basis of $C(S)$. The case $(s_k)_{k=0}^{\infty}$ is strictly increasing is quite similar. □

The standard example due to the geometric sequence is

$$S_q = \{1, q, q^2, \ldots\} \cup \{0\}, \tag{31}$$

where $0 < q < 1$. Now, by Theorem 3.2 it follows that the Lagrange polynomials $(l_k)_{k=0}^{\infty}$ with respect to $(s_k)_{k=0}^{\infty}$, where $s_k = q^k$, constitute a Lagrange basis of $C(S_q)$.

For instance, if one is rearranging the sequence $(\frac{1}{2^k})_{k=0}^{\infty}$ in the way that

$$s_k = \begin{cases} 1 & \text{if } k = 0, \\ 2^{-(k+1)/2} & \text{if } k \neq 0 \text{ and } \log_2(k+1) \in \mathbb{N}, \\ 2^{-(k+1)} & \text{else}, \end{cases} \tag{32}$$

then the Lagrange polynomials $(l_k)_{k=0}^{\infty}$ with respect to $(s_k)_{k=0}^{\infty}$ don't constitute a Lagrange basis of $C(S_{\frac{1}{2}})$. The proof is left to the reader.

If we set $S_q^{(i)} = S_q \setminus \{q^i\}$ and $(l_k)_{k=0}^{\infty}$ denotes the sequence of Lagrange polynomials with respect to the sequence $(q^k)_{k=0}^{\infty}$ then $C(S_q^{(i)})$ in companion with $(l_k)_{k=0}^{\infty}$ states an example where a representation (11) exists but is not unique. To show this first notice that any $f^{(i)} \in C(S_q^{(i)})$ could be easily extended to a function $f \in C(S_q)$, where $f|_{S_q^{(i)}} = f^{(i)}$ and $f(q^i)$ is arbitrary. A representation of f in $C(S_q)$ also represents $f^{(i)}$ in $C(S_q^{(i)})$. Choose $f_1, f_2 \in C(S_q)$ with $f_1(q^i) \neq f_2(q^i)$ and $f_1|_{S_q^{(i)}} = f_2|_{S_q^{(i)}}$ to show that the representation is not unique. Of course, by Theorem 3.2 there is a Lagrange basis of $C(S_q^{(i)})$ with respect to the sequence $(q^k)_{k=0, k\neq i}^{\infty}$.

Let

$$S^r = \{1, \frac{1}{2^r}, \frac{1}{3^r}, \ldots\} \cup \{0\}, \tag{33}$$

where $0 < r < \infty$. With $s_k = \frac{1}{(k+1)^r}$ we get $\lim_{k\to\infty} s_{k+1}/s_k = 1$. By Theorem 3.2 the Lagrange polynomials with respect to $(s_k)_{k=0}^{\infty}$ do not constitute a basis of $C(S^r)$.

4 A boundedness property for orthogonal polynomial sequences

The following boundary property for orthogonal polynomial sequences is important for many reasons, see for instance [12], and is used in Section 5.

Definition 4.1 *We say that a polynomial sequence $(R_k)_{k=0}^{\infty}$ fulfills property (B), if there exists $\xi \in S$ such that*

$$|R_k(x)| \leq R_k(\xi) = 1 \quad \textit{for all } x \in S,\ k \in \mathbb{N}_0. \tag{34}$$

There is a condition on the linearization coefficients which yields that property (B) holds with respect to a proper normalization of the system.

Lemma 4.2 *Assume that the linearization coefficients $g(i,j,k)$ belonging to the sequence $(P_k)_{k=0}^{\infty}$ are nonnegative for all $i,j,k \in \mathbb{N}_0$, then there exists a normalization $R_k = \gamma_k P_k$ such that property (B) holds.*

Proof. The assumption yields $g(i,i,2i) > 0$ for all $i \in \mathbb{N}_0$. Hence, by (10) it follows $P_0 > 0$ and $\lim_{x\to-\infty} P_{2i}(x) = \lim_{x\to\infty} P_{2i}(x) = \infty$ for all $i \in \mathbb{N}$. Regarding $P_1 P_{2i}$ we get $\lim_{x\to\infty} P_1(x) = \lim_{x\to\infty} P_{2i+1}(x)$ for all $i \in \mathbb{N}$.
All zeros of the polynomials P_k are in the open interval $(\min S, \max S)$, see [3]. Hence there are two cases to handle. Namely, $P_k(\min S) > 0$ for all $k \in \mathbb{N}_0$, or $P_k(\max S) > 0$ for all $k \in \mathbb{N}_0$. Depending on this put $\xi = \min S$ or $\xi = \max S$ and define

$$R_k(x) = \frac{P_k(x)}{P_k(\xi)} \quad \text{for all } k \in \mathbb{N}_0. \tag{35}$$

Then the linearization coefficients g_R of $(R_k)_{k=0}^{\infty}$ are also nonnegative because

$$g_R(i,j,k) = \frac{P_k(\xi)}{P_i(\xi)P_j(\xi)} g(i,j,k) \quad \text{for all } i,j,k \in \mathbb{N}_0, \tag{36}$$

and it holds

$$\sum_{k=|i-j|}^{i+j} g_R(i,j,k) = 1 \quad \text{for all } i,j \in \mathbb{N}_0. \tag{37}$$

Hence, a hypergroup structure is associated with the orthogonal polynomial sequence $(R_k)_{k=0}^{\infty}$ which yields property (B), see [10]. □

There are well-known criteria by Askey [1] or [18, 19] implying the nonnegativity of the linearization coefficients. In case of a discrete measure π and nonnegative linearization coefficients we refer to Koornwinder [9] and [13].

