Algebraic length and Poincaré series on reflection groups with applications to representations theory

Marek Bożejko* and Ryszard Szwarc**

Institute of Mathematics
Wrocław University
pl. Grunwaldzki 2/4
50–384 Wrocław, Poland
bozejko@math.uni.wroc.pl
and
szwarc@math.uni.wroc.pl

Summary. Let W be a reflection group generated by a finite set of simple reflections S. We determine sufficient and necessary condition for invertibility and positive definitness of the Poincaré series $\sum_w q^{\ell(w)} w$, where $\ell(w)$ denotes the algebraic length on W relative to S. Generalized Poincaré series are defined and similar results for them are proved.

In case of finite W, representations are constructed which are canonically associated with the algebraic length. For crystallographic groups (Weyl groups) these representations are decomposed into irreducible components. Positive definitness of certain functions involving generalized lengths on W is proved. The proofs don't make use of the classification of finite reflection groups. Examples are provided.

Key words: reflection group, Coxeter group, root system, Poincaré series

1 Introduction

We start with an account of definitions and results on reflection groups that we need in this paper. For the details we refer to the book by Humphreys [20], whose notation we follow, and to [2, Chapter VI].

A reflection in a finite dimensional vector space V, endowed with a nondegenerate symmetric bilinear form (x, y), is a linear operator which sends some nonzero vector to its negative and fixes the orthogonal complement to this vector. If s_x denotes the reflection about the vector x then s_x acts by the rule

* Partially supported by KBN (Poland) under grant 2 P03A 054 15.
** Partially supported by KBN (Poland) under grant 2 P03A 048 15.

$$s_x y = y - 2\frac{(x,y)}{(x,x)}x. \tag{1}$$

Let the vectors $\Delta = \{r_1, r_2, \ldots, r_n\}$ form a basis for V and let $S = \{s_1, s_2, \ldots s_n\}$ denote the set of corresponding reflections. A reflection group is a group W generated by S.

The set

$$\Phi = \{wr_i \mid w \in W,\, i = 1, 2 \ldots, n\}$$

is called a root system whenever it can be decomposed as $\Phi = \Pi \cup -\Pi$, where Π denotes the subset of Φ consisting of the vectors $\sum_1^n a_i r_i$, with $a_i \geq 0$. The elements of Π are called positive roots, while those of $-\Pi$ are called negative roots. The elements of Δ are called the simple roots and the elements of S are called simple reflections.

If the group W is finite (which means that Φ is a finite set) then the order $\mathrm{m}(i,j)$ of any product $s_i s_j$ has to be finite. It turns out that the numbers $\mathrm{m}(i,j)$ determine the group W up to an isomorphic equivalence. The group W can be described in algebraic way as generated by S with

$$s_i^2 = 1, \qquad (s_i s_j)^{\mathrm{m}(i,j)} = 1,$$

as the only relations among elements in S. A convenient way of encoding this information is the Coxeter graph of W. The set of generators is the vertex set of this graph. If $\mathrm{m}(s,s') \geq 3$ we join s and s' by an edge labelled with $\mathrm{m}(s,s')$.

Any element $w \in W$ is a product of simple reflections, say $w = s_{i_1} s_{i_2} \ldots s_{i_k}$. The smallest k for which this occur will be called the length $\ell(w)$ of w. The length has important geometric interpretation. There holds

$$\ell(w) = |\Pi \cap w^{-1}(-\Pi)|. \tag{2}$$

This means that $\ell(w)$ is precisely the number of positive roots sent by w to negative roots.

If we select some of the reflections $J \subseteq S$, then the group generated by them is also a reflection group denoted by W_J. The corresponding simple roots are $\Delta_J = \{r_i \mid s_i \in J\}$. The set

$$\Phi_J = \{wr_i \mid w \in W_J,\ s_i \in J\}$$

is a corresponding root system. The set Φ_J decomposes into subsets of positive Π_J and negative $-\Pi_J$ roots. The length of w as element in W_J coincides with the one defined by the ambient system (W,S). Moreover for elements in $w \in W_J$ we have

$$\Pi_J \cap w^{-1}(-\Pi_J) = \Pi \cap w^{-1}(-\Pi). \tag{3}$$

(see [20, Proposition 1.10]).

The group W is finite if and only if the bilinear form associated with the $n \times n$ matrix

$$a(i,j) = -\cos\frac{\pi}{\mathrm{m}(i,j)}$$

is positive definite. Moreover finite reflection groups are classified into 5 series and some exceptional cases.

Part of our result will hold only for so called crystallographic reflection groups. These are finite reflection groups such that $2a(i,j)$ is an integer for any i, j. It turns out that these are precisely the Weyl groups of simple Lie algebras. The finite reflections groups that are left behind are dihedral groups and two exceptional ones, known as H_3 and H_4.

In Section 2 we study the invertibility of Poincaré series of (W, S). This has many important applications in noncommutative probability theories. The Poincaré series (or Poincaré polynomial in case of finite W) is a formal power series of the form $\sum_{w\in W} q^{\ell(w)}w$, where q is a complex number. For finite Coxeter groups (W, S) we determine the values of Q for which $P_q(W)$ is invertible in the group algebra of W. In particular, for the permutation group S_n, we obtain a result of Zagier [28] that $P_q(S_n)$ is invertible if and only if $q^{(i-1)i} \neq 1$ for any $i = 1, 2, \ldots, n$. Zagier uses the invertibility properties of $P_q(S_n)$ in constructing models of q–oscillators in infinite statistics. On the other hand Bożejko et al. [6, 7, 8, 9] make use of positive definitness of $P_q(S_n)$ this time in constructing generalized Brownian motions and in finding realizations of noncommutative Gausssian processes in connection with quantum Young–Baxter equation (see also [22]). Dykema and Nica [16] use Fock realizations of canonical q–commutation relations found in [8]. They derive positivity properties of $P_q(S_n)$ to show stability properties of the generalized Cuntz algebras generated by q–oscillator (see [6, 8, 17, 22]). The positive definitness of $P_q(W)$ for infinite Coxeter groups was a key point in the proof [4] that such groups do not have Kazhdan property T.

