STUDIA MATHEMATICA, T. XCIV (1989)

Matrix coefficients of irreducible representations of free products of groups

by

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Abstract. An analytic series of functions φ_z on the free product of groups $A_1 \circ \ldots \circ A_N$ which are matrix coefficients of irreducible representations is presented. The functions φ_z are radial with respect to a certain length on the free product. Applications to the Fourier–Stieltjes algebra are also given.

1. Introduction. We will work with a group G which is the free product of groups $A_1, \ldots, A_N$. Each element x of G, $x \neq e$, can be uniquely represented as a reduced word, i.e.

$$x = a_1 \ldots a_n, \quad \text{where } a_i \in A_{n_i} \setminus \{e\},\ n_i \neq n_{i+1}.$$

This gives rise to the notion of *length* on G defined as $|x| = n$ and $|e| = 0$ for the identity element e in G. The functions f on G whose values $f(x)$ depend only on the length of x in G will be called *radial*.

Harmonic analysis on free products was studied by several authors. Iozzi and Picardello [3] considered the free product of finite cyclic groups of the same order. In this case the space of absolutely summable radial functions is a commutative Banach algebra with convolution. This allows one to study the spherical functions on G which are multiplicative functionals on the radial functions and to develop a representation theory related to them. Further results, e.g. a characterization of radial positive-definite functions, can be found in a paper by Młotkowski [4]. Let us also mention the work of Cartwright and Soardi [2] concerning the free product of two cyclic groups of different orders.

The case of the free product of groups $\mathbf{Z}_2$ is strictly connected to the geometry of homogeneous trees. We refer to the PhD Thesis of Steger [5] where this situation was deeply explored.

Let us also recall some recent results of Bożejko [1] who introduced the notion of free product of representations. His construction produces a new

1980 *Mathematics Subject Classification* (1985 *Revision*): 43A65, 22D12.

Key words and phrases: free products, uniformly bounded representations, irreducible representations.

unitary (or uniformly bounded) representation of the free product $A \circ B$ from two unitary (or uniformly bounded) representations of the groups A and B. His theory concerns the free product of matrix coefficients of representations as well.

In this paper we consider the case when $A_1, \ldots, A_N$ are infinite groups (e.g. the free group $\boldsymbol{Z} \circ \ldots \circ \boldsymbol{Z}$ is covered).

Let z be a complex number in the open unit disc. By [1] the radial function $G \ni x \mapsto z^{|x|}$ is a coefficient of a uniformly bounded representation of G in Hilbert space. By [1], Corollary 3.2, the function $x \mapsto t^{|x|}$ is positive-definite for real positive t. Here we will study the class of radial functions φ_z on G defined by

$$\varphi_z(x) = \begin{cases} 1 & \text{for } x = e, \\ \dfrac{(N-1)z+1}{Nz} z^{|x|} & \text{for } x \neq e, \end{cases}$$

where $|z| < 1$ (cf. [4], §2). Clearly the φ_z are again matrix coefficients of uniformly bounded representations of G. By [4] the function φ_t is positive-definite for t in the segment $[-1/(N-1), 1]$.

The origin of the functions φ_z is the following. Any spherical function on the free product $Z_k \circ \ldots \circ Z_k$ (N times) is of the form $c_z z^{|x|} + c_z'[(N-1)(k-1)z]^{-|x|}$ where c_z, c_z' are constants depending on k and N (cf. [3], Theorem 7). Now we obtain φ_z by letting $k \to \infty$. So the φ_z play the role of spherical functions on the free product of infinite groups.

Assuming that $A_1, \ldots, A_N$ are all infinite we are going to prove that each function φ_z is a matrix coefficient of an irreducible representation of G except $z = 0$ and $z = -1/(N-1)$. In particular, the positive-definite functions φ_t with $t \in (-1/(N-1), 1]$, $t \neq 0$, are extreme. Moreover, for nonreal z or $z \in (-1, -1/(N-1))$ the functions φ_z do not belong to the Fourier–Stieltjes algebra of G.

