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Sharp estimates for Jacobi matrices and chain sequences, II

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Abstract

Chain sequences are positive sequences $\{c_n\}$ of the form $c_n = g_n(1 - g_{n-1})$ for a nonnegative sequence $\{g_n\}$. They are very useful in estimating the norms of Jacobi matrices and for localizing the interval of orthogonality for orthogonal polynomials. We give optimal estimates for the chain sequences which are more precise than the ones obtained in the paper (Constructive Approx. 6 (1990) 363) and in our earlier paper (J. Approx. Theory 118 (2002) 94).

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1. Introduction

The concept of chain sequences has been introduced by Wall [8] in connection with continued fractions. These are sequences $\{c_n\}_{n=1}^{\infty}$ which can be represented in the form

$$c_n = g_n(1 - g_{n-1}), \quad n \geqslant 0, \tag{1}$$

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for a sequence $\{g_n\}_{n=0}^{\infty}$, such that $0 \leqslant g_n \leqslant 1$. Chain sequences have also been used to locate the interval of orthogonality for systems of orthogonal polynomials. Namely, let p_n be a system of orthogonal polynomials satisfying the recurrence relation of the form

$$xp_n = \gamma_n p_{n+1} + \beta_n p_n + \alpha_n p_{n-1},$$

for $n \geqslant 0$, where $\gamma_n > 0$, $\alpha_{n+1} > 0$, $\beta_n \in \mathbb{R}$. We use the convention $\alpha_0 = p_{-1} = 0$. Let μ denote an orthogonality measure. It might not be unique. Then the number a is located to the left of support of μ if and only if $a < \beta_n$ for $n \geqslant 0$ and the numbers

$$\frac{\alpha_{n+1}\gamma_n}{(a-\beta_n)(a-\beta_{n+1})}$$

form a chain sequence (see [2, Theorem 2.1, p. 108]). From this point of view, estimates of the chain sequences are important to obtain precise location of the interval of orthogonality.

The greatest constant chain sequence is $c_n = \frac{1}{4}$. Chihara [3] obtained some upper estimates for the chain sequences whose terms are greater than $\frac{1}{4}$ and stated the problem of determining sharper estimates. In [7], we sharpened Chihara's estimates by showing that if a chain sequence satisfies

$$c_n \geqslant \frac{1}{4} + \frac{1+\varepsilon_n}{16n^2} \tag{2}$$

for almost all n, where $\varepsilon_n \geqslant 0$, then (see (11) in [7])

$$\sup_n (\log n) \sum_{k=n+1}^{\infty} \frac{\varepsilon_k}{k} < +\infty.$$

On the other hand Jacobsen and Masson [4] found very precise convergence results for continued fractions which can be reformulated into chain sequence setting. Theorem 1 of [4] yields the following.

Theorem 1 (Jacobsen, Masson). *Let c_n be a chain sequence such that for $n \geqslant N$ there holds*

$$c_n \geqslant \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16n^2(\log^{[1]} n)^2} + \cdots + \frac{1+\varepsilon}{16n^2(\log^{[1]} n \log^{[2]} n \ldots \log^{[k]} n)^2},$$

where $\log^{[k]} x$ denotes the kth composition of the logarithm. Then $\varepsilon \leqslant 0$.

In this paper we are going to estimate the remainder in the formula above. Namely we will show the following.

Theorem 2. *Let c_n be a chain sequence such that*

$$c_n \geqslant \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16n^2(\log^{[1]} n)^2} + \cdots + \frac{1+\varepsilon_n}{16n^2(\log^{[1]} n \log^{[2]} n \ldots \log^{[k]} n)^2}$$

for $n > N$, *where* $\varepsilon_n \geqslant 0$. *Then*,

$$\lim_n \log^{[k+1]} n \sum_{j=n+1}^{\infty} \frac{\varepsilon_j}{j \log^{[1]} j \log^{[2]} \dots \log^{[k]} j} \leqslant 4. \tag{3}$$

The estimates for chain sequences can be immediately applied to Jacobi matrices according to the following well-known fact. The Jacobi matrix J defined as

$$J = \begin{pmatrix} 0 & \sqrt{c_1} & 0 & 0 & \dots \\ \sqrt{c_1} & 0 & \sqrt{c_2} & 0 & \dots \\ 0 & \sqrt{c_2} & 0 & \sqrt{c_3} & \ddots \\ 0 & 0 & \sqrt{c_3} & 0 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$$

is a contraction on the Hilbert space of square summable complex sequences if and only if the numbers $\{c_n\}_{n=1}^{\infty}$ form a chain sequence.