In this paper we study also positive definitness and invertibility of the generalized Poincaré series for (W, S). This is any function $P(\cdot)$ which is multiplicative in the following sense: if the product uv is reduced for $u, v \in W$, then $P(uv) = P(u)P(v)$. We also require that $P(s) = P(s')$ for $s, s' \in S$, whenever s and s' are conjugate in W. We show that the generalized Poincaré series $P = \sum P(w)\, w$ is positive definite if and only if $-1 \le P(s) \le 1$, for any $s \in S$ (see [7], where the finite Coxeter groups were considered).

The positive definitness of the Poincaré series is also connected with the problem of determining which locally compact groups are so called weakly amenable (see [14]). Roughly a group G is weakly amenable if the Fourier algebra $A(G)$ admits an approximate unit bounded with respect to completely bounded multiplier norm (this is a norm weaker than the norm of $A(G)$ but stronger than the multiplier norm on $A(G)$). Partial results were obtained for special groups (see [5, 21, 23, 24, 27]). In particular our Theorem 1 played a crucial role in [21]. We conjecture that all Coxeter groups are weakly amenable. We think that the results in this paper constitute a step towards proving this conjecture. Another step, from the geometrical viewpoint, has been made in [25].

The proof of invertibility of $P_q(W)$ relies on a peculiar geometrical property of the Coxeter complex associated with (W, S). Namely, if w_0 denotes the longest element in W, then

$$w_0 W_{J_1} \cap W_{J_2} = \emptyset,$$

where J_1 and J_2 are arbitrary proper subsets of S. In other words, any facet about the identity element of the group e is disjoint from any facet about the element w_0.

In Section 3 we construct representations of W associated with the length $\ell(w)$. We show that the correspondence

$$w \mapsto \tfrac{1}{2}|\Pi| - \ell(w)$$

is a positive definite function on W. As a sideffect we obtain another proof of positive definitness of the Poincaré series $P_q(W)$ in case of finite Coxeter group. It would be of great interest to find a canonical representation of W associted with $P_q(W)$. This has been done only for special cases (see [23, 24]).

The main result in Section 3.1 is a decomposition of the representations corresponding to the length function into irreducible components in case of crystallographic groups. For noncrystallographic case, the decomposition holds but it does not yield irreducible components. In Section 3.2 we compute the decomposition of the function $\varphi(w) = \frac{1}{2}|\Pi| - \ell(w)$ into irreducible positive definite functions. Th eprrof don't make use of classification of finite reflection groups. The key point is a characterization of kernels $k(x, y)$ defined on $\Phi \times \Phi$, invariant for the action of the group W.

Throughout the paper by $A \subset B$ we will mean that A is properly contained in B. Otherwise we will write $A \subseteq B$. The symbol $|A|$ will denote the number of elements in the set A. The symbol $A \triangle B$ will denote the set $(A \setminus B) \cup (B \setminus A)$.

2 Positive definitness and invertibility of the Poincaré series

Let (W, S) be a Coxeter system with finite generator set S. For an element $w \in W$ let $\ell(w)$ denote the length of w.

For a subset J of S the symbol W_J will denote the subgroup of W generated by J. Let

$$W^J = \{w \in W \mid \ell(ws) > \ell(w), \text{ for all } s \in J\}. \tag{4}$$

By [20, page 19] (see also [2, Problem IV.1.3]) any element $w \in W$ admits a unique decomposition

$$w = w_J w^J, \quad \ell(w) = \ell(w_J) + \ell(w^J), \tag{5}$$

where $w_J \in W_J$ and $w^J \in W^J$.

Let $\underline{q} = \{q_s\}_{s \in S}$ be a family of complex numbers such that $q_s = q_{s'}$, if s and s' are conjugate in W. For an element $w \in W$, let

$$\underline{q}^w = q_{s_1} q_{s_2} \dots q_{s_n} \quad \text{if} \quad w = s_1 s_2 \dots s_n,\ \ell(w) = n.$$

By [2, Proposition 1.5.5] the function $w \mapsto \underline{q}^w$ is well defined. Observe that if $q_s = q$ for all $s \in S$, then $\underline{q}^w = q^{\ell(w)}$. The function $\underline{q}^w$ is multiplicative in the following sense.

$$\underline{q}^{w_1 w_2} = \underline{q}^{w_1} \underline{q}^{w_2}\ , \text{ if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2).$$

The conjugation relation yields a decomposition of S into equivalence classes say

$$S = A_1 \cup A_2 \cup A_m. \tag{6}$$

Hence $\underline{q}$ takes m values $q_1, q_2, \dots, q_m$. Let $w = s_1 s_2 \dots s_n$ be a reduced representation of w. Then

$$\underline{q}^w = \prod_{i=1}^{m} q_i^{\ell_i(w)}, \tag{7}$$

where

$$\ell_i(w) = |\{j \mid s_j \in A_i\}|. \tag{8}$$

By aforementioned [2, Proposition 1.5.5] the function $\ell_i(w)$ does not depend on representation of w in reduced form. This will also follow from the next proposition.