2. Preliminaries. Let G be a locally compact group. Denote by $BH(G)$ the space of all matrix coefficients of uniformly bounded representations of G in Hilbert space, i.e. the space of functions φ on G such that there exists a representation π of G in a Hilbert space H and two vectors $\zeta, \eta \in H$ satisfying

$$(1) \qquad \varphi(x) = \langle \pi(x)\zeta, \eta \rangle, \qquad \sup_{x \in G} \|\pi(x)\| < +\infty.$$

With a given representation π of G in a Hilbert space we may associate the conjugate representation π^* defined by $\pi^*(x) = \pi(x^{-1})^*$. If π is uniformly bounded then π^* is also uniformly bounded with the same bound. If π is unitary then of course $\pi = \pi^*$.

PROPOSITION 1. *Any function φ in $BH(G)$ can be represented as in* (1) *with ζ a cyclic vector for the representation π and η a cyclic vector for π^*.*

Proof. By definition $\varphi(x) = \langle \sigma(x)\xi, \eta \rangle$, where σ is a uniformly bounded representation in H_0 and $\xi, \eta \in H_0$. We may assume that ξ is a cyclic vector for π (if not we restrict σ to the invariant subspace generated by ξ and project η onto this subspace). Next let $H = \overline{\operatorname{span}} \{\sigma^*(x)\eta \colon x \in G\}$ and $\zeta = P_H \xi$ where P_H is the orthogonal projection of H_0 onto H. Put $\pi(x) = (\sigma^*(x^{-1})|_H)^*$. Then π is a uniformly bounded representation of G in H and

$$\langle \pi(x)\zeta, \eta \rangle = \langle \zeta, \sigma^*(x^{-1})\eta \rangle = \langle \xi, \sigma^*(x^{-1})\eta \rangle = \langle \sigma(x)\xi, \eta \rangle = \varphi(x).$$

By construction η is a cyclic vector for π^*. Assume that for some $u \in H$ and every x in G we have $\langle \pi(x)\zeta, u \rangle = 0$. Then $0 = \langle \zeta, \sigma^*(x^{-1})u \rangle = \langle \xi, \sigma^*(x^{-1})u \rangle = \langle \sigma(x)\xi, u \rangle$. Since ξ is a cyclic vector for σ this shows $u = 0$, i.e. ζ is cyclic for π.

DEFINITION. The function φ in $BH(G)$ is called *extreme* if no uniformly bounded representation π of G in H such that $\varphi(x) = \langle \pi(x)\zeta, \eta \rangle$, where ζ is cyclic for π and η is cyclic for π^*, has a nontrivial closed invariant subspace.

Remark. If φ is a positive-definite function then the definition above implies that φ is extreme in the usual sense.

3. Free products of groups. Let $A_1, \ldots, A_N$ be discrete groups. Denote by G their free product $A_1 \circ \ldots \circ A_N$. Each element $x \neq e$ in G can be uniquely expressed as a reduced word

$$x = a_1 \ldots a_n, \quad \text{where } a_i \in A_{n_i} \setminus \{e\},\ n_i \neq n_{i+1}.$$

We define the *length* of x by $|x| = n$ and $|e| = 0$ for the identity element e in G.

For a complex number z in the open unit disc define the functions ψ_z and φ_z on G by

$$(2) \qquad \psi_z(x) = z^{|x|}, \qquad \varphi_z(x) = \begin{cases} 1 & \text{for } x = e, \\ \dfrac{(N-1)z+1}{Nz} z^{|x|} & \text{for } x \neq e \end{cases}$$

(see [4], §2). Clearly we have

$$(3) \qquad \psi_z(x) = \frac{Nz}{(N-1)z+1} \varphi_z(x) + \frac{1-z}{(N-1)z+1} \delta_e(x),$$

where δ_e is the characteristic function of the one-point set $\{e\}$. By [1] (§5, p. 180) the functions ψ_z belong to $BH(G)$. As obviously δ_e is in $BH(G)$ we obtain

