PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 87, Number 4, April 1983

CONVOLUTION OPERATORS OF WEAK TYPE (2, 2) WHICH ARE NOT OF STRONG TYPE (2, 2)

RYSZARD SZWARC

ABSTRACT. It is well known that if G is a locally compact and amenable group then the Banach spaces of operators of weak type (2, 2) and of strong type (2, 2) commuting with the right translations on G are the same. In contrast we show that if G is a nonabelian free group then there exists an operator of weak type (2, 2) commuting with the right translations on G which is not of strong type (2, 2).

Introduction. Let G be a locally compact group. We let $L_2^2(G)$, $L_2^{2,\infty}(G)$ be the Banach spaces of convolution operators of respectively strong type (2, 2), weak type (2, 2). By convolution operator we mean a linear operator from one space of functions on G that is closed under right translations to another such space, that commutes with right translations. We always have: $L_2^2(G) \subset L_2^{2,\infty}(G)$. M. Cowling proved that if G is an amenable group then these two spaces coincide [**2**, Theorem 5.4]. In this note we show that if G is a nonabelian free group then $L_2^2(G) \neq L_2^{2,\infty}(G)$.

Preliminaries. Let G be a locally compact group and m a fixed left invariant Haar measure on G. For a function f in $L^1(G) + L^\infty(G)$ let f^* denote the unique right continuous positive function on $\mathbf{R}^+$ which is of the same distribution as the function $|f|$. By f^{**} we denote the function defined on $\mathbf{R}^+$ by

$$f^{**}(t) = \left(\frac{1}{t}\right)\int_0^t f^*(s)\,ds.$$

The Lorentz space $L^{p,q}(G)$ is the Banach space of m-measurable functions f on G for which

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty \left(t^{1/p}f^{**}(t)\right)^q dt/t\right)^{1/q}, & 1 < p \leq \infty,\ 1 \leq q < \infty, \\ \sup_{t>0}\left(t^{1/p}f^{**}(t)\right), & 1 < p < \infty,\ q = \infty, \end{cases}$$

is finite. Up to equivalence of norms $L^{p,p}(G) = L^p(G)$. If $p > 1$ then the conjugate Banach space of $L^{p,1}$ is $L^{p',\infty}$, where $1/p + 1/p' = 1$. The duality is given by the formula

$$(*) \qquad (f, g) = \int_G f(x)g(x)\,dm(x), \qquad f \in L^{p,1},\ g \in L^{p',\infty}.$$

Received by the editors March 29, 1982.
1980 *Mathematics Subject Classification*. Primary 43A15, 22D15.
Key words and phrases. Free group, convolution operator, strong type, weak type.

0002-9939/82/0000-0757/$01.50

We need also the following

PROPOSITION [4, p. 259]. *Suppose T is a linear operator which maps characteristic functions χ_E, $m(E) < \infty$, into a Banach space B and $\|T\chi_E\| \leq c\|\chi_E\|_{p,1}$ where c is independent of E. Then there exists a unique linear extension of T to a continuous map of $L^{p,1}$ into B and $\|Tf\| \leq c_1 c\|f\|_{p,1}$, where c_1 depends only on p.*

A linear operator T: $L^p(G) \to L^{p,\infty}(G)$ is called of weak type (p, p) if it is bounded. In contrast, by definition T: $L^p(G) \to L^p(G)$ is bounded if and only if T is of strong type (p, p). The spaces of operators commuting with the right translations on G which are of weak type (p, p), of strong type (p, p) we denote respectively $L_p^{p,\infty}(G)$, $L_p^p(G)$. Generally, following [2], if T is a continuous linear operator from $L^{p,q}$ to $L^{p,r}$ $(1 < p < \infty,\ 1 \leq q, r \leq \infty)$ commuting with the right translations we will say that T belongs to $L_{p,q}^{p,r}$.

The case of the free groups. Let G be a free group on r generators, $r \geq 2$. For $x \in G$ let $|x|$ denote the usual length of the word x that is the number of factors in generators or their inverses which are needed to write x as a reduced word. Let χ_n denote the characteristic function of the words of length n. Let us regard χ_n as the element of $L_2^{2,\infty}(G)$ or $L_2^2(G)$ acting by left convolution. By [1, 5]:

$$(**) \qquad \|\chi_n\|_{L_2^2} = 2r(2r-1)^{-1}\left[1 + n(r-1)r^{-1}\right](2r-1)^{n/2}.$$

LEMMA. *$\|\chi_n\|_{L_2^{2,\infty}} \leq b[3(n+1)]^{1/2}(2r-1)^{n/2}$, where b is a constant independent of n.*

PROOF. First let us compute the norm $\|\chi_n\|_{L_{2,1}^2}$. By Proposition it suffices to act only on characteristic functions. Let g be the characteristic function of a finite set $E \subset G$. Without loss of generality we may assume that g is supported by the words longer than $2n$, because we may take $g * \delta_x = \chi_{Ex}$ instead of g, where x is a sufficiently long word. Let $h = \chi_n * g$, $h_m = (\chi_n * g)\chi_m$ for $m = 0, 1, \ldots,$ and $g_k = g\chi_k$ for $k = 0, 1, \ldots$. Then

$$\|h_m\|_2^2 = \left\| \sum_{k=0}^{\infty} (\chi_n * g_k)\chi_m \right\|_2^2.$$

It is easy to see that $h_m = 0$ for $m \leq n$ and $(\chi_n * g_k)\chi_m = 0$ for all k except $k = m-n, m-n+2, \ldots, m+n$. Since that for $m > n$:

$$\begin{aligned}\|h_m\|_2^2 &= \left\| \sum_{k=0}^{n} (\chi_n * g_{m-n+2k})\chi_m \right\|_2^2 \\ &= \sum_{k=0}^{n} \|(\chi_n * g_{m-n+2k})\chi_m\|_2^2 + \sum_{\substack{l,k=0 \\ k<l}}^{n} \left\langle (\chi_n * g_{m-n+2k})\chi_m, (\chi_n * g_{m-n+2l})\chi_m \right\rangle.\end{aligned}$$