Proposition 1.

(i) Two simple reflections s and s' in S are conjugate in W if and only if there exists $w \in W$ such that $wr = r'$, where r and r' are the corresponding simple roots.

(ii) For any $w \in W$ and $i = 1, 2, \dots, m$

$$\ell_i(w) = |\Pi_i \cap w^{-1}(-\Pi_i|,$$

where $\Pi_i = \Pi \cap \{wr_j \mid s_j \in A_i\}$.

Proof. Part (i) follows immediately from the identity

$$s_{wr} = w s_r w^{-1}.$$

Let $w = s_1 s_2 \dots s_n$ be a reduced representation of w. Define roots θ_i, $i = 1, 2, \dots, n$ by the rule

$$\theta_i = s_n s_{n-1} \cdots s_{i+1}(r_i).$$

It is easy to check that

$$\Pi \cap w^{-1}\Pi = \{\theta_1, \theta_2, \dots, \theta_n\},$$

(see [20, Exercise 5.6.1, page 115]). Observe that $s_j \in A_i$ if and only if $\theta_j \in \Pi_i$. This yields the conclusion.

Lemma 1. *Assume a group W acts transitively on a set Ω, and there exists a subset $A \subset \Omega$, such that $(wA)\triangle A$ is a finite set for any $w \in G$. Then the function $w \mapsto q^{|(wA)\triangle A|}$ is positive definite on W, for any $-1 \le q \le 1$.*

Proof. Observe that

$$|(v^{-1}wA)\triangle A| = |(wA)\triangle(vA)| = \sum_{x\in\Omega} |\chi_{wA} - \chi_{vA}|^2.$$

By [1, page 81] the correspondence $w \mapsto |(wA)\triangle A|$ is so called negative definite function on W. Thus by Schoenberg's theorem (see [1, Theorem 2.2, page 74]) we get the conclusion for $0 \le q \le 1$. Observe that the real valued function

$$w \mapsto (-1)^{|(wA)\triangle A|}$$

is multiplicative on W. Hence it is positive definite on W. Using Schur's theorem, that the product of positive definite functions yields another such function, completes the proof of the lemma.

Theorem 1. *Let (W, S) be a Coxeter system and let $\underline{q} = \{q_s\}_{s\in S}$ be such that $-1 \le q_s \le 1$ and $q_s = q_{s'}$, whenever s, s' are conjugate in W. Then the function $w \mapsto \underline{q}^w$ is positive definite.*

Proof. By (7) it suffices to show that $w \mapsto q_i^{\ell_i(w)}$ is positive definite. This follows immediately from Lemma 1, Proposition 1 and the fact that

$$\ell_i(w) = |\Pi_i \cap w^{-1}(-\Pi)| = \frac{1}{2}|(w\Pi_i)\triangle\Pi_i|.$$

The functions $w \mapsto \ell_i(w)$ will play essential role in the next section.

The function $w \mapsto \underline{q}^w$ will be called the Poincaré series of W and denoted by $P_{\underline{q}}(W)$. It can be expressed as the power series

$$P_{\underline{q}}(W) = \sum_{w\in W} \underline{q}^w\, w.$$

In the sequel we will identify $P_{\underline{q}}(W)$ with the convolution operator by this function on $\ell^2(W)$. We are interested when this operator is invertible.

For a subset $A \subset W$, let

$$P_{\underline{q}}(A) = \sum_{w\in A} \underline{q}^w\, w\ .$$

By (4) and (5) we immediately get

$$P_{\underline{q}}(W) = P_{\underline{q}}(W_J)P_{\underline{q}}(W^J)\ . \tag{9}$$

The following formula has important consequences (cf. [20, Proposition 1.11]).

Proposition 2.
(i) Let (W,S) be a finite Coxeter group. Then

$$\underline{q}^{w_0}w_0 = \sum_{J\subseteq S}(-1)^{|J|}P_{\underline{q}}(W^J)\ ,$$

where w_0 is the unique longest element in W.
(ii) If $P_{\underline{q}}(W)$ is an invertible operator, then

$$\sum_{J\subset S}(-1)^{|J|}P_{\underline{q}}(W_J)^{-1} = \{\underline{q}^{w_0}\,w_0 - (-1)^{|S|}\,e\}P_{\underline{q}}(W)^{-1}\ .$$

Proof. By (9) the operator $P_{\underline{q}}(W_J)$ is invertible, if $P_{\underline{q}}(W)$ is invertible. Thus it suffices to show (i). We have

$$\begin{aligned}\sum_{J\subseteq S}(-1)^{|J|}P_{\underline{q}}(W^J) &= \sum_{J\subseteq S}(-1)^{|J|}\left(\sum_{w\in W^J}\underline{q}^w\lambda(w)\right)\\ &= \sum_{w\in W}\left(\sum_{\substack{J\\ w\in W^J}}(-1)^{|J|}\right)\underline{q}^w\lambda(w)\ .\end{aligned}$$

Let

$$J_w = \{s\in S\mid\ \ell(ws)>\ell(w)\}.$$

Observe that $w\in W^J$ if and only if $J\subseteq J_w$. Therefore

$$\sum_{\substack{J\\ w\in W^J}}(-1)^{|J|} = \sum_{J\subset J_w}(-1)^{|J|} = (1-1)^{|J_w|} = \begin{cases}0 & \text{if } J_w\neq\emptyset\\ 1 & \text{if } J_w=\emptyset\end{cases}$$

However $J_w=\emptyset$ if and only if $w=w_0$. Thus

$$\sum_{J\subset S}(-1)^{|J|}P_{\underline{q}}(W_J)^{-1} = \underline{q}^{w_0}w_0.$$

The subset $I\subseteq S$ will be called connected if I is connected in the Coxeter graph of (W,S). If W_I is finite group the unique longest element of W_I will be denoted by $w_0(I)$.