For related results on chain sequences we refer to [1], [5], and [6].

2. The proofs

For a sequence $\{a_n\}$ we define $\Delta a_n = a_{n-1} - a_n$.

Lemma 1. *Fix a positive integer* N. *Assume sequences* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *satisfy* $\lim_{n\to\infty} a_n = 0$, $\Delta a_n \geqslant 0$ *and there exists a sequence* u_n *such that* $u_n > 0$ *and*

$$u_{n-1} - u_n \geqslant \frac{1}{2} b_n + \frac{1}{2} \frac{\Delta a_n}{a_{n-1} a_n} u_{n-1} u_n, \tag{4}$$

for $n \geqslant N$. *Then*,

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{j=n+1}^{\infty} b_j \leqslant 4. \tag{5}$$

Assuming additionally that

$$b_n \geqslant \Delta a_n, \quad n \geqslant N \tag{6}$$

yields $u_n \geqslant a_n$ *for* $n \geqslant N$. *Furthermore if* $u_n = a_n(1 + v_n)$ *then*

$$v_{n-1} - v_n \geqslant \frac{b_n - \Delta a_n}{a_{n-1} + a_n} + \frac{\Delta a_n}{a_{n-1} + a_n} v_{n-1} v_n, \quad n \geqslant N. \tag{7}$$

Proof. We have

$$u_{n-1} - u_n \geqslant \frac{1}{2} b_n.$$

Summing up the terms from n, where $n > N$, to infinity gives

$$u_n \geqslant \frac{1}{2} \sum_{j=n+1}^{\infty} b_j, \quad n \geqslant N. \tag{8}$$

In particular the series $\sum b_j$ is convergent. On the other hand (4) implies

$$\frac{1}{u_n} - \frac{1}{u_{n-1}} \geqslant \frac{1}{2} \left(\frac{1}{a_n} - \frac{1}{a_{n-1}} \right).$$

Thus

$$\frac{1}{u_n} - \frac{1}{u_N} \geqslant \frac{1}{2} \left(\frac{1}{a_n} - \frac{1}{a_N} \right), \quad n \geqslant N. \tag{9}$$

Multiplying inequalities (8) and (9) sidewise and taking the limit gives (5).

Let γ be a constant such that

$$u_n = \gamma a_n + r_n, \quad \text{where } r_n \geqslant 0, \ n \geqslant N. \tag{10}$$

Then by (4) and (6) we obtain

$$\gamma \Delta a_n + r_{n-1} - r_n \geqslant \frac{1}{2} \Delta a_n + \frac{1}{2} \gamma^2 \Delta a_n.$$

Therefore

$$r_{n-1} - r_n \geqslant \frac{1}{2} (\gamma - 1)^2 \Delta a_n.$$

Since $a_n \to 0$ we get $r_n \geqslant \frac{1}{2}(\gamma - 1)^2 a_n$. Let γ be the greatest constant for which (10) holds. The last inequality implies that $\gamma = 1$. Thus $u_n \geqslant a_n$ for $n \geqslant N$.

Let $u_n = a_n(1 + v_n)$. Plugging it into (4) and performing straightforward transformations results in (7). □

Let $\log^{[l]}(x)$ denote the lth composition of the logarithm.

Lemma 2. *Let*

$$a_n^{[0]} = \frac{1}{n}, \tag{11}$$

$$a_n^{[l]} = \frac{1}{\log^{[l]}(n + l)}. \tag{12}$$

Then

$$\frac{\Delta a_n^{[l]}}{a_{n-1}^{[l]} + a_n^{[l]}} \geqslant \frac{1}{2} \frac{\Delta a_n^{[l+1]}}{a_{n-1}^{[l+1]} a_n^{[l+1]}}. \tag{13}$$

Proof. For $l=0$ we have

$$\frac{\Delta a_n^{[0]}}{a_{n-1}^{[0]}+a_n^{[0]}} \geqslant \frac{1}{2}\frac{\Delta a_n^{[0]}}{a_{n-1}^{[0]}} = \frac{1}{2n},$$

and

$$\frac{1}{2}\frac{\Delta a_n^{[1]}}{a_{n-1}^{[1]}a_n^{[1]}} = \frac{1}{2}[\log(n+1)-\log n] \leqslant \frac{1}{2n}.$$