By a simple computation (or see [1]) we have

$$(\chi_n * g_{m-n+2k})\chi_m \leq (\chi_n * \chi_{m-n+2k})\chi_m \leq (2r-1)^k \chi_m$$

and

$$\langle \chi_m, (\chi_n * g_{m-n+2l})\chi_m \rangle = \langle \chi_m, \chi_n * g_{m-n+2l} \rangle = \langle \chi_n * \chi_m, g_{m-n+2l} \rangle$$
$$= \langle (\chi_n * \chi_m) g_{m-n+2l}, g_{m-n+2l} \rangle \leqslant (2r-1)^{n-l} \| g_{m-n+2l} \|_2^2.$$

So we obtain that

$$\langle (\chi_n * g_{m-n+2k})\chi_m, (\chi_n * g_{m-n+2l})\chi_m \rangle \leqslant (2r-1)^{n+k-l} \| g_{m-n+2l} \|_2^2.$$

Hence

$$\| h_m \|_2^2 \leqslant \sum_{k=0}^{n} (2r-1)^n \| g_{m-n+2k} \|_2^2 + 2 \sum_{\substack{k,l=0 \\ k<l}}^{n} (2r-1)^{n+k-l} \| g_{m-n+2l} \|_2^2$$

$$\sum_{\substack{k,l=0 \\ k<l}}^{n} (2r-1)^{n+k-l} \| g_{m-n+2l} \|_2^2$$
$$= (2r-1)^n \sum_{l>0}^{n} \| g_{m-n+2l} \|_2^2 \sum_{k=0}^{l-1} (2r-1)^{k-l}$$
$$\leqslant (2r-1)^n \sum_{l=0}^{n} \| g_{m-n+2l} \|_2^2.$$

The last inequality follows from the fact that $2r-1 \geqslant 3$ when $r \geqslant 2$. Now we conclude that

$$\| h_m \|_2^2 \leqslant 3(2r-1)^n \sum_{k=0}^{n} \| g_{m-n+2k} \|_2^2.$$

Furthermore

$$\| h \|_2^2 = \sum_{m=n+1}^{\infty} \| h_m \|_2^2 \leqslant 3(2r-1)^n \sum_{m=n+1}^{\infty} \sum_{k=0}^{n} \| g_{m-n+2k} \|_2^2$$
$$= 3(2r-1)^n \sum_{k=0}^{n} \sum_{m=n+1}^{\infty} \| g_{m-n+2k} \|_2^2 = 3(n+1)(2r-1)^n \| g \|_2^2.$$

Hence $\| \chi_n * g \|_2 \cdot \| g \|_2^{-1} \leqslant [3(n+1)]^{1/2}(2r-1)^{n/2}$. By Proposition we obtain that $\| \chi_n \|_{L_2^{2,1}} \leqslant c_1 [3(n+1)]^{1/2} (2r-1)^{n/2}$ and by duality $(*)$

$$\| \chi_n \|_{L_2^{2,\infty}} \leqslant c_1 c_2 [3(n+1)]^{1/2} (2r-1)^{n/2}.$$

THEOREM. *Let G be a free group on r ($r \geqslant 2$) generators. Then the Banach spaces of the operators of strong type $(2,2)$ and of weak type $(2,2)$ commuting with the right translations on G are not the same.*

PROOF. By the preceding Lemma and $(**)$ we have that $\| \chi_n \|_{L_2^{2,\infty}} \cdot \| \chi_n \|_{L_2^2}^{-1} \to 0$, so the norms $\| \cdot \|_{L_2^{2,\infty}}$ and $\| \cdot \|_{L_2^2}$ are not equivalent on $L_2^2(G)$. Hence the Banach spaces $L_2^2(G)$ and $L_2^{2,\infty}(G)$ must be different.

REMARKS. Every nonabelian free group contains the free group on two generators as the subgroup. So the foregoing theorem holds for all nonabelian free groups. In this case by standard application of Banach-Steinhaus theorem and closed graph theorem we may easily attain that $L_2^{2,\infty}(G)$ is not an algebra under convolution.

ACKNOWLEDGEMENT. I would like to thank Professor Marek Bożejko for discussions and many valuable suggestions.

REFERENCES

1. J. M. Cohen, *Operator norms on free groups*, Boll. Un. Mat. Ital. (to appear).

2. M. Cowling, *Some applications of Grothendieck's theory of topological tensor product in harmonic analysis*, Math. Ann. **232** (1978), 273–285.

3. U. Haagerup, *An example of a non-nuclear C^*-algebra which has the metric approximation property*, Invent. Math. **50** (1979), 279–293.

4 R. Hunt, *On $L(p, q)$ spaces*, Enseigenement Math. **12** (1966), 249–276.

5. T. Pytlik, *Radial functions on free groups and decomposition of the regular representations into irreducible components*, J. Reine Angew. Math. **326** (1981), 124–135.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCŁAW, PL. GRUNWALDZKI 2/4, PL 50-384 WROCŁAW, POLAND