KACZMARZ ALGORITHM IN HILBERT SPACE AND TIGHT FRAMES

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Abstract. We prove that any tight frame in Hilbert space can be obtained by the Kaczmarz algorithm. An explicit way of constructing this correspondence is given. The uniqueness of the correspondence is determined.

1. Introduction

Let $\{e_n\}_{n=0}^{\infty}$ be a linearly dense sequence of unit vectors in a Hilbert space $\mathcal{H}$. Define

$$\begin{aligned} x_0 &= \langle x, e_0\rangle e_0, \\ x_n &= x_{n-1} + \langle x - x_{n-1}, e_n\rangle e_n. \end{aligned}$$

The formula is called the Kaczmarz algorithm ([2]).

It can be shown that if vectors the g_n are given by the recurrence relation

$$g_0 = e_0, \quad g_n = e_n - \sum_{i=0}^{n-1} \langle e_n, e_i\rangle g_i \tag{1}$$

then g_0 is orthogonal to g_n, for any $n \geq 1$ and

$$x_n = \sum_{i=0}^{n} \langle x, g_i\rangle e_i. \tag{2}$$

By (1) the vectors $\{g_n\}_{n=0}^{\infty}$ are linearly dense in $\mathcal{H}$. Also by definition of the algorithm the vectors $x - x_n$ and e_n are orthogonal to each other. Hence

$$\begin{aligned} \|x\|^2 &= \|x - x_0\|^2 + |\langle x, g_0\rangle|^2, \\ \|x - x_{n-1}\|^2 &= \|x - x_n\|^2 + |\langle x, g_n\rangle|^2, \quad n \geq 1. \end{aligned} \tag{3}$$

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For $n \geq 1$ let S_n denote the finite dimensional operator defined by the rule

$$S_n y = \sum_{j=0}^{n} \langle y, e_j \rangle g_j, \quad y \in \mathcal{H}. \tag{4}$$

Observe that the formulas (1) and (2) can be restated as

$$(I - S_{n-1})e_n = g_n \tag{5}$$

$$(I - S_n^*)x = x - x_n. \tag{6}$$

Moreover by (3) it follows that

$$\|x - x_n\|^2 = \|(I - S_n^*)x\|^2 = \|x\|^2 - \sum_{j=0}^{n} |\langle x, g_j \rangle|^2. \tag{7}$$

In particular

$$\sum_{n=0}^{\infty} |\langle x, g_n \rangle|^2 \leq \|x\|^2, \quad x \in \mathcal{H}. \tag{8}$$

The sequence $\{e_n\}_{n=0}^{\infty}$ is called effective if $x_n \to x$ for any $x \in \mathcal{H}$. By virtue of (7) this is equivalent to $\|x\|^2 = \sum_{n=0}^{\infty} ||\langle x, g_n \rangle|^2$ for any $x \in \mathcal{H}$, which means $\{g_n\}_{n=0}^{\infty}$ is a normalized tight frame. We refer to [3] and [4] for more information on Kaczmarz algorithm.

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2. Bessel sequences

Definition 1. *A sequence of vectors $\{g_n\}_{n=0}^{\infty}$ in a Hilbert space $\mathcal{H}$ will be called a Bessel sequence if (8) holds. The sequence $\{g_n\}_{n=0}^{\infty}$ will be called a normalized Bessel sequence if in addition $\|g_0\| = 1$.*

Let P_n denote the orthogonal projection onto $e_n^{\perp}$ the orthogonal complement to the vector e_n. By [4, (1)] we have

$$I - S_n^* = P_n P_{n-1} \ldots P_0, \tag{9}$$

$$I - S_n = P_0 \ldots P_{n-1} P_n. \tag{10}$$

Theorem 1. *For any normalized Bessel sequence $\{g_n\}_{n=0}^{\infty}$ in a Hilbert space $\mathcal{H}$ there exists a sequence $\{e_n\}_{n=0}^{\infty}$ of unit vectors such that (1) holds. In other words any normalized Bessel sequence can be obtained through Kaczmarz algorithm.*

