Weak type translation invariant operators on groups and amenability

Let G be a discrete group. Consider a symmetric probability measure μ on G, i.e.

$$\mu = \sum_{x \in G} \mu(x) \delta_x, \qquad \mu(x) \geq 0,$$

$$\sum_{x \in G} \mu(x) = 1, \qquad \mu(x^{-1}) = \mu(x).$$

The left convolution operator $\lambda(\mu)$ with μ is bounded on $\ell^2(G)$ and

$$\|\lambda(\mu)(f)\|_2 = \|\mu * f\|_2 \leq \|f\|_2, \quad f \in \ell^2(G).$$

$$\begin{aligned}\|\mu * f\|_2 &= \left\|\sum_{x\in G} \mu(x)[\delta_x * f]\right\|_2 \\ &\leq \sum_{x\in G} \mu(x)\|\delta_x * f\|_2 = \|f\|_2.\end{aligned}$$

Thus $\|\lambda(\mu)\|_{2\to 2} \leq 1$.

Kesten (1959) showed that a discrete group G is amenable iff for any symmetric probability measure μ on G we have $\|\lambda(\mu)\|_{2\to 2} = 1$. He showed that G is amenable if condition is satisfied for one measure μ such that $\operatorname{supp}\mu$ generates G algebraically. In particular let G be generated by $g_1, g_2, \ldots, g_k$ and $\mu = \frac{1}{2k}\sum_{i=1}^{k}(\delta_{g_i} + \delta_{g_i^{-1}})$. Then G is amenable iff $\|\lambda(\mu)\|_{2\to 2} = 1$.

The group G is called amenable if there exists a linear functional m on $\ell^\infty_{\mathbb{R}}(G)$ such that

(1) $\inf_{x \in G} f(x) \leq m(f) \leq \sup_{x \in G} f(x)$,

(2) $m({}_x f) = m(f)$, where ${}_x f(y) = f(x^{-1}y)$.

m is called a left invariant mean. Then the functional $M(f) = m(m(f_x))$ satisfies (1), (2) and is also right invariant, where $f_x(y) = f(yx)$.

Følner condition*

For any number $\varepsilon > 0$ and any finite set $K \subset G$ there exists a finite set $N \subset G$ such that

$$|xN \triangle N| < \varepsilon |N|, \qquad x \in K.$$

Hence N is almost K invariant.

Example. Let $G = \mathbb{Z}$, $K = [-k, k]$ and $N = [-n, n]$. Then

$$|(x + N) \triangle N| \leq 2k < \frac{k}{n}|N|.$$

For $\varepsilon > 0$ take $n \geq \frac{k}{\varepsilon}$.

*I learnt character ø from Christina Kuttler

G is amenable iff the Følner condition holds.

One direction is easy. Assume G satisfies the Følner condition and the group G is countable. Then $G = \bigcup_{n=1}^{\infty} K_n$, $K_n \subset K_{n+1}$. For $\varepsilon = \frac{1}{n}$ let N_n denote the corresponding almost K_n invariant set. Define

$$m_n(f) = \frac{1}{|N_n|} \sum_{x \in N_n} f(x).$$

Then any accumulation point of the functionals m_n leads to left invariant mean.

Assume that G is amenable. Let μ be a probability measure with finite support K. For $\varepsilon = \eta^2 > 0$ choose N with respect to ε and K. Then

$$\|\mu * \chi_N - \chi_N\|_2 = \left\| \sum_{x \in K} \mu(x)[\chi_{xN} - \chi_N] \right\|_2$$
$$\leq \sum_{x \in K} \mu(x) \|\chi_{xN} - \chi_N\|_2 = \sum_{x \in K} \mu(x) \|\chi_{xN \triangle N}\|_2$$
$$= \sum_{x \in K} \mu(x) |xN \triangle N|^{1/2} \leq \eta |N|^{1/2} = \eta \|\chi_N\|_2.$$

Therefore

$$\|\mu * \chi_N\|_2 \geq (1-\eta)\|\chi_N\|_2$$

which implies that $\|\lambda(\mu)\|_{2\to 2} \geq 1-\eta$, hence $\|\lambda(\mu)\|_{2\to 2} = 1$. Observe that we showed that if the group is amenable then

$$1 = \|\lambda(\mu)\|_{2\to 2} = \sup_{N\subset G} \frac{\|\mu * \chi_N\|_2}{\|\chi_N\|_2},$$

i.e. the operator norm is attained at characteristic functions of finite sets.