For a subset $J\subseteq S$ let

$$T(J) = \{\underline{q}\mid\ (\underline{q}^{w_0(I)})^2 = 1 \text{ for some connected } I\subseteq J\}.$$

Proposition 3. *Let (W,S) be a finite Coxeter system. If $\underline{q}\notin T(S)$, then the convolution with $P_{\underline{q}}(W)$ is an invertible operator on $\ell^2(W)$.*

Proof. We prove the assertion by induction on $n = |S|$. For $n = 1$ we have $S = \{s\}$ and $P(W) = 1 + qs$. Hence $P_{\underline{q}}(W)^{-1} = (1 - q^2)^{-1}(1 - qs)$ exists as long as $q^2 \neq 1$.

Assume S is not connected. Then $S = S' \cup S''$, $s's'' = s''s'$ for any $s' \in S'$ and $s'' \in S''$. Hence $W = W_{S'}W_{S''}$. This implies

$$P_{\underline{q}}(W) = P_{\underline{q}}(W_{S'})P_{\underline{q}}(W_{S''}).$$

Thus, with no loss of generality, we can restrict ourselves to the case S is connected. By definition we have

$$T(J) \subseteq T(S).$$

Therefore, if $\underline{q} \notin T(S)$, then $\underline{q} \notin T(J)$ for $J \subset S$. By induction hypothesis the inverse $P_{\underline{q}}(W_J)^{-1}$ exists for each $J \subset S$. Thus by Proposition 2

$$\begin{aligned} \underline{q}^{w_0}w_0 - (-1)^{|S|}e &= \sum_{J \subset S}(-1)^{|J|}P_{\underline{q}}(W^J) \\ &= \left[\sum_{J \subset S}(-1)^{|J|}P_{\underline{q}}(W_J)^{-1}\right] P_{\underline{q}}(W) \, . \end{aligned}$$

Using the fact that $w_0^2 = 1$ (see [20, page 16]) gives

$$\begin{aligned} (\underline{q}^{w_0})^2 - 1 &= (\underline{q}^{w_0}w_0 + (-1)^{|S|}e)(\underline{q}^{w_0}w_0 - (-1)^{|S|}e) \\ &= (\underline{q}^{w_0}w_0 + (-1)^{|S|}e)\left[\sum_{J \subset S}(-1)^{|J|}P_{\underline{q}}(W_J)^{-1}\right] P_{\underline{q}}(W). \qquad (10) \end{aligned}$$

Since by assumption $(\underline{q}^{w_0})^2 \neq 1$ thus thus $P_{\underline{q}}(W)$ is left invertible, and also right invertible as it is a finite dimensional operator.

The converse implication will be shown in Theorem 3. To this end we need more information about parabolic subgroups of (W, S). A Coxeter group (W, S) is called irreducible if the Coxeter graph is connected; i.e. the set S cannot be decomposed into two disjoint subsets S_1 and S_2 commuting with each other.

The following result is interesting for its own sake.

Theorem 2. *If (W, S) is a finite irreducible Coxeter group, and J, J' are proper subsets of S, then*

$$w_0 W_J \cap W_{J'} = \emptyset \, .$$

We start with a lemma.

Lemma 2. *Let $\{s_1, s_2, \ldots, s_n\} = S$ be such that $\{s_1, s_2, \ldots, s_k\}$ is connected for any $1 \leq k \leq n$. Then the root $r = s_n s_{n-1} \ldots s_2(r_1)$ is positive and*

$$r = \sum_{i=1}^{n} \alpha_i r_i, \quad \alpha_i > 0.$$

Proof. We use induction on n. Assume

$$s_{n-1}\dots s_2(r_1)=\sum_{i=1}^{n-1}\alpha_i r_i \quad \alpha_i>0.$$

Then

$$\begin{aligned} r = s_n s_{n-1}\dots s_2(r_1) &= s_n\left(\sum_{i=1}^{n-1}\alpha_i r_i\right)\\ &=\sum_{i=1}^{n-1}\alpha_i r_i - 2\sum_{i=1}^{n-1}\alpha_i\frac{(r_i,r_n)}{(r_n,r_n)}r_n. \end{aligned}$$

We have $(r_i,r_n)\le 0$ (see [20]) and $(r_i,r_n)<0$ for at least one value of i, where $i=1,2,\dots,n-1$. Thus the lemma follows.

