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CONVOLUTION OPERATORS OF WEAK TYPE (p, p) WHICH ARE NOT OF STRONG TYPE (p, p)

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ABSTRACT. We give an example of a locally compact group G for which, for every p with $2 < p < \infty$, there exists an operator of weak type (p, p) commuting with the right translations on G which is not of strong type (p, p). This gives a negative solution of E. M. Stein's problem.

Let G be a locally compact group. For $1 < p < \infty$ we let $L_p^p(G)$, $L_p^{p,\infty}(G)$ be the Banach spaces of convolution operators of strong type (p, p), weak type (p, p). We always have $L_p^p(G) \subset L_p^{p,\infty}(G)$. E. M. Stein posed the problem whether $L_p^p(G) = L_p^{p,\infty}(G)$ for every locally compact group. M. Zafran [8] showed that if $1 < p < 2$ then $L_p^p(G) \neq L_p^{p,\infty}(G)$ for many classical abelian groups. Later M. Cowling and J. Fournier [3] extended this result on all infinite locally compact groups using the method of Rudin-Shapiro measures. In the case $p = 2$, M. Cowling [2] showed that amenability of G implies the equality $L_2^2(G) = L_2^{2,\infty}(G)$. He conjectured that the equality also holds for $p > 2$ at least for amenable groups, but there were no examples in support of, or in contradiction, to this. It is still not known whether Cowling's conjecture holds for amenable groups. It is not solved even for the group of integers. The author [7] has given the first example of a locally compact group (nonabelian free group) for which $L_2^2(G) \neq L_2^{2,\infty}(G)$. In this note we prove that for nonabelian free groups and $p > 2$: $L_p^p(G) \neq L_p^{p,\infty}(G)$.

For a locally compact group G let $L^{p,q}(G)$ denote the usual Lorentz space on G. We recall that for $p > 1$ and $1 \leqslant q < \infty$ the conjugate Banach space of $L^{p,q}(G)$ is $L^{p',q'}(G)$, where $1/p + 1/p' = 1 = 1/q + 1/q'$, and duality is given by scalar product. We need the following simple fact:

PROPOSITION [4, 6]. *Suppose T is a linear operator which maps characteristic functions χ_E, $m(E) < \infty$, into a Banach space B and $\|T\chi_E\| \leqslant c\|\chi_E\|_{p,1}$, where c is independent of E. Then there exists a unique linear extension of T to a continuous map of $L^{p,1}(G)$ into B and $\|Tf\| \leqslant c_p c\|f\|_{p,1}$.*

We will call the bounded operator from L^p to $L^{p,\infty}$ (from L^p to L^p) weak (strong) type (p, p).

THEOREM. *Let G be a nonabelian free group on finitely many generators. Then for every $p > 2$ there exists the convolution operator of weak type (p, p) which is not of strong type (p, p).*

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PROOF. By duality it suffices to show that for $1 < p < 2$ there exists a function f such that $f * L^{p,1} \subset L^p$ and $f * L^p \not\subset L^p$. Let us take into consideration the radial functions on G that mean the complex functions f for which $f(x)$ depends only on the length of the word x (that is, number of letters of the word in its reduced form). Let $L_r^{p,q}$ denote the space of radial functions in $L^{p,q}$. Now we prove

LEMMA. *Let* $1 < p < 2$. *For every radial function* f *the following conditions are equivalent*:

(1) *For every function* g in $L^{p,1}$, $\|f * g\|_p \leq c\|g\|_{p,1}$;

(2) $f \in L^p$.

Before proving the Lemma we recall T. Pytlik's [5] result that if $1 < p < 2$, then every nonnegative radial function f is in $L^{p,1}$ if and only if for every g in L^p, $\|f * g\|_p \leq c\|g\|_p$. This result and the Lemma immediately prove the Theorem because $L_r^p \neq L_r^{p,1}$. It remains to prove the Lemma.

PROOF OF THE LEMMA. Let $1 < p < 2$. Choose q such that $1 < p < q < 2$. By Pytlik's result we have that for $g \in L^q$ and $f \in L_r^{q,1}$, $\|f * g\|_q \leq c\|f\|_{q,1}\|g\|_q$. If we assume that $f, g \in L^1$ then $\|f * g\|_1 \leq \|g\|_1\|f\|_1$. This means that the map $f \to f * g$ is a bounded operator from $L_r^{q,1}$ to L^q and from L_r^1 to L^1. Then by the Calderón interpolation theorem [1, Corollary 3.3.12] we have that, for every f in L_r^p, $\|f * g\|_p \leq d_p\|g\|_q^{1-t}\|g\|_1^t\|f\|_p$, where $1/p = (1 - t)/q + t$. If g is the characteristic function of a set in G then $\|g\|_q^{1-t}\|g\|_1^t = \|g\|_{p,1}$. So for the characteristic function g we have $\|f * g\|_p \leq d_p\|g\|_{p,1}\|f\|_p$. By the Proposition this formula holds for every function g in $L^{p,1}$. Hence if $f \in L_r^p$, then the map $g \to f * g$ is bounded from $L^{p,1}$ to L^p with the norm less than $c_p d_p\|f\|_p$. On the other hand, $\|f * \delta_e\|_p = \|f\|_p$, so if f is radial, then $f * L^{p,1} \subset L^p$ if and only if $f \in L^p$. This completes the proof.

REMARK. One may notice that $L_r^{p,1} = L^1(\mathbf{N}, \omega_n^{1/p})$, where $\omega_n = 2k(2k-1)^{n-1}$ (k = number of generators in G). By this fact the Theorem may also be proved from the interpolation theorem of Stein and Weiss with change of measures.

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