Proof. We will construct the sequence $\{e_n\}_{n=0}^{\infty}$ recursively. Set $e_0 = g_0$. Assume the unit vectors $e_1, \ldots, e_{N-1}$ have been constructed such that the formula (1) holds for $n = 0, \ldots, N-1$. We want to solve in y the equations

$$(I - S_{N-1})y = g_N, \quad \|y\| = 1. \tag{11}$$

By (10) we have $(I - S_{N-1})e_{N-1} = 0$, i.e. the operator $I - S_{N-1}$ admits nontrivial kernel. Hence the solvability of (11) is equivalent to that of

$$(I - S_{N-1})y = g_N, \quad \|y\| \leq 1. \tag{12}$$

By the Fredholm alternative the equation $(I - S_{N-1})y = g_N$ is solvable if and only if g_N is orthogonal to $\ker(I - S_{N-1}^*)$. We will check that this condition holds. Let $x \in \ker(I - S_{N-1}^*)$. Then by (7) and (8) we have

$$0 = \|(I - S_{N-1}^*)x\|^2 = \|x\|^2 - \sum_{j=0}^{N-1} |\langle x, g_j\rangle|^2 \geq \sum_{j=N}^{\infty} \langle x, g_j\rangle|^2.$$

In particular $\langle x, g_N\rangle = 0$, i.e. $g_N \perp \ker(I - S_{N-1}^*)$.

Let y denote the unique solution to

$$(I - S_{N-1})y = g_N, \qquad y \perp \ker(I - S_{N-1}).$$

The proof will be complete if we show that $\|y\| \leq 1$. Again by the Fredholm alternative we have $y \in \operatorname{Im}(I - S_{N-1}^*)$. Let $y = (I - S_{N-1}^*)x$ for some $x \in \mathcal{H}$. We may assume that $x \perp \ker(I - S_{N-1}^*)$. In particular $\langle x, g_0\rangle = 0$, as (9) yields $g_0 \in \ker(I - S_{N-1}^*)$. By (7) we have

$$\|y\|^2 = \|(I - S_{N-1}^*)x\|^2 = \|x\|^2 - \sum_{j=1}^{N-1} |\langle x, g_j\rangle|^2.$$

One the other hand

$$\|y\|^2 = \langle x, (I - S_{N-1})y\rangle = \langle x, g_N\rangle.$$

Therefore

$$\|y\|^2 - \|y\|^4 = \|x\|^2 - \sum_{j=1}^{N} |\langle x, g_j\rangle|^2 \geq 0,$$

which implies $\|y\| \leq 1$. $\square$

Corollary 1. *For any normalized tight frame $\{g_n\}_{n=0}^{\infty}$ in a Hilbert space $\mathcal{H}$ there exists an effective sequence $\{e_n\}_{n=0}^{\infty}$ of unit vectors such that (1) holds, i.e. any normalized tight frame can be obtained through Kaczmarz algorithm.*

For a sequence $\{e_n\}_{n=0}^\infty$ of unit vectors the normalized Bessel sequence $\{g_n\}_{n=0}^\infty$ is determined uniquely. However a given normalized Bessel sequence may correspond to many sequences of unit vectors due to two reasons. First of all for certain N the dimension of the space $\ker(I - S_{N-1})$ may exceed 1. Secondly, if we fix a unit vector u in $\ker(I - S_{N-1})$ the vector e_N can be defined as $e_N = y + \lambda u$ for any complex number such that $|\lambda|^2 + \|y\|^2 = 1$. In what follows we will indicate properties which guarantee one to one correspondence between $\{e_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$.