We return to the proof of Theorem 2. We will use the fact that w_0 sends all positive roots to negative roots. This implies

$$\Pi\cap(w_0w)^{-1}(-\Pi)=\Pi\cap w^{-1}\Pi.$$

Hence

$$\Pi=[\Pi\cap w^{-1}(-\Pi)]\cup[\Pi\cap(w_0w)^{-1}(-\Pi)]. \tag{11}$$

Consider the positive root constructed in Lemma 2. If r belongs to the first summand in (11) then it is a linear combination of those r_i, for which $s_i\in J$ (see Introduction). This is a contradiction. Similarly r cannot belong to the second summand, if $w_0w\in W_{J'}$. □

Remark. Theorem 2 does not hold if the group (W,S) is not irreducible. For example we can take $W=\mathbb{Z}_2\times\mathbb{Z}_2$ and $S=\{s_1,s_2\}$. Let $W_J=\{e,s_1\}$ and $W_{J'}=\{e,s_2\}$. Then $w_0=s_1s_2$ and

$$w_0W_J\cap W_{J'}=\{s_2\}.$$

Theorem 3. *Let (W,S) be a finite Coxeter group. Then the convolution operator $P_{\underline{q}}(W)$ is invertible if and only if $\underline{q}\notin T(S)$.*

Proof. It suffices to consider the case when the system (W,S) is irreducible. The "if" part has been shown in Proposition 3. We only need to show that if $P_{\underline{q}}(W)$ is invertible, then $\underline{q}\notin T(S)$. Assume for a contradiction that $\underline{q}\in T(S)$. Then there exists a connected subset $J\subset S$, such that $(\underline{q}^{w_0(J)})^2=1$. We will show that $P_{\underline{q}}(W_J)$ is not invertible, which implies that $P_{\underline{q}}(W)$ is not so either, in view of (9). Thus with no loss of generality we may assume that $J=S$, i.e. $(\underline{q}^{w_0})^2=1$. By (10) we get

$$(\underline{q}^{w_0}w_0+(-1)^{|S|}e)\left[\sum_{J\subset S}(-1)^{|J|}P_{\underline{q}}(W_J)^{-1}\right]P_{\underline{q}}(W)=0.$$

Since $P_{\underline{q}}(W)$ is invertible we have

$$(\underline{q}^{w_0} w_0 + (-1)^{|S|} e) \left[\sum_{J \subset S} (-1)^{|J|} P_{\underline{q}}(W_J)^{-1} \right] = 0. \qquad (12)$$

We will show that (12) is impossible. The support of $P_{\underline{q}}(W_J)^{-1}$ is a subset of W_J. This is based on the general fact that the convolution inverse to a function f on a group is supported by a subgroup generated by the support of f. Therefore

$$\mathrm{supp}(f) = G \subset \bigcup_{J \subset S} W_J \ .$$

The equation (12) implies $w_0 G \cap G \neq \emptyset$, which yields

$$w_0 W_J \cap W_{J'} \neq \emptyset$$

for some $J, J' \subset S$. In view of Theorem 2, this is a contradiction.

Remark. In view of Theorems 1 and 3 we get that for every choice of $q_s \in (-1,1)$, where $s \in S$, the convolution operator with $P_{\underline{q}}(W)$ is strictly positive definite; i.e.

$$(P_{\underline{q}}(W) * f, f) \geq c(f,f),$$

for some $c > 0$, where $(f,g) = \sum_{w \in W} f(w) g(w)$.

If (W,S) has the property that all generators in S are conjugate, then $q_s = q$ does not depend on s, and $\underline{q}^w = q^{\ell(w)}$.

Corollary 1. *Let $W = S_n$ be the permutation group generated by the set of transpositions $S = \{(i, i+1) \mid i = 1, 2, \ldots, n\}$. The convolution operator with the Poincaré polynomial*

$$P_q(S_n) = \sum_{g \in S_n} q^{\ell(g)} g$$

is invertible if and only if

$$q^{\frac{i(i-1)}{2}} \neq 1 \quad \textit{for every } i = 1, 2, \ldots, n.$$

3 Canonical representations associated with length

3.1 Irreducible root systems and invariant kernels

Let Φ be a root system in an Euclidean space V (see [20, Section 1.2]). Any vector x in Φ defines the reflection s_x according to (1). The group generated by all reflections s_x will be denoted by W. The elements of W permute the vectors in Φ. We will assume that the system Φ is irreducible; i.e. it cannot be decomposed into two nonempty root systems which are orthogonal to each other with respect to the inner product in V.

Vectors in Φ may have different lengths. For a real number λ let Φ_λ consist of all roots in Φ of length λ, and let V_λ be the linear span of Φ_λ. Since the elements of W act on V by isometries they permute the vectors in Φ_λ.

Lemma 3. *Let $k_\lambda = \dim V_\lambda$ and $d_\lambda = |\Phi_\lambda|$. Then*

$$\sum_{z\in\Phi_\lambda} (x,z)(y,z) = \lambda^2 \frac{d_\lambda}{k_\lambda}(x,y)$$

for any $x, y \in V_\lambda$.

Proof. It suffices to prove the formula for $x, y \in \Phi_\lambda$ since both its sides represent bilinear forms on V_λ. We define a new inner product in V_λ as follows.

$$[x,y] = \sum_{z\in\Phi_\lambda} (x,z)(y,z).$$

Any element $g \in W$ permute the roots in Φ_λ, hence

$$[gx, gy] = [x,y] \qquad \text{for any} \qquad x, y \in V_\lambda,\ g \in W.$$

By [2, Prop V.1.11] the group W acts transitively on Φ_λ, beacuse the system Φ is irreducible. This implies

$$[x,x] = [y,y] \qquad \text{for any} \qquad x, y \in \Phi_\lambda.$$

Let $c = \lambda^{-2}[x,x]$ for $x \in \Phi_\lambda$. Then since $\lambda^2 = (x,x)$ we get

$$\begin{aligned}[x,y] &= [s_x x, s_x y] = [-x, y - 2\frac{(x,y)}{(x,x)}x] \\ &= -[x,y] + 2c(x,y),\end{aligned}$$

for $x, y \in \Phi_\lambda$. Thus

$$[x,y] = c(x,y) \qquad \text{for} \qquad x, y \in \Phi_\lambda. \tag{13}$$

The equality extends linearly to all $x, y \in V_\lambda$.