In the next section we use the fact that in case of property (B) it holds

$$|K_n(x,y)| \leq K_n(\xi,\xi) \quad \text{for all } n \in \mathbb{N}_0,\ x,y \in S. \tag{38}$$

5 Little q-Jacobi polynomials

In all that follows we keep $0 < q < 1$ fixed and S_q is defined by (31). For $\alpha > -1$ we define a probability measure $\pi^{(\alpha)}$ on S_q by

$$\pi^{(\alpha)}(\{q^j\}) = (q^{\alpha+1})^j(1-q^{\alpha+1}), \;\; \pi^{(\alpha)}(\{0\}) = 0. \tag{39}$$

The orthogonal polynomial sequence $(R_k^{(\alpha)})_{k=0}^{\infty}$ with respect to $\pi^{(\alpha)}$ are special little q-Jacobi polynomials, see [8]. They fulfill the following orthogonality relation

$$\begin{aligned} \int_{S_q} R_k^{(\alpha)} R_l^{(\alpha)}\, d\pi^{(\alpha)} &= \sum_{j=0}^{\infty} R_k^{(\alpha)}(q^j) R_l^{(\alpha)}(q^j)(q^{\alpha+1})^j(1-q^{\alpha+1}) \\ &= \frac{(q^{\alpha+1})^k(1-q^{\alpha+1})}{1-q^{2k+\alpha+1}} \left(\prod_{i=1}^{k} \frac{1-q^i}{1-q^{\alpha+i}} \right)^2 \delta_{k,l}. \end{aligned} \tag{40}$$

Starting with

$$R_0^{(\alpha)} = 1 \text{ and } R_1^{(\alpha)}(x) = 1 - \frac{1-q^{\alpha+2}}{1-q^{\alpha+1}} x \tag{41}$$

they are defined by the three term recurrence relation (3) with coefficients

$$a_k = \frac{1-q^{\alpha+2}}{1-q^{\alpha+1}} A_k, \tag{42}$$

$$b_k = 1 - \frac{1-q^{\alpha+2}}{1-q^{\alpha+1}} (A_k + C_k), \tag{43}$$

$$c_k = \frac{1-q^{\alpha+2}}{1-q^{\alpha+1}} C_k, \tag{44}$$

where

$$A_k = q^k \frac{(1-q^{k+\alpha+1})(1-q^{k+\alpha+1})}{(1-q^{2k+\alpha+1})(1-q^{2k+\alpha+2})} \tag{45}$$

$$C_k = q^{k+\alpha} \frac{(1-q^k)(1-q^k)}{(1-q^{2k+\alpha})(1-q^{2k+\alpha+1})}. \tag{46}$$

In case of $\alpha \geq 0$ the orthogonal polynomial sequence $(R_k^{(\alpha)})_{k=0}^{\infty}$ has nonnegative linearization coefficients, see [9] ($\alpha = 0$) and [5] ($\alpha > 0$).

Theorem 5.1 *If $0 \leq \alpha$, then the sequence $(R_k^{(\alpha)})_{k=0}^{\infty}$ of little q-Jacobi polynomials constitutes a basis of $C(S_q)$.*

Proof. The nonnegativity of the linearization coefficients implies property (B) with $\xi = 0$. Let $(p_k^{(\alpha)})_{k=0}^{\infty}$ denote the corresponding orthonormal polynomial sequence. Using (40) we get

$$\sqrt{\frac{1-q^{2k+\alpha+1}}{(q^{\alpha+1})^k(1-q^{\alpha+1})}}\prod_{i=1}^{k}\frac{1-q^{\alpha+i}}{1-q^i} = p_k^{(\alpha)}(0) \geq \max_{x\in S_q}|p_k^{(\alpha)}(x)|. \tag{47}$$

Note that $S_q \subset [0,1]$. In order to prove

$$\sup_{x\in S_q}\int_{[0,1]}|K_n(x,y)|\,d\pi^{(\alpha)}(y) \leq C \quad \text{for all } n \in \mathbb{N}_0, \tag{48}$$

we split the integration domain into two parts $[0,\epsilon]$ and $[\epsilon,1]$. For the first it holds

$$\int_{[0,\epsilon]}|K_n(x,y)|\,d\pi^{(\alpha)}(y) \leq K_n(0,0)\pi^{(\alpha)}([0,\epsilon]). \tag{49}$$