Definition 2. *A sequence of unit vectors $\{e_n\}_{n=0}^\infty$ will be called stable if the vectors $\{e_n\}_{n=N}^\infty$ are linearly dense for any N. A normalized Bessel sequence $\{g_n\}_{n=0}^\infty$ will be called stable if the vectors $\{g_0\} \cup \{g_n\}_{n=N}^\infty$ are linearly dense for any N.*

Proposition 1. *Let sequences $\{e_n\}_{n=0}^\infty$ and $\{g_n\}_{n=0}^\infty$ satisfy (1). The sequence $\{g_n\}_{n=0}^\infty$ is stable if and only if $\{e_n\}_{n=0}^\infty$ is stable and $\langle e_n, e_{n+1}\rangle \neq 0$ for any $n \geq 0$.*

Proof. Assume $\{g_n\}_{n=0}^\infty$ is stable. First we will show that the kernel of $I - S_{N-1}$ is one dimensional and thus consists of the multiples of the vector e_{N-1} (see (10). Assume for a contradiction that $\dim\ker(I - S_{N-1}) \geq 2$. By the Fredholm alternative we get $\dim\ker(I - S_{N-1}^*) \geq 2$. Hence there exists a nonzero vector x such that $x \perp g_0$ and $(I - S_{N-1}^*)x = 0$. By (3) we obtain

$$\|x\|^2 = \sum_{n=1}^{N-1} |\langle x, g_n\rangle|^2.$$

This and the condition (8) imply that x is orthogonal to all the vectors g_0 and $\{g_n\}_{n=N}^\infty$, which contradicts the stability assumption.

Assume $\langle e_{N-1}, e_N\rangle = 0$ for some $N \geq 1$. Then by (10) we have e_{N-1}, $e_N \in \ker(I - S_{N-1})$ which is a contradiction as the kernel is one dimensional.

Concerning stability of $\{e_n\}_{n=0}^\infty$ assume a vector y is orthogonal to all the vectors $\{e_n\}_{n=N}^\infty$. In particular y is orthogonal to e_N. Since $\ker(I - S_N) = \mathbb{C}e_N$ by the Fredholm alternative y belongs to $\mathrm{Im}\,(I - S_N^*)$. Let $y = (I - S_N^*)x$ for some $x \in \mathcal{H}$. We may assume that $x \perp g_0$ as $g_0 \in \ker(I - S_N^*)$. By (9), since y is orthogonal to e_n for $n \geq N$, we get $y = (I - S_n^*)x = (I - S_N^*)x$ for $n \geq N$. On the other hand by (2) and (6) we obtain that $\langle x, g_n\rangle = 0$ for $n > N + 1$. Since $x \perp g_0$ by stability assumptions we obtain $x = 0$ and thus $y = 0$.

For the converse implication assume $\{e_n\}_{n=0}^\infty$ is stable and $\langle e_n, e_{n+1}\rangle \neq 0$. By the inequality (see [3])

$$\|x - x_n\| \geq |\langle e_{n-1}, e_n\rangle| \|x - x_{n-1}\|$$

we get that $x - x_n \neq 0$ for any $x \perp e_0$. Since $x - x_n = (I - S_n^*)x$ the kernel of $I - S_n^*$ consists only of the multiples of $e_0 = g_0$.

Let x be orthogonal to $\{g_0\} \cup \{g_n\}_{n \geq N+1}$ for some $N \geq 1$. By (2) we obtain that $x_n = x_N$ for $n \geq N$. By the definition of the Kaczmarz algorithm we get $x - x_N \perp e_n$ for $n \geq N+1$. Now stability of $\{e_n\}_{n=0}^\infty$ implies $x - x_N = 0$. By (6) we obtain $(I - S_N^*)x = 0$. This implies $x = 0$ since the kernel is one dimensional and consists of the multiples of g_0. □

For sequences $\{e_n\}_{n=0}^\infty$ and $\{\sigma_n e_n\}_{n=0}^\infty$, where σ_n are complex numbers of absolute value 1, the Kaczmarz algorithm coincide. Therefore we will restrict our attention to *admissible* sequences of unit vectors $\{e_n\}_{n=0}^\infty$ such that $\langle e_n, e_{n+1}\rangle \geq 0$.