Let $l>0$. By the mean value theorem we have

$$\frac{\Delta a_n^{[l]}}{a_{n-1}^{[l]}+a_n^{[l]}} \geqslant \frac{1}{2}\frac{\Delta a_n^{[l]}}{a_{n-1}^{[l]}} = \frac{1}{2}\frac{\log^{[l]}(n+l)-\log^{[l]}(n+l-1)}{\log^{[l]}(n+l)}$$
$$= \frac{1}{2\xi\log^{[1]}\xi\log^{[2]}\xi\ldots\log^{[l-1]}\xi\log^{[l]}(n+l)},$$

where $n+l-1<\xi<n+l$. On the other hand we have

$$\frac{1}{2}\frac{\Delta a_n^{[l+1]}}{a_{n-1}^{[l+1]}a_n^{[l+1]}} = \frac{1}{2}[\log^{[l+1]}(n+l+1)-\log^{[l+1]}(n+l)]$$
$$= \frac{1}{2\eta\log^{[1]}\eta\log^{[2]}\eta\ldots\log^{[l-1]}\eta\log^{[l]}\eta},$$

where $n+l<\eta<n+l+1$. Thus the lemma follows. □

Lemma 3. *Fix a positive integer k and a sequence ε_n of nonnegative numbers. Let*

$$b_n^{[k+1]} = a_n^{[0]}a_n^{[1]}a_n^{[2]}\ldots a_n^{[k-1]}a_{n-1}^{[k]}\varepsilon_n$$

$$\begin{aligned} b_n^{[l]} = {} & a_n^{[0]}a_n^{[1]}a_n^{[2]}\ldots a_n^{[l-1]}(a_{n-1}^{[l]})^2 + a_n^{[0]}a_n^{[1]}a_n^{[2]}\ldots a_n^{[l-1]}(a_{n-1}^{[l]})^2(a_{n-1}^{[l+1]})^2 \\ & + \cdots + a_n^{[0]}a_n^{[1]}a_n^{[2]}\ldots a_n^{[l-1]}(a_{n-1}^{[l]})^2(a_{n-1}^{[l+1]})^2\ldots(a_{n-1}^{[k]})^2 \\ & + a_n^{[0]}a_n^{[1]}a_n^{[2]}\ldots a_n^{[l-1]}(a_{n-1}^{[l]})^2(a_{n-1}^{[l+1]})^2\ldots(a_{n-1}^{[k]})^2\varepsilon_n, \end{aligned} \tag{14}$$

where $0\leqslant l\leqslant k$. Then for $0\leqslant l\leqslant k$ there holds

$$\frac{b_n^{[l]}-\Delta a_n^{[l]}}{a_{n-1}^{[l]}+a_n^{[l]}} \geqslant \frac{1}{2}b_n^{[l+1]}. \tag{15}$$

Proof. By the mean value theorem there is ξ with $n+l-1<\xi<n+l$ such that

$$\Delta a_n^{[l]} = \frac{1}{\xi\log^{[1]}\xi\ldots\log^{[l-1]}\xi[\log^{[l]}\xi]^2}$$
$$\leqslant a_0^{[0]}a_n^{[1]}\ldots a_n^{[l-1]}(a_{n-1}^{[l]})^2.$$

Therefore

$$\frac{b_n^{[l]} - \Delta a_n^{[l]}}{a_{n-1}^{[l]} + a_n^{[l]}} \geqslant \frac{b_n^{[l]} - \Delta a_n^{[l]}}{2a_{n-1}^{[l]}} \geqslant \frac{1}{2} b_n^{[l+1]}. \qquad \square$$

Now we are in a position to prove Theorem 2. By assumption we have

$$c_{n-1} \geqslant \frac{1}{4} + \frac{1}{16} b_n^{[0]} \tag{16}$$

for $n > N+1$ (see (11), (12), (14)) and

$$c_{n-1} = g_n(1 - g_{n-1}), \quad n \geqslant 2 \tag{17}$$

for a sequence g_n such that $0 \leqslant g_n \leqslant 1$, and $n \geqslant 1$. Since

$$g_n(1 - g_{n-1}) \geqslant \frac{1}{4} \geqslant g_{n-1}(1 - g_{n-1}), \tag{18}$$

the sequence g_n is nondecreasing for $n > N+1$. This and (18) imply that $g_n \leqslant \frac{1}{2}$. Therefore

$$g_n = \frac{1}{2} - u_n^{[0]}, \quad n > N+1 \tag{19}$$

for a sequence of nonnegative numbers $u_n^{[0]}$. Substituting this into (16), (17) and simplifying the terms give

$$\begin{aligned} u_{n-1}^{[0]} - u_n^{[0]} &\geqslant \frac{1}{2} b_n^{[0]} + \frac{1}{2} u_{n-1}^{[0]} u_n^{[0]} \\ &= \frac{1}{2} b_n^{[0]} + \frac{1}{2} \frac{\Delta a_n^{[0]}}{a_{n-1}^{[0]} a_n^{[0]}} u_{n-1}^{[0]} u_n^{[0]}. \end{aligned} \tag{20}$$