By the Christoffel-Darboux formula (9) and property (B) we get

$$|K_n(x,y)| \leq \sqrt{c_{n+1}a_n}\,\frac{p_{n+1}^{(\alpha)}(0)|p_n^{(\alpha)}(y)| + p_n^{(\alpha)}(0)|p_{n+1}^{(\alpha)}(y)|}{|x-y|}, \quad x \neq y. \tag{50}$$

Hence, setting $\lambda_n = \sqrt{c_{n+1}a_n}$ and applying $|x-y| \geq (1-q)\,y$, $x \neq y$, it follows

$$\begin{aligned}\int_{[\epsilon,1]}|K_n(x,y)|\,d\pi^{(\alpha)}(y) \quad &\leq \quad \frac{\lambda_n p_{n+1}^{(\alpha)}(0)}{1-q}\int_{[\epsilon,1]}\frac{|p_n^{(\alpha)}(y)|}{y}\,d\pi^{(\alpha)}(y) &(51)\\ &+ \quad \frac{\lambda_n p_n^{(\alpha)}(0)}{1-q}\int_{[\epsilon,1]}\frac{|p_{n+1}^{(\alpha)}(y)|}{y}\,d\pi^{(\alpha)}(y) &(52)\\ &+ \quad \sum_{k=0}^{n}(p_k^{(\alpha)}(x))^2\pi^{(\alpha)}(\{x\}). &(53)\end{aligned}$$

By (8) we obtain

$$\sum_{k=0}^{n}(p_k^{(\alpha)}(x))^2\pi^{(\alpha)}(\{x\}) \leq 1. \tag{54}$$

It is simple to derive that

$$K_n(0,0) = \mathcal{O}(q^{-(\alpha+1)n}). \tag{55}$$

Now, we fix $\epsilon = q^n$ to get

$$\pi^{(\alpha)}([0,\epsilon]) = \mathcal{O}(q^{(\alpha+1)n}), \tag{56}$$

which yields a uniform bound for the integral on the left-hand side of (49) not depending on $x \in S$. Next, note that

$$\lambda_n = \mathcal{O}(q^n), \tag{57}$$

and by (47) we get

$$p_n^{(\alpha)}(0) = \mathcal{O}(q^{-(\alpha+1)\frac{n}{2}}). \tag{58}$$

In order to obtain a uniform bound for the integral on the left-hand side of (51) it remains to prove

$$\int_{[\epsilon,1]} \frac{|p_n^{(\alpha)}(y)|}{y}\, d\pi^{(\alpha)}(y) = \mathcal{O}(q^{(\alpha-1)\frac{n}{2}}). \tag{59}$$

For that purpose let $k \in \mathbb{N}_0$ with $\alpha < 2k+1$.
By the Cauchy-Schwarz inequality we get

$$\int_{[\epsilon,1]} \frac{|p_n^{(\alpha)}(y)|}{y}\, d\pi^{(\alpha)}(y) \le \left(\int_{[\epsilon,1]} \frac{d\pi^{(\alpha)}(y)}{y^{2(k+1)}}\right)^{\frac{1}{2}} \left(\int_{[0,1]} (p_n^{(\alpha)}(y)y^k)^2\, d\pi^{(\alpha)}(y)\right)^{\frac{1}{2}}. \tag{60}$$

By simple means it follows

$$\int_{[\epsilon,1]} \frac{d\pi^{(\alpha)}(y)}{y^{2(k+1)}} = \mathcal{O}(q^{(\alpha-2k-1)n}). \tag{61}$$

The three term recurrence formula for $p_n^{(\alpha)}$ is

$$yp_n^{(\alpha)} = -\Lambda_n p_{n+1}^{(\alpha)} + (A_n + C_n)p_n^{(\alpha)} - \Lambda_{n-1}p_{n-1}^{(\alpha)}, \tag{62}$$

where $\Lambda_n = \sqrt{C_{n+1}A_n}$, see (45) and (46). So the coefficients behave like q^n. The minus sign comes from the fact that $p_n(0) > 0$. By applying the recurrence relation k times we get

$$y^k p_n^{(\alpha)} = \sum_{n-k}^{n+k} d(k,n,i)p_i^{(\alpha)}, \tag{63}$$

where each coefficient $d(k,n,i)$ behaves like q^{kn}. Therefore, by orthogonality

$$\int_{[0,1]} (p_n^{(\alpha)}(y)y^k)^2\, d\pi^{(\alpha)}(y) = \sum_{n-k}^{n+k} (d(k,n,i))^2 = \mathcal{O}(q^{2kn}). \tag{64}$$

So we have shown (59) and the proof is complete. □

The little q-Legendre case ($\alpha = 0$) is also investigated in [16].
So in case of S_q we are able to give orthogonal Faber basis for $C(S_q)$. For the set S^r, see (33), the existence of an orthogonal Faber basis or even a Faber basis for $C(S^r)$ seems still to be open.

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