Theorem 2. *Let $\{g_n\}_{n=0}^\infty$ be a stable normalized Bessel sequence. Then there exists a unique admissible sequence $\{e_n\}_{n=0}^\infty$ of unit vectors such that (1) holds. Moreover the sequence $\{e_n\}_{n=0}^\infty$ is stable.*

Proof. The proof will go by induction. The vector e_0 is determined by $e_0 = g_0$. Assume the vectors $e_0, \ldots, e_{N-1}$ were determined uniquely. We have to show that the problem

$$(I - S_{N-1})y = g_N, \quad \|y\| = 1, \quad \langle y, e_{N-1}\rangle \geq 0$$

has the unique solution y.

By the proof of Proposition 2 the kernel of $I - S_{N-1}$ is one dimensional and thus consists of the multiples of the vector e_{N-1}. By the proof of Theorem 1 there exists the unique solution y_N to the problem

$$(I - S_{N-1})y = g_N, \quad y \perp \ker(I - S_{N-1})$$

and $\|y_N\| \leq 1$. Moreover by this proof $\|y_N\| = 1$ if and only if

$$\|x\|^2 - \sum_{j=1}^{N} |\langle x, g_j\rangle|^2 = 0,$$

where $y_N = (I - S_{N-1}^*)x$ and $x \perp \ker(I - S_{N-1}^*)$. This leads to a contradiction because by inequality (8) we get that x is orthogonal to all the vectors g_0 and $\{g_n\}_{n=N}^\infty$. Hence $\|y_N\| < 1$.

At this stage we know that any solution to the equation

$$(I - S_{N-1})y = 0$$

is of the form

$$y = y_N + \lambda e_{N-1}, \quad \lambda \in \mathbb{C}$$

because $\ker(I - S_{N-1}) = \mathbb{C}e_{N-1}$. Since $\|y_N\| < 1$ and $y_N \perp e_{N-1}$ there exists a unique solution y satisfying $\|y\| = 1$ and $\langle y, e_{N-1}\rangle \geq 0$ namely the one corresponding to $\lambda = \sqrt{1 - \|y_N\|^2}$. □

Corollary 2. *Let $\{g_n\}_{n=0}^\infty$ be a stable normalized tight frame. Then there exists a unique admissible effective sequence $\{e_n\}_{n=0}^\infty$ of unit vectors such that (1) holds. Moreover the sequence $\{e_n\}_{n=0}^\infty$ is stable.*

3. Algorithm

The proof of Theorem 1 can also be given by using Gram matrix of the sequence $\{g_n\}_{n=0}^\infty$. This argument can be used for constructing an underlying sequence of unit vectors $\{e_n\}_{n=0}^\infty$. This will be done below.

The next lemma follows from [1, Lemma 3.5.1].

Lemma 1. *The collection $\{g_n\}_{n=0}^\infty$ is a Bessel sequence if and only if the Gram matrix $G = \{\langle g_i, g_j\rangle\}_{i,j=0}^\infty$ corresponds to a contraction operator on $\ell^2(\mathbb{N})$. The sequence $\{g_n\}_{n=0}^\infty$ is a tight frame if and only if it is linearly dense and G corresponds to a projection on $\ell^2(\mathbb{N})$.*

Let $\{e_n\}_{n=0}^\infty$ be a sequence of unit vectors in a Hilbert space $\mathcal{H}$ and let $\{g_n\}_{n=0}^\infty$ be the corresponding normalized Bessel sequence. Let M be the strictly lower triangular part of the Gram matrix of the sequence $\{e_n\}_{n=0}^\infty$ and U strictly lower triangular matrix defined by

$$(I + U)(I + M) = I.$$

By [4] the matrix U is a contraction on the Hilbert space $\ell^2(\mathbb{N})$.