PROPOSITION 2. *For any z, $|z| < 1$, the function φ_z belongs to $BH(G)$.*

THEOREM. *Let G be the free product of groups $A_1, \ldots, A_N$. If $A_1, \ldots, A_N$ are all infinite then the function φ_z is extreme* (see the definition in § 2) *for any complex $z \in D = \{|z| < 1\}$ except $z = 0$ and $z = -1/(N-1)$.*

Proof. By Propositions 1, 2 we may express φ as $\varphi(x) = \langle \pi(x)\zeta, \eta\rangle$, where π is a uniformly bounded representation of G in a Hilbert space H and ζ, η are cyclic vectors for π, π^* respectively. For any $i = 1, \ldots, N$ let $\{a_{n,i}\}_{n=1}^{\infty}$ denote a sequence of distinct elements of the group A_i. For any natural n define the operator T_n on H by

$$T_n = \frac{1}{Nz[(N-1)z+1]}\frac{1}{n}\sum_{k=1}^{n}\Big[z\sum_{i=1}^{N}\pi(a_{k,i}) + \sum_{i\neq j=1}^{N}\pi(a_{k,i}a_{k,j})\Big].$$

LEMMA 1. *For any x, y in G we have*

$$\lim_n \langle T_n\pi(x)\zeta, \pi^*(y)\eta\rangle = \varphi_z(x)\varphi_z(y).$$

Proof of Lemma 1. As π is uniformly bounded the contribution of each single term of T_n after the summation sign $\sum_{k=1}^{n}$ is infinitesimal. We will make computations keeping this in mind. Let $t_n(x, y) = \langle T_n\pi(x)\zeta, \pi^*(y)\eta\rangle$. Then

$$\begin{aligned} t_n(x, y) &= \langle \pi(y^{-1})T_n\pi(x)\zeta, \eta\rangle \\ &= \frac{1}{Nz[(N-1)z+1]}\frac{1}{n}\sum_{k=1}^{n}\Big[z\sum_{i=1}^{N}\varphi_z(y^{-1}a_{k,i}x) + \sum_{i\neq j=1}^{N}\varphi_z(y^{-1}a_{k,i}a_{k,j}x)\Big] \\ &= \frac{1}{N^2z^2}\frac{1}{n}\sum_{k=1}^{n}\Big(\sum_{i=1}^{N} z^{|y^{-1}a_{k,i}x|+1} + \sum_{i\neq j=1}^{N} z^{|y^{-1}a_{k,i}a_{k,j}x|}\Big) + o(1) \end{aligned}$$

as n tends to infinity. We will consider four possible cases.

1. $x, y \neq e$ and $|y^{-1}x| < |x|+|y|$. This means the reduced words corresponding to x and y start with elements of the same group, say A_i. Then

$$\begin{aligned} t_n(x, y) &= \frac{1}{N^2z^2}[z^{|x|+|y|} + (N-1)z^{|x|+|y|+2} + 2(N-1)z^{|x|+|y|+1} \\ &\qquad + (N-2)(N-1)z^{|x|+|y|+2}] + o(1) \\ &= \left[\frac{(N-1)z+1}{Nz}\right]^2 z^{|x|+|y|} + o(1) = \varphi_z(x)\varphi_z(y) + o(1). \end{aligned}$$

2. $x, y \neq e$ and $|y^{-1}x| = |x|+|y|$. This means x, y start in A_i, A_j respectively and $i \neq j$. Then

$$\begin{aligned} t_n(x, y) &= \frac{1}{N^2z^2}\{2z^{|x|+|y|+1} + (N-2)z^{|x|+|y|+2} + z^{|x|+|y|} \\ &\qquad + 2(N-2)z^{|x|+|y|+1} + [(N-2)^2+N-1]z^{|x|+|y|+2}\} + o(1) \\ &= \left[\frac{(N-1)z+1}{Nz}\right]^2 z^{|x|+|y|} + o(1) = \varphi_z(x)\varphi_z(y) + o(1). \end{aligned}$$