If the group G is amenable the same holds (with the same proof) for any $1 \le p \le +\infty$, i.e.

$$1 = \|\lambda(\mu)\|_{p\to p} = \sup_{N\subset G} \frac{\|\mu * \chi_N\|_p}{\|\chi_N\|_p},$$

which means the operator norm is also attained at characteristic functions.

Consider a general σ-finite measure space (Ω, ω) and $1 < p < +\infty$. For $f \in L^p(\Omega, \omega)$ and $t > 0$ we have

$$t^p \, \omega\{x : |f(x)| > t\} \leq \int_\Omega |f(x)|^p d\omega(x).$$

Functions for which the left hand side is bounded form a linear space

$$L^{p,\infty}(\Omega, \omega) = \left\{ f : \sup_{t>0} t^p \omega\{x : |f(x)| > t\} < +\infty \right\}.$$

called the weak L^p space. This space contains $L^p(\Omega, \omega)$.

For $p' = p/(p-1)$ the predual of $L^{p',\infty}(\Omega,\omega)$ with respect to the standard inner product is denoted by $L^{p,1}(\Omega,\omega)$. We have

$$L^{p,1}(\Omega,\omega) \subset L^{p}(\Omega,\omega) \subset L^{p,\infty}(\Omega,\omega).$$

For $p > 1$ these spaces are normed. The spaces $L^{1,1}(\Omega,\omega)$ and $L^{1,\infty}(\Omega,\omega)$ can also be defined but they are not normed.

The bounded linear operator $T : L^p(\Omega, \omega) \to L^{p,\infty}(\Omega, \omega)$ is called of weak type $(p.p)$. Any operator mapping L^p into itself is called of strong type (p, p). We will use the following fact.
The linear operator T is bounded from $L^{p,1}$ into L^p if and only if

$$\|T\|_{(p,1)\to p} = \sup_{E\subset\Omega} \frac{\|T\chi_E\|_p}{\|\chi_E\|_p} < +\infty.$$

$L^{p,q}$ spaces have been introduced by Lorentz (see J. Bergh, J. Löfström, Interpolation Spaces). By duality and by symmetry of μ we have

$$\begin{aligned} \|\lambda(\mu)\|_{p\to p} &= \|\lambda(\mu)\|_{p'\to p'}, \\ \|\lambda(\mu)\|_{p\to(p,\infty)} &= \|\lambda(\mu)\|_{(p',1)\to p'}, \end{aligned}$$

for any group G.

It is convenient to switch to the dual space because we have easy expressions for the operator norms.

In case the group G is discrete and amenable we showed that

$$\|\lambda(\mu)\|_{p'\to p'} = \|\lambda(\mu)\|_{(p',1)\to p'} = 1.$$

Hence for these groups convolution operators with **nonnegative functions** of weak type (p,p) and of strong type (p,p) coincide. The same is true for general amenable groups.

Example. Consider the Hilbert transform

$$(Hf)(x) = \mathrm{pv}\,\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(x-y)}{y}\,dy$$
$$= \lim_{\delta\to 0^+} \frac{1}{\pi}\int_{|y|>\delta} \frac{f(x-y)}{y}\,dy,$$

for $f \in C_c(\mathbb{R})$. It can be shown that $\widehat{Hf}(\zeta) = -i\,\mathrm{sgn}\,(\zeta)\,\widehat{f}(\zeta)$. Hence H is a an isometry on $L^2(\mathbb{R})$. The operator H is not bounded on L^1, because $1/x$ is not absolutely integrable.

But H is of weak type (1,1), i.e. it maps L^1 into $L^{1,\infty}$. By Marcinkiewicz interpolation theorem H is bounded on L^p for $1 < p < 2$. By duality H is also bounded on L^p for $p > 2$. Of course this operator commutes with translations.

The operator $I + iH$ restricts Fourier transform to positive half axis and is bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. Hence the operator restricting Fourier transform to the interval $[-1, 1]$ is also bounded.

Similar result for $\mathbb{R}^n$, $n \geq 2$, is not true due to the famous result of Ch. Fefferman (1971) that restriction of the Fourier transform to the unit ball is bounded only on $L^2(\mathbb{R}^n)$. In order to make it bounded the multiplier has to be smoothed suitably

$$\widehat{M_\delta f}(\xi) = (1 - |\xi|^2)_+^\delta \hat{f}(\xi).$$

The range of p for which M_δ is bounded depends on δ.

