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Note

A Lower Bound for Orthogonal Polynomials with an Application to Polynomial Hypergroups

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We give a lower bound for solutions of linear recurrence relations of the form $za_n = \sum_{k=n-N}^{n+N} \alpha_{k,n} a_k$, whenever z is not in the l^p-spectrum of the corresponding banded operator. In particular if P_n are polynomials orthonormal with respect to a measure μ supported in a bounded interval the sequence $P_n(x)^2 + P_{n+1}(x)^2$ is bounded from below by $(1+\varepsilon)^n$, for $x \notin \operatorname{supp} \mu$. We give an application to polynomial hypergroups.

INTRODUCTION

Given a probability measure μ on the real line $\mathbb{R}$ such that all its moments are finite. Let $\{P_n\}_{n=0}^{\infty}$ be a system of orthonormal polynomials obtained from the sequence of consecutive monomials $1, x, x^2, \ldots$ by the Gram–Schmidt procedure. It is well known that P_n obey a three-term recurrence formula of the form

$$xP_n = \lambda_n P_{n+1} + \beta_n P_n + \lambda_{n-1} P_{n-1}, \tag{1}$$

where λ_n are positive coefficients while β_n are real ones. We are going to study the growth of $P_n(z)$ for z not in the support of the measure μ. By a well known theorem by Poincaré if $\lambda_n \to \lambda$ and $\beta_n \to \beta$ then

$$\frac{P_n(z)}{P_{n-1}(z)} \to u + \sqrt{u^2-1},$$

where $u = (2\lambda)^{-1}(z-\beta)$, and $\sqrt{u^2-1}$ is that branch of the square root for which $|u + \sqrt{u^2-1}| > 1$. In particular $P_n(z)$ have exponential growth.

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In general exponential lower estimates have been proved for z not in the convex hull of the support of the measure. For example if supp $\mu \subset [-1, 1]$, then

$$\liminf_{n \to \infty} \sqrt[n]{|P_n(z)|} \geqslant |z + \sqrt{z^2 - 1}|$$

for $z \notin [-1, 1]$ (see [1, Theorem 7.1, p. 117; 5, Theorem 1.1.4, p. 4]).

In Section 1 we give a lower estimate for the solutions of recurrence relations associated with finite band-width difference operators. When applied to orthogonal polynomials this yields an exponential lower estimate of $P_n(z)$ for z outside the set of orthogonality. In Section 2 we give an application to hypergroup theory. We show the symmetry of the discrete polynomial hypergroups under mild conditions on the weight function.

1. Lower Bounds for Polynomials

Let L be a linear operator acting on sequences $\{a_n\}_{n=0}^{\infty}$ according to

$$La_n = \sum_{k=n-N}^{n+N} \alpha_{k,n} a_k, \tag{2}$$

where $\alpha_{k,n}$ are complex coefficients such that $\alpha_{k,n} = 0$, for $k < 0$.

Theorem 1. *Let* $\sup\{|\alpha_{k,n}| : n \neq k \in \mathbb{N}\} < +\infty$. *Let* $b = \{b_n\}_{n \geqslant 0}$ *be a nonzero solution of*

$$Lb = zb$$

where $z \in \mathbb{C}$ *does not belong to the* l^p*-spectrum* $\sigma_p(L)$ *of* L. *Then for* $0 < p < \infty$

$$\liminf_{n \to \infty} \sqrt[n]{|b_{n+1-N}|^p + |b_{n+2-N}|^p + \cdots + |b_{n+N}|^p} > 1.$$

Proof. Let $z \notin \sigma_p(L)$. Then there is a constant $\eta > 0$ such that

$$\|(zI - L)a\|_p \geqslant \eta \, \|a\|_p. \tag{3}$$

Let $b = \{b_n\}_{n \geqslant 0}$ be a nonzero solution of $Lb = zb$. Define a sequence $b^{\{n\}}$ by

$$b_m^{\{n\}} = \begin{cases} b_m, & \text{if } 0 \leqslant m \leqslant n; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that

$$(zI-L)\, b_m^{\{n\}} = \begin{cases} \sum_{n+1}^{m+N} \alpha_{k,m} b_k, & \text{for } n-N<m\leqslant n; \\ -\sum_{m-N}^{n} \alpha_{k,m} b_k, & \text{for } n<m\leqslant n+N; \\ 0, & \text{otherwise.} \end{cases}$$