It suffices to determine the constant c. Let $\{e_i\}_{i=1}^{k_\lambda}$ be an orthonormal basis for V_λ relative the inner product $(\ \cdot,\ \cdot)$. Then by (13) we have

$$\begin{aligned} ck_\lambda &= \sum_{i=1}^{k_\lambda} c(e_i, e_i) = \sum_{i=1}^{k_\lambda} [e_i, e_i] \\ &= \sum_{z\in\Phi_\lambda} \sum_{i=1}^{k_\lambda} (e_i, z)^2 = \sum_{z\in\Phi_\lambda} (z,z) = \lambda^2 d_\lambda. \end{aligned}$$

Let $\mathcal{H}_\lambda$ be the linear space of all real valued functions on Φ_λ with the property $f(-x) = -f(x)$, for $x \in \Phi_\lambda$. We endow $\mathcal{H}_\lambda$ with the inner product

$$(f_1, f_2)_{\mathcal{H}} = \sum_{x \in \Phi_\lambda} f_1(x) f_2(x). \tag{14}$$

The action

$$\pi_\lambda(g) f(x) = f(g^{-1}x)$$

yields a unitary representation of W on $\mathcal{H}_\lambda$. We are going to decompose this representation into irreducible components. In the next subsection we will also study its matrix coefficients.

By Schur's lemma the problem of decomposition of a given unitary representation can be reduced to studying linear selfadjoint operators commuting with this representation. Any selfadjoint operator K on $\mathcal{H}_\lambda$ commuting with the action of W on $\mathcal{H}_\lambda$ corresponds to a real valued matrix $k(x,y)$, $x, y \in \Phi_\lambda$, such that

$$k(gx, gy) = k(x,y), \tag{15}$$

$$k(x,y) = k(y,x), \tag{16}$$

$$k(-x,y) = -k(x,y). \tag{17}$$

The correspondence is given by

$$k(x,y) = (Ke_x, e_y),$$

where

$$e_v(w) = 2^{-1/2}(\delta_{v,w} - \delta_{-v,w}), \qquad v, w \in \mathcal{H}_\lambda,$$

where

$$\delta_{v,w} = \begin{cases} 1 & \text{if } v = w, \\ 0 & \text{if } v \neq w. \end{cases}$$

From now on we assume that the root system Φ is crystallographic; i.e. the quantity

$$\mathrm{n}(x,y) = 2\frac{(x,y)}{(y,y)}$$

takes only integer values. This property implies (see [2, Proposition VI.1.12]) that the roots can take two different lengths at most. Moreover the following lemma will be very useful (see [2, Section VI.1.3]).

Lemma 4 **([2]).** *Let $x, y \in \Phi_\lambda$. Then $\mathrm{n}(x,y)$ can take only the values 0, ± 1 and ± 2. In particular $\mathrm{n}(x,y) = \pm 2$ if and only if $x = \pm y$.*

Proposition 4. *Let a matrix $k(x,y)$ satisfy (15), (16) and (17). Then there exist real constants α and β such that*

$$k(x,y) = \alpha(x,y) + \beta\{\delta_{x,y} - \delta_{-x,y}\}.$$

Proof. First we will show that the value $k(x,y)$ depends only on (x,y) or equivalently on $\mathrm{n}(x,y)$. By (17) we may restrict ourselves to the case $n(x,y) \geq 0$. We have three possibilities.

(1) $\mathrm{n}(x,y) = 0$.

Then $s_x y = y$ and

$$-k(x,y) = k(-x,y) = k(s_x x, s_x y) = k(x,y).$$

Thus $k(x,y) = 0$.

(2) $n(x,y) = 2$.

Thus by Lemma 2 we get $x = y$. Therefore it suffices to make sure that $k(x,x)$ does not depend on x. This holds true because W acts transitively on Φ_λ (see [2, Prop V.1.11]) and $k(gx,gx) = k(x,x)$ for $g \in W$.

(3) $\mathrm{n}(x,y) = 1$.

Let $\mathrm{n}(x',y') = 1$. We have to show that $k(x',y') = k(x,y)$. In view of the properties of $k(x,y)$ and $\mathrm{n}(x,y)$ satisfy it suffices to show that there exists $w \in W$ such that

$$wx = \pm x' \quad \text{and} \quad wy = \pm y'.$$

Firstly, there exists g such that $gx = x'$. Let $y'' = gy$. If $y'' = y'$ we are done. Thus we may assume $y'' \neq y'$. We have

$$\mathrm{n}(x',y'') = \mathrm{n}(gx,gy) = \mathrm{n}(x,y) = \mathrm{n}(x',y') = 1.$$

This implies $y'' \neq -y'$. Thus $y'' \neq \pm y'$. We will further break the resaoning into three subcases.

(3a) $\mathrm{n}(y',y'') = 1$.

Letting $h = s_{y''}s_{y'}s_{y''}$ gives

$$h = s_{y'-y''}, \qquad hx' = x', \qquad hy' = y''.$$

Let $w = hg$. Then $wx = hx' = x'$ and $wy = hy'' = y'$.

(3b) $n(y',y'') = 0$.

Letting $h = s_{y'}s_{x'}s_{y''}s_{x'}s_{y'}s_{x'}$ gives

$$h = s_{y'+y''-x'}s_{x'}, \qquad hx' = -x', \qquad hy'' = -y'.$$

Let $w = hg$. Then $wx = hx' = -x'$ and $wy = hy'' = -y'$.