Lemma 2. *For any i, j we have*

$$\langle g_i, g_j\rangle = \langle (I - UU^*)\delta_j, \delta_i\rangle_{\ell^2(\mathbb{N})}$$

Proof. Let

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ldots \\ m_{10} & 0 & 0 & 0 & 0 & \ldots \\ m_{20} & m_{21} & 0 & 0 & 0 & \ldots \\ m_{30} & m_{31} & m_{32} & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ldots \\ c_{10} & 0 & 0 & 0 & 0 & \ldots \\ c_{20} & c_{21} & 0 & 0 & 0 & \ldots \\ c_{30} & c_{31} & c_{32} & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

By [4, (6)] we have

$$g_i = e_i + \sum_{k=0}^{i-1} c_{ik} e_k. \tag{13}$$

Set $c_{nn} = m_{nn} = 1$ and let $1 \le k \le n$. By taking inner product with g_j in (13) we get

$$\langle g_i, g_j\rangle = \sum_{k=0}^{i} c_{ik} \sum_{l=0}^{j} \overline{c}_{jl} \langle e_k, e_l\rangle$$
$$= \langle (I+U)(I+M+M^*)(I+U^*)\delta_j, \delta_i\rangle_{\ell^2(\mathbb{N})}.$$

Taking into account relations between the matrices M and U gives

$$(I+U)(I+M+M^*)(I+U^*) = I - UU^*. \tag{14}$$

The product of these matrices is well defined since U^* leaves the space $F(\mathbb{N}) = \operatorname{span}\{\delta_n \mid n \ge 0\})$ invariant. Therefore

$$\langle g_i, g_j\rangle = \langle (I - UU^*)\delta_j, \delta_i\rangle_{\ell^2(\mathbb{N})}.$$

□

Remark. Lemma 2 can be used to give a shorter and simpler proof of Theorem 1 in [4]. Indeed, by Lemma 1, a linearly dense sequence $\{g_n\}_{n=0}^{\infty}$ constitutes a normalized tight frame if and only if the matrix $G = \{\langle g_i, g_j\rangle\}_{i,j=0}^{\infty}$ is a projection. In view of Lemma 2 the latter is equivalent to U being a partial isometry. Moreover in this case we have

$$\dim \mathcal{H} = \sum_{n=0}^{\infty} \|g_n\|^2 = \operatorname{Tr}(I - UU^*).$$

We are ready now to give an alternative proof of Theorem 1. Let $\{g_i\}_{i=0}^{\infty}$ be a normalized Bessel sequence. By Lemma 1 the matrix $A := I - G$ is positive definite. Moreover $A(0,i) = A(i,0) = 0$ because $\|g_0\| = 1$ and $g_0 \perp g_i$ for $i \ge 1$. Let $\tilde{A}$ denote the truncated matrix obtained from A by removing the first row and the first column. Clearly $\tilde{A}$ corresponds to a positive definite contraction on $\ell^2(\mathbb{N}^+)$. The next lemma is probably known and provides infinite dimensional version of the so called Cholesky decomposition of positive definite matrices.

Lemma 3. *For any positive definite matrix $B = \{b(i,j)\}_{i,j=1}^{\infty}$ there exists a lower triangular matrix $V = \{v(i,j)\}_{i,j=1}^{\infty}$ such that $B = VV^*$.*

Proof. By the well known fact there exists a Hilbert space $\mathcal{M}$ and a linearly dense sequence of vectors $\{h_i\}_{i=1}^{\infty}$ in $\mathcal{M}$ such that

$$b(i,j) = \langle h_i, h_j\rangle.$$

By applying the Gram-Schmid procedure to this sequence we obtain an orthonormal sequence $\{\eta_i\}_{i=1}^{N}$, where $N = \dim \mathcal{M}$, such that $h_i \in$

span $\{\eta_1, \eta_2, \ldots, \eta_i\}$ for $i < N+1$. In particular there are coefficients v_{ik} for $i \geq k$ and $N+1 > k$ for which we have

$$h_i = \sum_{k=1}^{i} v_{ik}\eta_k.$$

Set $v_{ik} = 0$ for $i > k$ and for $k > N$. Then

$$b(i,j) = \langle h_i, h_j \rangle = \sum_{k,l=1}^{\min(i,j)} v_{ik}\overline{v}_{jk} = (VV^*)(i,j).$$