3. $x \neq e$, $y = e$ (or $x = e$ and $y \neq e$). Then

$$\begin{aligned} t_n(x, e) &= \frac{1}{N^2z^2}[z^{|x|+1} + (N-1)z^{|x|+2} + (N-1)z^{|x|+1} \\ &\qquad + (N-1)^2 z^{|x|+2}] + o(1) \\ &= \frac{(N-1)z+1}{Nz}z^{|x|} + o(1) = \varphi_z(x)\varphi_z(e) + o(1). \end{aligned}$$

4. $x = y = e$. Then

$$t_n(e, e) = \frac{1}{N^2z^2}[Nz^2 + (N-1)Nz^2] = \varphi_z(e)\varphi_z(e).$$

Proof of the theorem. Since the norms $\|T_n\|$ are uniformly bounded and ζ, η are cyclic vectors for π, π^* respectively, by Lemma 1 the sequence of operators T_n is weakly converging to the operator T defined by

$$\langle T\pi(x)\zeta, \pi^*(y)\eta\rangle = \varphi_z(x)\varphi_z(y) = \langle \pi(x)\zeta, \eta\rangle\langle\zeta, \pi^*(y)\eta\rangle.$$

Thus T is one-dimensional and $T = \zeta\otimes\eta$.

From now on we argue in a routine way. Assume that M is a nontrivial closed subspace invariant for the representation π. Then $T_nM \subset M$ and so $TM \subset M$. If $TM \neq 0$ then $TM = \{C\zeta\}$. This gives $\zeta \in M$ and since ζ is cyclic for π it follows that $M = H$. If $TM = 0$ then for any x in G and u in M we have $0 = \langle \pi(x)u, \eta\rangle = \langle u, \pi^*(x^{-1})\eta\rangle$. This implies $u = 0$ because η is cyclic for π^*. Hence $M = 0$. This completes the proof.

Remark. Observe that for $t = 0$ or $t = -1/(N-1)$ the corresponding functions are not extreme. In fact, $\varphi_0 = N^{-1}(\chi_{A_1} + \ldots + \chi_{A_N})$, where χ_{A_i} is the characteristic function of the subgroup A_i in G, and $\varphi_{-1/(N-1)} = \delta_e$.

Recall that by [4], Theorem 1, the function φ_t is positive-definite if $t \in [-1/(N-1), 1]$. So by Theorem 1 we obtain

COROLLARY 1. *Let G be the free product of infinite groups $A_1, \ldots, A_N$. Then φ_t is an extreme positive-definite function on G for any $t \in (-1/(N-1), 1]$ except $t = 0$.*

COROLLARY 2. *Let G be the free product of infinite groups $A_1, \ldots, A_N$. Then the function $\psi_t(x) = t^{|x|}$ does not belong to the Fourier–Stieltjes algebra $B(G)$ for any $t \in (-1, -1/(N-1))$.*

Before the proof of Corollary 2 we need a simple lemma.

LEMMA 2. *Let φ be a matrix coefficient of an irreducible unitary representation of the group G. Then φ is a positive-definite function on G if and only if $\varphi(e) \geqslant 0$ and $\varphi(x^{-1}) = \overline{\varphi(x)}$.*