(3c) $n(y',y'') = -1$.

We have

$$\mathrm{n}(s_{x'}y'',y') = \mathrm{n}(y''-x',y') = \mathrm{n}(y'',y') - \mathrm{n}(x',y') = -2.$$

In view of Lemma 2 this implies $s_{x'}y'' = -y'$. Let $w = s_{x'}g$. Then $wx = s_{x'}x' = -x'$ and $wy = s_{x'}y'' = -y'$.

We are now in position to determine the constants α and β. Assume $x \not\perp y$, $x \neq \pm y$ and let $\alpha = (x,y)^{-1}k(x,y)$, if such a pair exists, and $\alpha = 0$, otherwise. Set $\beta = k(x,x) - \alpha(x,x)$. The lemma is then satisfied.

Proposition 1 implies that the space of operators commuting with action of W on V_λ is at most two dimensional. Hence the representation π_λ can be decomposed into two irreducible subrepresentations or it is irreducible itself. The latter holds if and only if the positive roots in Φ_λ are mutually orthogonal.

Let P_λ be defined on $\mathcal{H}_\lambda$ by

$$P_\lambda f(x) = \frac{k_\lambda}{\lambda^2 d_\lambda} \sum_{z\in\Phi_\lambda} (x,z)f(z). \tag{18}$$

Lemma 3 implies that the operator P_λ is a projection. Clearly it commutes with the action of W. Let us determine the subspace $P_\lambda\mathcal{H}_\lambda$.

Lemma 5. $P_\lambda\mathcal{H}_\lambda = \{f \in \mathcal{H}_\lambda \mid f(\,\cdot\,) = (\,\cdot\,,\ w)$ *for some* $w \in V_\lambda\}$.

Proof. For $f \in \mathcal{H}_\lambda$ we have

$$P_\lambda f(x) = (x,w), \quad \text{where} \quad w = \frac{k_\lambda}{\lambda^2 d_\lambda} \sum_{z\in\Phi_\lambda} f(z)\, z.$$

Conversely, if $f(x) = (x,w)$ for some $w \in V_\lambda$, then by Lemma 3

$$P_\lambda f(x) = \frac{k_\lambda}{\lambda^2 d_\lambda} \sum_{z\in\Phi_\lambda} (x,z)(z) = (x,w) = f(x).$$

□

Lemma 3 implies that $P_\lambda\mathcal{H}_\lambda$ is isomorphic to V_λ and the representation π_λ restricted to $P_\lambda\mathcal{H}_\lambda$ is equivalent to the action of W on V_λ. If Φ_λ contains two roots which are not parallel nor perpendicular, then $P_\lambda \neq I$ and $\mathcal{H}_\lambda$ has nontrivial decomposition

$$\mathcal{H}_\lambda = (I - P_\lambda)\mathcal{H}_\lambda \oplus P_\lambda\mathcal{H}_\lambda.$$

Thus we arrived at the following.

Theorem 4. *The representation* π_λ *of* W *on* $\mathcal{H}_\lambda$ *has two irreducible subspaces* $P_\lambda\mathcal{H}_\lambda$ *and* $(I - P_\lambda)\mathcal{H}_\lambda$*, provided that* Φ_λ *contains two roots* x *and* y *such that* $x \neq \pm y$ *and* $x \not\perp y$*. Otherwise the representation* π_λ *is irreducible itself. The representation* π_λ *restricted to* $P_\lambda\mathcal{H}_\lambda$ *is equivalent to the action of* W *on* V_λ.

3.2 Matrix coefficients of π_λ

For an irreducible root system Φ in V let Δ denote the set of simple roots. Φ decomposes into two disjoint subset Π and $-\Pi$ consisting of positive and negative roots, respectively. The group W is generated by the reflections s_x, where $x \in \Delta$. As was mentioned in the Introduction the algebraic length of elements $g \in W$ with respect to the generators s_x, $x \in \Delta$ can be expressed in geometric manner as

$$\ell(g) = |g\Pi \cap -\Pi|. \tag{19}$$

This formula has been used in [4] to show that the correspondence $g \mapsto \ell(g)$ is a negative definite function on W. We will extend this result in the case of finite Coxeter groups by showing that the function $g \mapsto \frac{1}{2}|\Pi| - \ell(g)$ is positive definite. We will also give a decomposition of it into pure positive definite functions; i.e. positive definite functions which are coefficients of irreducible representations.

Let $\mathcal{H}$ consists of all real valued functions on Φ such that $f(-x) = -f(x)$, for $x \in \Phi$, endowed with the inner product (14). The group W acts on $\mathcal{H}$ by isometries

$$\pi(g)f(x) = f(g^{-1}x).$$

Obviously the representations π_λ defined in Section 2 are subreprepresentations of π.

Proposition 5. *Let $\xi = \frac{1}{2}(\chi_\Pi - \chi_{-\Pi})$, where χ_A denotes the indicator function of a set A. Then*

$$(\pi(g)\xi, \xi)_{\mathcal{H}} = \tfrac{1}{2}|\Pi| - \ell(g).$$

In particular the mapping $g \mapsto \frac{1}{2}|\Pi| - \ell(g)$, is positive definite function on W.