□

By Lemma 3 there is a lower triangular matrix $V = \{v_{ij}\}_{i,j=1}^{\infty}$ such that $\widetilde{A} = VV^*$. Let $U = \{c_{ij}\}_{i,j=0}^{\infty}$ be the strictly lower triangular matrix obtained from V by adding a zero row and a zero column, i.e.

$$c_{ij} = \begin{cases} v_{i-1,j-1} & ij > 0, \\ 0 & ij = 0. \end{cases}$$

In this way we obtain

$$I - G = A = UU^*. \tag{15}$$

Since G corresponds to a contraction on $\ell^2(\mathbb{N})$ so does U. Let $M = \{m_{ij}\}_{i,j=0}^{\infty}$ be the strictly lower triangular matrix determined by $(I+M)(I+U) = I$. Set $m_{ii} = 1$ and define (cf. (1))

$$e_i = \sum_{k=0}^{i} m_{ik} g_k.$$

We claim that e_i are unit vectors and moreover $\langle e_i, e_j \rangle = m_{i,j}$ for $i \geq j$. This will give (1) and thus conclude another proof of Theorem 1. By (15) we have

$$\begin{aligned} \langle e_i, e_j \rangle = \sum_{k=0}^{i}\sum_{l=0}^{j} m_{ik}\overline{m}_{jl}\langle g_k, g_l \rangle &= \sum_{k=0}^{i}\sum_{l=0}^{j} m_{ik}\overline{m}_{jl}(I - UU^*)(k,l) \\ &= (I+M)(I-UU^*)(I+M^*)(i,j) \end{aligned}$$

On the other hand (14) yields

$$(I+M)(I-UU^*)(I+M^*) = I + M + M^*.$$

In particular for $i \geq j$ we obtain

$$\langle e_i, e_j \rangle = \begin{cases} m_{i,j} & i > j, \\ 1 & i = j. \end{cases}$$

This way of proving Theorem 1 provides an algorithm for constructing a sequence of unit vectors $\{e_n\}_{n=0}^{\infty}$ for a given normalized Bessel sequence $\{g_n\}_{n=0}^{\infty}$. Indeed, it suffices to determine an algorithm for proving Lemma 3. When B is strictly positive definite then the solution can be given by the so called Cholesky algorithm. When B is not necessarily strictly positive definite this algorithm fails and we have to find a different way of constructing the decomposition.

We will construct a sequence of indices $\{n_k\}_{k=1}^{N}$ in the following way. Let n_1 be the smallest number i such that $b_{ii} > 0$. If such number does not exist then $B = 0$. Assume $n_1, n_2, \ldots, n_k$ have been constructed in such a way that the determinant

$$\Delta_k = \det(b_{n_i n_j})_{i,j=1}^{k} > 0.$$

Then let n_{k+1} be the smallest number such that

$$\det(b_{n_i n_j})_{i,j=1}^{k+1} > 0.$$

If such number does not exist the procedure terminates and $N = n_k$.

The matrix B gives rise to a positive definite hermitian form on the space $F(\mathbb{N}_+) = \operatorname{span}\{\delta_n \,|\, n \geq 1\}$ by the rule

$$\langle x, y\rangle = \sum_{i,j=1}^{\infty} b(i,j) x_i \overline{y}_j.$$

Lemma 4. *For any n there exist i with $n_i \leq n < n_{i+1}$ and numbers λ_{nk} for $1 \leq k \leq i$, such that*

$$\left\langle \delta_n - \sum_{k=1}^{i} \lambda_{nk}\delta_{n_k}, \delta_m \right\rangle = 0, \quad m \geq 1. \tag{16}$$

Proof. If $n = n_i$ for some i, then statement follows as $\delta_n = \delta_{n_i}$. Otherwise we have $n_i < n < n_{i+1}$ for some i. By plugging in $m = n_1, n_2, \ldots, n_i$ to (16) we obtain a system of linear equations

$$\sum_{k=1}^{i} \lambda_{nk} b_{n_k n_l} = b_{n n_l}, \quad l = 1, 2, \ldots, i.$$