M. Zafran (1975) showed that for $G = \mathbb{R}, \mathbb{T}, \mathbb{Z}$ and $1 < p < 2$ there are translation invariant operators of weak type (p,p) which are not bounded on $L^p(G)$, i.e. are not of strong type (p,p). By amenability these operators cannot be convolutions with nonnegative distribution. M. Cowling and J. Fournier (1976) extended this result on all infinite groups. Cowling (1979) showed that if the group G is amenable then the weak type (2,2) coincides with strong type (2,2) for translation invariant operators.

Three problems remained: determine if weak and strong type (p, p) coincide for translation invariant operators in

1. the case $p > 2$ for any infinite group, even for $G = \mathbb{R}, \mathbb{T}, \mathbb{Z}$,
2. the case $p = 2$ for nonamenable groups,
3. the case $1 < p < 2$ for nonamenable groups and convolution with nonnegative distributions (by Cowling and Fournier the notions are different if we do not impose nonnegativity).

Let $\mathbb{F}_k$ be a free group on $k \geq 2$ generators. (Sz1983) There are convolution operators of weak type (2,2) which are not of strong type (2,2).

(Sz1983) There are convolution operators of weak type (p, p) which are not of strong type (p, p) for $p > 2$.

Conjecture. The group G is amenable iff the weak and strong type (2,2) coincide for translation invariant operators.

Let $\mathbb{F}_k = \mathrm{gp}\{g_1, g_2, \ldots, g_k\}$. The group consists of reduced words in generators and their inverses. This representation is unique. The number of letters in reduced form defines length function on $\mathbb{F}_k$. Let χ_n denote the characteristic function of words of length n. There are $2k(2k-1)^{n-1}$ such words. as we have $2k$ choices for the first letter and $2k-1$ choices for every consecutive one. J. Cohen (1982) showed that

$$\|\lambda(\chi_n)\|_{2\to 2} \approx n(2k-1)^{n/2} \approx n\|\chi_n\|_2.$$

(Sz1983)

$$\|\lambda(\chi_n)\|_{2\to(2,\infty)} \approx \sqrt{n}(2k-1)^{n/2} \approx \sqrt{n}\|\chi_n\|_2.$$

Hence the norms are not equivalent, i.e. the corresponding spaces must be different.

Let's turn to the case $p > 2$. By duality we are interested in comparing the norms $\|\lambda(\mu)\|_{p\to p}$ and $\|\lambda(\mu\|_{(p,1)\to p}$ for $1 < p < 2$.

For $1 < p < 2$, T. Pytlik (1982) showed that

$$\|\lambda(\chi_n)\|_{p\to p} \approx \|\chi_n\|_p.$$

But $\|\chi_n * \delta_e\|_p = \|\chi_n\|_p \|\delta_e\|_p$, hence

$$\|\lambda(\chi_n)\|_{(p,1)\to p} \geq \|\chi_n\|_p.$$

Therefore

$$\|\lambda(\chi_n)\|_{p\to p} \approx \|\lambda(\chi_n)\|_{(p,1)\to p}.$$

Pytlik showed also that for $f_n \geq 0$ we have

$$\left\| \sum_{n=0}^{\infty} f_n \lambda(\chi_n) \right\|_{p\to p} \approx \sum_{n=0}^{\infty} f_n \|\chi_n\|_p \approx \left\| \sum_{n=0}^{\infty} f_n \chi_n \right\|_{p,1} .$$

Basing on this and using interpolation machinery one can show that (Sz1983) we have

$$\left\| \sum_{n=0}^{\infty} f_n \lambda(\chi_n) \right\|_{(p,1)\to p} \approx \left(\sum_{n=0}^{\infty} |f_n|^p \|\chi_n\|_p^p \right)^{1/p} .$$

By comparing these two results one can see that the spaces of convolution operators from L^p into itself and from $L^{p,1}$ into L^p do not coincide for $1 < p < 2$. By duality, for any $p > 2$, there exist convolution operators, with non-negative function, of weak type (p, p) which are not of strong type (p, p).

(Sz 2004.11.27) For $1 < p < 2$ we have

$$\left\| \sum_{n=0}^{\infty} f_n \lambda(\chi_n) \right\|_{p \to (p,\infty)} \approx \left(\sum_{n=0}^{\infty} |f_n|^p \|\chi_n\|_p^p \right)^{1/p} .$$

By Pytlik result and duality we have for $f_n \geq 0$

$$\left\| \sum_{n=0}^{\infty} f_n \lambda(\chi_n) \right\|_{p \to p} \approx \sum_{n=0}^{\infty} f_n \|\chi_n\|_p$$

Hence strong and weak type (p,p) do not coincide for convolution operators with nonnegative functions.