Denote $\alpha = \sup\{\alpha_{k,n} : n \neq k \in \mathbb{N}\}$. Then

$$\|(zI-L)\, b^{\{n\}}\|_p^p \leqslant \alpha^p \sum_{m=n-N+1}^{n} \left(\sum_{k=n+1}^{m+N} |b_k| \right)^p + \alpha^p \sum_{m=n+1}^{n+N} \left(\sum_{k=m-N}^{n} |b_k| \right)^p.$$

The first sum can be majorized by using the Hölder inequality, if $p>1$, or the triangle inequality, if $0<p\leqslant 1$, as follows.

$$\begin{aligned} &\sum_{m=n-N+1}^{n} \left(\sum_{k=n+1}^{m+N} |b_k| \right)^p \\ &\quad \leqslant N^r \sum_{m=n-N+1}^{n} \sum_{k=n+1}^{m+N} |b_k|^p \\ &\quad = N^r \sum_{k=n+1}^{n+N} (n+1-k+N)\, |b_k|^p \leqslant N^{r+1} \sum_{k=n+1}^{n+N} |b_k|^p \end{aligned}$$

where $r=p/q$ or $r=0$ according to $p>1$ or $0<p\leqslant 1$. In a similar way we obtain an estimate for the second sum.

$$\sum_{m=n+1}^{n+N} \left(\sum_{k=m-n}^{n} |b_k| \right)^p \leqslant N^{r+1} \sum_{k=n+1-N}^{n} |b_k|^p.$$

Combining these estimates gives

$$\|(zI-L)\, b^{\{n\}}\|_p^p \leqslant \alpha^p N^{r+1} \sum_{k=n+1-N}^{n+N} |b_k|^p. \tag{4}$$

Let $B_n = \sum_{k=n+1-N}^{n+N} |b_k|^p$. Then by (3) and (4) we get

$$\alpha^p N^{r+1} B_n \geqslant \eta^p \sum_{k=0}^{n} |b_k|^p \geqslant \frac{\eta^p}{2N} \sum_{k=0}^{n-N} B_k.$$

For a fixed natural number $0\leqslant r\leqslant N-1$, consider the sequence $c_{n,r} = B_{Nn+r}$. Then we have

$$c_{n,r} \geqslant \varepsilon \sum_{k=0}^{n-1} c_{k,r}, \qquad \text{where} \quad \varepsilon = \frac{1}{2} N^{-r-2} \alpha^{-p} \eta^p.$$

By a simple induction argument one can show that $c_{n,r} \geqslant \varepsilon(1+\varepsilon)^{n-m-1} c_{m,r}$. Take m to be the first index such that $c_{m,r} > 0$. Then $\liminf \sqrt[n]{c_{n,r}} > 1$. ∎

Remark. It is worthwhile observing that the diagonal coefficients $\alpha_{n,n}$ do not need to be bounded for the theorem to hold. Thus L can be an unbounded operator with respect to any l^p-norm.

Let $P_n(x)$ be polynomials orthonormal with respect to the measure μ and satisfying the recurrence formula (1). Then the sequence $a = \{P_n(z)\}_{n \geqslant 0}$, is a solution of the equation $La = za$, where

$$La_n = \lambda_n a_{n+1} + \beta_n a_n + \lambda_{n-1} a_{n-1}.$$

L is a symmetric operator on the space $l^2(\mathbb{N})$ of square summable sequences and its spectrum can be identified with the support of the measure μ, whenever L has a unique self-adjoint extension. In particular by Carleman's condition this holds if the sequence λ_n is bounded. Thus by Theorem 1 we get the following.

COROLLARY 1. *Let P_n be orthonormal polynomials relative to the measure μ. Assume that the coefficients λ_n in (1) are bounded. Let $z \notin \operatorname{supp} \mu$. Then*

$$\liminf_{n \to \infty} \sqrt[n]{|P_n(z)|^2 + |P_{n+1}(z)|^2} > 1.$$

Remark 1. The corollary generalizes [4, Proposition 8.3, p. 81], where the authors showed $\limsup_{n \to \infty} \sqrt[n]{|P_n(z)|} > 1$. In our case we cannot replace the conclusion of the corollary by $\liminf \sqrt[n]{|P_n(z)|} > 1$. In fact, if $0 \notin \operatorname{supp} \mu$ and the measure μ is symmetric about 0, then $P_{2n-1}(0) = 0$.