The main determinant of this system is Δ_i. Therefore the system has the unique solution $\lambda_{n,1}, \ldots, \lambda_{n,i}$. By definition of n_{i+1} we have

$$\begin{vmatrix} b_{n_1 n_1} & b_{n_1 n_2} & \cdots & b_{n_1 n_i} & b_{n_1 n} \\ b_{n_2 n_1} & b_{n_2 n_2} & \cdots & b_{n_2 n_i} & b_{n_2 n} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ b_{n_i n_1} & b_{n_i n_2} & \cdots & b_{n_i n_i} & b_{n_i n} \\ b_{n n_1} & b_{n n_2} & \cdots & b_{n n_i} & b_{nn} \end{vmatrix} = 0.$$

As $\Delta_i > 0$ the first i rows of this matrix are linearly independent. Therefore the last row of this matrix is a linear combination of the first i. The coefficients must coincide with $\lambda_{n1}, \ldots, \lambda_{ni}$. In particular considering the last entry of the rows gives

$$\sum_{k=1}^{i} \lambda_{nk} b_{n_k n} = b_{nn}.$$

This is equivalent to (16) with $m = n$.

Since (16) is valid for $m = n, n_1, \ldots, n_i$ then

$$\left\langle \delta_n - \sum_{k=1}^{i} \lambda_{nk} \delta_{n_k}, \delta_n - \sum_{k=1}^{i} \lambda_{nk} \delta_{n_k} \right\rangle = 0.$$

By Schwarz inequality this implies (16) for any m. □

Define the sequence of vectors $\{\eta_i\}_{i=1}^N$ by the formula

$$\eta_1 = \frac{1}{\sqrt{\Delta_1}} \delta_{n_1}, \qquad \eta_i = \frac{1}{\sqrt{\Delta_{i-1}\Delta_i}} \begin{vmatrix} b_{n_1 n_1} & b_{n_1 n_2} & \cdots & b_{n_1 n_i} \\ b_{n_2 n_1} & b_{n_2 n_2} & \cdots & b_{n_2 n_i} \\ \vdots & \vdots & \ldots & \vdots \\ b_{n_{i-1} n_1} & b_{n_{i-1} n_2} & \cdots & b_{n_{i-1} n_i} \\ \delta_{n_1} & \delta_{n_2} & \cdots & \delta_{n_i} \end{vmatrix}$$

It can be checked easily that

$$\langle \eta_i, \eta_j \rangle = \delta_i^j. \tag{17}$$

Obviously from the definition we have

$$\eta_i = \frac{\sqrt{\Delta_i}}{\sqrt{\Delta_{i-1}}} \delta_{n_i} + \sum_{k=1}^{i-1} \alpha_{ik} \delta_{n_k}$$

for some explicitly given coefficients α_{ik}. Therefore

$$\delta_{n_i} = \sum_{k=1}^{i} \beta_{ik} \eta_k, \tag{18}$$

for some coefficients β_{ik}. By (16) and (18) we get that for any n there exist i with $n_i \le n < n_{i+1}$ and numbers v_{nk} for $1 \le k \le i$, such that

$$\left\langle \delta_n - \sum_{k=1}^{i} v_{nk} \eta_k, \delta_m \right\rangle = 0, \quad m \ge 1.$$

Setting $v_{nk} = 0$ for $i < k \le n$ gives

$$\left\langle \delta_n - \sum_{k=1}^{n} v_{nk} \eta_k, \delta_m \right\rangle = 0, \quad m \ge 1. \tag{19}$$

Therefore by (16) and (19) we have

$$0 = \left\langle \sum_{k=1}^{n} v_{n,k}\eta_k \,,\, \delta_m - \sum_{k=1}^{n} v_{mk}\eta_k \right\rangle = \langle \delta_n, \delta_m \rangle - \sum_{k=0}^{\min(n,m)} v_{nk}\overline{v}_{mk}$$

$$= b_{nm} - \sum_{k=0}^{\min(n,m)} v_{nk}\overline{v}_{mk}.$$

Therefore $B = VV^*$ where

$$V = \begin{pmatrix} v_{11} & 0 & 0 & 0 & 0 & \dots \\ v_{21} & v_{22} & 0 & 0 & 0 & \dots \\ v_{31} & v_{32} & v_{33} & 0 & 0 & \dots \\ v_{41} & v_{42} & v_{43} & v_{44} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

By analyzing the entire construction we may conclude the coefficients v_{nk} can be computed in an algorithmic way.