Proofs. Let $1 < p < p_0 < 2$. Functions of the form $f = \sum_{n=0}^{\infty} f_n \chi_n$ will be called radial. Let $E \subset \mathbb{F}_k$ be finite. Consider right hand side convolution operators

$$\begin{aligned} \varrho(\chi_E) :\ & L^1_r(\mathbb{F}_k) \to L^1(\mathbb{F}_k), \\ \varrho(\chi_E) :\ & L^{p_0,1}_r(\mathbb{F}_k) \to L^{p_0}(\mathbb{F}_k) \end{aligned}$$

For radial function f we have

$$\|f * \chi_E\|_1 \le \|\chi_E\|_1 \|f\|_1$$

$$\begin{aligned}\|f * \chi_E\|_{p_0} &\le \sum_{n=0}^{\infty} |f_n| \, \|\chi_n * \chi_E\|_{p_0} \\ &\le \|\chi_E\|_{p_0} \sum_{n=0}^{\infty} |f_n| \, \|\chi_n\|_{p_0} \le C \|\chi_E\|_{p_0} \|f\|_{p_0,1}.\end{aligned}$$

By Calderón interpolation theorem we get

$$\varrho(\chi_E): \; L^p_r(\mathbb{F}_k) \to L^p(\mathbb{F}_k)$$

$$\|f * \chi_E\|_p \le C(p)\|\chi_E\|_1^{\theta}\|\chi_E\|_{p_0}^{1-\theta}\|f\|_p$$

where

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{p_0}, \quad 0 < \theta < 1.$$

Hence

$$\|f * \chi_E\|_p \le C(p)\|f\|_p\|\chi_E\|_p.$$

This implies that for radial function f we have

$$\|\lambda(f)\|_{(p,1)\to p} \leq C(p)\|f\|_p.$$

On the other hand

$$\|\lambda(f)\|_{(p,1)\to p} \geq \|f * \delta_e\|_p = \|f\|_p.$$

Therefore for radial functions the operator norm from $L^{p,1}$ to L^p coincides with L^p norm.

Case $p = 2$. Instead of estimating $\|\lambda(\chi_n)\|_{2\to(2,\infty)}$ we will estimate $\|\lambda(\chi_n)\|_{(2,1)\to 2}$.

$\|\chi_n * \chi_E\|_2^2 = \langle \chi_n * \chi_E, \chi_n * \chi_E \rangle = \langle \chi_n * \chi_n * \chi_E, \chi_E \rangle$
Let $q = 2k - 1$. Then
$\chi_n * \chi_n = \chi_{2n} + q\chi_{2n-2} + \ldots + q^{n-1}\chi_2 + (q+1)q^{n-1}\chi_0$.
Lemma.

$$\langle \chi_{2j} * \chi_E, \chi_E \rangle \leq q^j \|\chi_E\|_2^2.$$

Hence

$$\frac{\|\chi_n * \chi_E\|_2^2}{\|\chi_E\|_2^2} \leq nq^n + (q+1)q^{n-1} \leq Cn\|\chi_n\|_2^2.$$

Proof of Lemma. Define an operator P by the rule

$$\langle P\delta_x, \delta_y \rangle = \begin{cases} \langle \chi_{2j} * \delta_x, \delta_y \rangle & \text{if } |x| \geq |y| \\ 0 & \text{if } |x| < |y|. \end{cases}$$

Then

$$\langle \chi_{2j} * \delta_x, \delta_y \rangle \leq \langle P\delta_x, \delta_y \rangle + \langle \delta_x, P\delta_y \rangle.$$

$$\langle \chi_{2j} * \chi_E, \chi_E \rangle \leq 2\langle P\chi_E, \chi_E \rangle = 2\langle \chi_E, P^*\chi_E \rangle \leq 2|E| \, \|P^*\chi$$

$$\langle \delta_x, P^* \chi_E \rangle = \langle P\delta_x, \chi_E \rangle \leq \|P\delta_x\|_1.$$

Next

$$P\delta_x = \sum_{\substack{|w|=2j \\ |wx|\leq|x|}} \delta_{wx}.$$

Let $w = w_1 w_2$ where $|w_1| = |w_2| = j$. The conditions $|w| = 2j$ and $|wx| \leq |x|$ imply that w_2 is determined by the first j letters of x.

Hence we have as many terms as choices for w_1, i.e. at most q^j. Thus

$$\|P\delta_x\|_1 \leq q^j.$$

Therefore $\|P^*\chi_E\|_\infty \leq q^j$ and

$$\langle \chi_{2j} * \chi_E, \chi_E \rangle \leq 2q^j|E| = 2q^j\|\chi_E\|_2^2.$$