Lemmas 1 and 3 imply that $u_n^{[0]}$ is of the form $u_n^{[0]} = a_n^{[0]}(1 + u_n^{[1]})$. Next Lemmas 1–3 combined yield

$$u_{n-1}^{[1]} - u_n^{[1]} \geqslant \frac{1}{2} b_n^{[1]} + \frac{1}{2} \frac{\Delta a_n^{[1]}}{a_{n-1}^{[1]} a_n^{[1]}} u_{n-1}^{[1]} u_n^{[1]}. \tag{21}$$

By iterating this procedure we obtain the existence of a sequence of positive numbers $u_n^{[k+1]}$ such that

$$u_{n-1}^{[k+1]} - u_n^{[k+1]} \geqslant \frac{1}{2} b_n^{[k+1]} + \frac{1}{2} \frac{\Delta a_n^{[k+1]}}{a_{n-1}^{[k+1]} a_n^{[k+1]}} u_{n-1}^{[k+1]} u_n^{[k+1]}. \tag{22}$$

Then Lemma 1 yields

$$\lim_{n \to \infty} \frac{1}{a_n^{[k+1]}} \sum_{j=n+1}^{\infty} b_j^{[k+1]} \leqslant 4. \tag{23}$$

The latter is equivalent to (3). $\square$

Theorem 2 implies Theorem 1 immediately. Indeed, assume $\varepsilon \geqslant 0$. We have

$$\sum_{j=n+1}^{\infty} \frac{1}{j \log^{[1]} j \log^{[2]} \dots \log^{[k]} j} = +\infty .$$

Therefore $\varepsilon = 0$.

Theorem 2 is sharp, i.e. there exist chain sequences a_n satisfying

$$c_n \geqslant \frac{1}{4} + \frac{1}{16n^2} + \frac{1}{16n^2(\log^{[1]} n)^2} + \cdots + \frac{1}{16n^2(\log^{[1]} n \log^{[2]} n \dots \log^{[k]} n)^2},$$

for n big enough. Indeed, let

$$g_n = \frac{n}{2n+1} - \frac{1}{2(2n+1)\log^{[1]} n} - \frac{1}{2(2n+1)\log^{[1]} n \log^{[2]} n} - \cdots - \frac{1}{2(2n+1)\log^{[1]} n \log^{[2]} n \dots \log^{[k]} n}.$$

For n big enough, say $n \geqslant N$, the numbers g_n are well defined and $0 \leqslant g_n \leqslant 1$. Let $c_n = g_n(1 - g_{n-1})$ for $n \geqslant N$ and $c_n = 0$ for $n < N$. Then c_n is a chain sequence. Moreover we have

$$c_n = \frac{1}{4} + \frac{1}{4(4n^2-1)} + \frac{1}{4(4n^2-1)} \sum_{i=1}^{k} \gamma_i,$$

where

$$\begin{aligned}\gamma_i = {} & \frac{2n}{\log^{[1]}(n-1) \dots \log^{[i]}(n-1)} - \frac{2n}{\log^{[1]} n \dots \log^{[i]}(n)} \\ & - \frac{1}{\log^{[1]} n \dots \log^{[i]} n} \sum_{j=1}^{i-1} \frac{1}{\log^{[1]}(n-1) \dots \log^{[j]}(n-1)} \\ & - \frac{1}{\log^{[1]}(n-1) \dots \log^{[i]}(n-1)} \sum_{j=1}^{i-1} \frac{1}{\log^{[1]}(n) \dots \log^{[j]}(n)} \\ & - \frac{1}{\log^{[1]}(n-1) \dots \log^{[i]}(n-1) \log^{[1]}(n) \dots \log^{[i]}(n)}.\end{aligned}$$

This implies

$$\gamma_i = \frac{1}{(\log^{[1]}(n))^2 \dots (\log^{[i]}(n))^2} + o\left(\frac{1}{n}\right).$$

Therefore the sequence c_n satisfies the assumptions of Theorem 2 with $\varepsilon_n = 0$.

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