4. Equivalent sequences

Any sequence of unit vectors $\{e_n\}_{n=0}^{\infty}$ leads by Kaczmarz algorithm to a normalized Bessel sequence $\{g_n\}_{n=0}^{\infty}$.

Definition 3. *Two normalized Bessel sequences $\{g_n\}_{n=0}^{\infty}$ and $\{g'_n\}_{n=0}^{\infty}$ will be called equivalent if there is a unitary operator V such that $g'_n = Vg_n$ for $n \geq 0$. Similarly two sequences $\{e_n\}_{n=0}^{\infty}$ and $\{e'_n\}_{n=0}^{\infty}$ of unit vectors will be called equivalent if there is a unitary operator V such that $e'_n = Ve_n$ for $n \geq 0$.*

It is easy to see that if the sequences $\{e_n\}_{n=0}^{\infty}$ and $\{e'_n\}_{n=0}^{\infty}$ are equivalent so are the corresponding sequences of normalized Bessel sequences $\{g_n\}_{n=0}^{\infty}$ and $\{g'_n\}_{n=0}^{\infty}$, with the same unitary operator V. The converse is not true, as the normalized Bessel sequences do not correspond to sequences of unit vectors in one-to-one fashion. Nevertheless by Lemma 1, the equivalence relation between normalized Bessel sequences can be described in terms of Gram matrices of the corresponding sequences of unit vectors.

Assume sequences of unit vectors $\{e_n\}_{n=0}^{\infty}$ and $\{e'_n\}_{n=0}^{\infty}$ are associated with normalized Bessel sequences $\{g_n\}_{n=0}^{\infty}$ and $\{e'_g\}_{n=0}^{\infty}$, respectively. Let M and M' be the strictly lower triangular part of the Gram matrices of sequences $\{e_n\}_{n=0}^{\infty}$ and $\{e'_n\}_{n=0}^{\infty}$, respectively. Let U and U' be strictly lower triangular matrices defined by

$$(I + U)(I + M) = (I + U')(I + M') = I.$$

By [4] the matrices U and U' are contractions on the Hilbert space $\ell^2(\mathbb{N})$

Corollary 3. *The sequences $\{g_n\}_{n=0}^\infty$ and $\{g'_n\}_{n=0}^\infty$ are equivalent if and only if $UU^* = U'U'^*$.*

Proof. By Lemma 1 we get that $UU^* = U'U'^*$ if and only if $\langle g_i, g_j\rangle = \langle g'_i, g'_j\rangle$ for any $i, j \geq 0$. Obviously the latter, along with the linear density of vectors $\{g_n\}_{n=0}^\infty$ and $\{g'_n\}_{n=0}^\infty$, is equivalent to the existence a unitary operator U such that $g'_i = Ug_i$. □

Remark. It would be of interest to determine when two sequences of unit vectors $\{e_n\}_{n=0}^\infty$ and $\{e'_n\}_{n=0}^\infty$ lead to the same normalized Bessel sequence $\{g_n\}_{n=0}^\infty$.

References

[1] O.Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.

[2] S. Kaczmarz, Approximate solution of systems of linear equations, Bull. Acad. Polon. Sci. Lett. A, 35 (1937), 355–357 (in German); English transl.: Internat. J. Control 57(6) (1993), 1269–1271.

[3] S. Kwapień, J. Mycielski, On the Kaczmarz algorithm of approximation in infinite-dimensional spaces, Studia Math. 148 (2001), 75–86.

[4] R. Haller, R. Szwarc,Kaczmarz algorithm in Hilbert space, Studia Math. 169.2 (2005).

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