Proof. Clearly, we only have to prove that the conditions are sufficient. Let $\varphi(x) = \langle \pi(x)\zeta, \eta \rangle$, where π is an irreducible unitary representation of G in a Hilbert space H. Assume $\varphi(x^{-1}) = \overline{\varphi(x)}$. This implies $\langle \pi(x)\zeta, \eta \rangle = \langle \pi(x)\eta, \zeta \rangle$ for x in G, which means that

$$\operatorname{Tr}[\pi(x)(\zeta \otimes \eta)] = \operatorname{Tr}[\pi(x)(\eta \otimes \zeta)] \quad \text{for } x \in G,$$

where $\zeta \otimes \eta$ denotes the one-dimensional operator given by $(\zeta \otimes \eta) u = \langle u, \eta \rangle \zeta$. Since span $\{\pi(x)\colon x \in G\}$ is σ-weakly dense in $B(H)$ we have $\zeta \otimes \eta = \eta \otimes \zeta$. This yields that ζ, η are linearly dependent, hence φ is positive-definite because $\langle \zeta, \eta \rangle = \varphi(e) \geqslant 0$.

Proof of Corollary 2. Suppose that ψ_t belongs to $B(G)$. Then by (3) also φ_t belongs to $B(G)$. Hence $\varphi_t(x) = \langle \pi(x)\zeta, \eta \rangle$, where π is a unitary representation of G. By Proposition 1 we may assume that ζ, η are cyclic vectors. Therefore by the Theorem π is irreducible. Since $\varphi_t(x^{-1}) = \overline{\varphi_t(x)}$ and $\varphi_t(e) = 1$, by Lemma 2 the function φ_t is positive-definite. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of distinct elements of A_1. Then for any natural n

$$0 \leqslant \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \varphi_t(a_j^{-1} a_i) = \frac{1}{n-1} + \frac{(N-1)t+1}{N},$$

which implies $(N-1)t+1 \geqslant 0$.

Remark. Just as well, we can apply Lemma 2 to prove that φ_t is positive-definite for t positive. In fact, by [1], Corollary 3.2, the function $x \mapsto t^{|x|}$ is positive-definite for $0 \leqslant t \leqslant 1$. Hence φ_t belongs to $B(G)$. Combining the Theorem and Lemma 2 shows that φ_t is positive-definite.

Now we turn to the case of z nonreal.

PROPOSITION 3. *Let G be the free product of two infinite groups A and B. Then for any nonreal number z in the open unit disc the function $\psi_z(x) = z^{|x|}$ does not belong to the Fourier–Stieltjes algebra of G.*

Proof. Suppose ψ_z lies in $B(G)$. Then φ_z is in $B(G)$, too (with $N = 2$). Let $\varphi_z(x) = \langle \pi(x)\zeta, \eta \rangle$, where π is a unitary representation and ζ, η are cyclic vectors for π (cf. Prop. 1). Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be two sequences of distinct elements in A and B respectively. Define the operators T_n, $n = 1, 2, \ldots$, by

$$T_n = \frac{1}{2n} \sum_{k=1}^{n} \left(\pi(a_k) + \pi(b_k) + \pi(a_k^{-1}) + \pi(b_k^{-1})\right).$$

As in the proof of the Theorem, the sequence T_n is weakly converging to some operator T. We are going to compute T. For $y \neq e$ we have

$$\langle T_n \zeta, \pi(y)\eta \rangle = \frac{1}{2n} \sum_{k=1}^{n} \left(\varphi_z(y^{-1} a_k) + \varphi_z(y^{-1} b_k) + \varphi_z(y^{-1} a_k^{-1}) + \varphi_z(y^{-1} b_k^{-1})\right)$$
$$= \varphi_z(y) + z\varphi_z(y) + o(1) = (1+z)\varphi_z(y) + o(1)$$

and $\langle T_n \zeta, \eta \rangle = 1 + z$. Thus

$$\langle T\zeta, \pi(y)\eta \rangle = (1+z)\varphi_z(y) = (1+z)\langle \zeta, \pi(y)\eta \rangle,$$

which gives $T\zeta = (1+z)\zeta$ because η is a cyclic vector for π. On the other hand, since π is unitary, T is selfadjoint. Hence $1+z$ has to be a real number.

Remark. Obviously Proposition 3 holds for the free product of more than two groups provided that two of them are infinite.

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Received September 24, 1987
Revised version June 24, 1988 (2365)