Remark 2. Since L is a symmetric linear operator, with spectrum $\operatorname{supp} \mu$, the constant η from (3) is equal to $\operatorname{dist}(z, \operatorname{supp} \mu)$. Analyzing the proof of Theorem 1 gives a more explicit estimate:

$$\liminf_{n \to \infty} \sqrt[n]{|P_n(z)|^2 + |P_{n+1}(z)|^2} \geqslant 1 + \frac{\operatorname{dist}(z, \operatorname{supp} \mu)^2}{2\lambda^2}, \qquad \lambda = \limsup_{n \to \infty} \lambda_n.$$

2. Application to Polynomial Hypergroups

We start by briefly describing polynomial hypergroups. We refer to [6, pp. 159–162] for details (see also [3]). Let $\{P_n\}_{n \geqslant 0}$ be polynomials orthonormal with respect to a measure μ on the real line. We assume that the support of the measure μ is bounded from the right hand side, say by 1. This is equivalent to requiring that $P_n(1)$ be positive numbers. Let $R_n(x)$ denote polynomials normalized at 1, i.e., $R_n(x) = P_n(x)/P_n(1)$. Assume also that the linearization coefficients $c(n, m, k)$ in the product formula

$$R_n(x)\, R_m(x) = \sum_{k=|n-m|}^{n+m} c(n, m, k)\, R_k(x)$$

are nonnegative. Let $\omega_n = P_n(1)^2$. By using the coefficients $c(n, m, k)$ we can define the convolution $*$ of the two sequences a and b according to

$$(a * b)(k) = \sum_{n, m=0}^{\infty} c(n, m, k)\, \omega_n \omega_m a_n b_m.$$

With this operation $l^1(\omega_n)$, the space of sequences absolutely summable with respect to the weight ω_n, becomes a Banach algebra. This structure is a polynomial hypergroup, and μ is called the Plancherel measure, while ω_n is called the Haar measure of this hypergroup. The maximal ideal space of this hypergroup can be identified with the set

$$\mathcal{M} = \{z \in \mathbb{C} : \sup_n |R_n(z)| \leqslant 1\} = \{z \in \mathbb{C} : |P_n(z)| \leqslant P_n(1),\ n \in \mathbb{N}\}$$

We are interested in the relation between $\mathcal{M}$ and $\operatorname{supp} \mu$. We always have $\operatorname{supp} \mu \subset \mathcal{M}$, (see [3, Theorem 7.3C, p. 41; 6, Theorem 1]).

THEOREM 2. *Let $P_n(x)$ be polynomials orthonormal relative to μ, $\operatorname{supp} \mu \subset (-\infty, 1]$, having nonnegative linearlization coefficients. If*

$$\liminf_{n \to \infty} \sqrt[n]{P_n(1)^2 + P_{n+1}(1)^2} \leqslant 1$$

then

$$\operatorname{supp} \mu = \{z \in \mathbb{C} : |P_n(z)| \leqslant P_n(1),\ n \in \mathbb{N}\}.$$

In other words the maximal ideal space of $l^1(\omega_n)$ coincides with $\operatorname{supp} \mu$. *In particular the algebra $l^1(\omega_n)$ is symmetric.*

Proof. It suffices to show the inclusion from the right to the left as the opposite inclusion always holds true (cf. [6, Theorem 1]). Assume that $|P_n(z)| \leqslant P_n(1)$, $n \in \mathbb{N}$. Then

$$\liminf \sqrt[n]{|P_n(z)|^2 + |P_{n+1}(z)|^2} \leqslant \liminf \sqrt[n]{P_n(1)^2 + P_{n+1}(1)^2} \leqslant 1.$$

In view of Corollary 1 this implies $z \in \operatorname{supp} \mu$. ∎

COROLLARY 2 ([2, 7, 8]). *If $P_n(1)$ has subexponential growth, then the maximal ideal space of the Banach algebra $l^1(\omega_n)$ coincides with the support of the measure μ; i.e., the algebra is symmetric.*

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