Proof. We have

$$\begin{aligned} 4(\pi(g)\xi, \xi)_{\mathcal{H}} = {} & |g\Pi \cap \Pi| + |g(-\Pi) \cap (-\Pi)| \\ & -|g\Pi \cap (-\Pi)| - |g(-\Pi) \cap \Pi|. \end{aligned}$$

We have

$$\begin{aligned} |g\Pi \cap \Pi| &= |g(-\Pi) \cap (-\Pi)| = |\Pi| - |g\Pi \cap (-\Pi| = |\Pi| - \ell(g), \\ |g(-\Pi) \cap \Pi| &= |g\Pi \cap (-\Pi)| = \ell(g). \end{aligned}$$

This gives the conclusion.

The set Φ_λ decomposes in natural way into the subsets Π_λ and $-\Pi_\lambda$ of positive and negative, respectively, roots of length λ. Let us define the length function on W relative to λ as

$$\ell_\lambda(g) = |g\Pi_\lambda \cap -\Pi_\lambda|.$$

Similarly to Proposition 2 we get the following.

Proposition 6. *Let $\xi_\lambda = \frac{1}{2}(\chi_{\Pi_\lambda} - \chi_{-\Pi_\lambda})$. Then*

$$(\pi_\lambda(g)\xi_\lambda, \xi_\lambda)_{\mathcal{H}} = \frac{1}{2}|\Pi_\lambda| - \ell_\lambda(g).$$

In particular the mapping $g \mapsto \frac{1}{2}|\Pi_\lambda| - \ell_\lambda(g)$, is positive definite function on W.

Remark. By [2, Proposition V.1.11] if two simple roots r_i and r_j have equal length then there exists $w \in W$ such that $wr_i = r_j$. This implies $ws_iw^{-1} = s_j$; i.e. the reflections s_i and s_j are conjugate to each other. The converse is also true. In this way the set Δ_λ corresponds to a conjugate class A_i in (6). Thus the length ℓ_λ coincides with one of the functions ℓ_i defined in Proposition 1.

Using Schur's theorem and power series expansion of e^x gives that the exponent of a positive function is again such function. Thus Proposition 6 implies that for every choice of $0 < q < 1$ the correspondence

$$w \mapsto q^{\ell_\lambda(w)}$$

is positive definite. This gives an alternate proof of Theorem 1 in the case of finite reflection groups.

From now on we assume that the root system Φ is crystallographic. By [2] Φ can have roots of at most two different lengths. If $\Phi = \Phi_\lambda \cup \Phi_{\lambda'}$ then $\ell(g) = \ell_\lambda(g) + \ell_{\lambda'}(g)$ and $\pi = \pi_\lambda \oplus \pi_{\lambda'}$. If $\Phi = \Phi_\lambda$, then of course $\ell(g) = \ell_\lambda(g)$ and $\pi = \pi_\lambda$. Thus $\pi = \pi_\lambda$ or it decomposes into representations π_λ and $\pi_{\lambda'}$. In Section 3.1 we solved the problem of decomposing the representation π_λ. Now we will compute the corresponding decomposition of the positive definite functions

$$g \mapsto \tfrac{1}{2}|\Pi_\lambda| - \ell_\lambda(g).$$

Lemma 6. *Let v_λ be the sum of all positive roots of length λ; i.e.*

$$v_\lambda = \sum_{x \in \Pi_\lambda} x.$$

Then

$$(r, v_\lambda) = \begin{cases} 0 & r \in \Delta \setminus \Delta_\lambda, \\ \lambda^2 & r \in \Delta_\lambda, \end{cases} \tag{20}$$

where Δ_λ is the set of simple roots of length λ.

Proof. Let $r \in \Delta \setminus \Delta_\lambda$. Then $s_r v_\lambda = v_\lambda$, because s_r permutes the roots in Π_λ. Hence

$$(r, v_\lambda) = (r, s_r v_\lambda) = (s_r r, v_\lambda) = -(r, v_\lambda).$$

Thus $(r, v_\lambda) = 0$.

Let $r \in \Delta_\lambda$. Then $s_r v_\lambda = v_\lambda - 2r$, beacuse $s_r r = -r$. Hence

$$(r, v_\lambda) = (s_r r, s_r v_\lambda) = -(r, v_\lambda) + 2(r, r).$$

Therefore

$$(r, v_\lambda) = (r, r) = \lambda^2.$$

Any root x in Φ can be uniquely represented as a linear combination of simple roots

$$x = \sum_{r \in \Delta} \alpha_r(x) r. \tag{21}$$

The coefficients $\alpha_r(x)$, $r \in \Delta$, are all nonnegative or all nonpositive according to whether the root x is positive or negative. Define the function $\mathrm{n}_\lambda(x)$ for $x \in \Phi_\lambda$ by

$$\mathrm{n}_\lambda(x) = \sum_{r \in \Delta_\lambda} \alpha_r(x),$$

where $\Delta_\lambda = \{r \in \Delta \mid \|r\| = \lambda\}$.

Lemma 7. *For $x \in \Phi_\lambda$ we have*

$$\sum_{z \in \Pi_\lambda} (x, z) = \lambda^2 \mathrm{n}_\lambda(x).$$

Proof. By Lemma 4 and by (21) we have

$$\begin{aligned}\sum_{z \in \Pi_\lambda} (x, z) = (x, v_\lambda) &= \sum_{r \in \Delta} \alpha_r(x)(r, v_\lambda) \\ &= \sum_{r \in \Delta_\lambda} \lambda^2 \alpha_r(x) = \lambda^2 \mathrm{n}_\lambda(x).\end{aligned}$$

Theorem 5. *The mappings*

$$\varphi_\lambda(g) = \mathrm{n}_\lambda(g v_\lambda) \tag{22}$$

$$\psi_\lambda(g) = \tfrac{1}{4} d_\lambda - \ell_\lambda(g) - \frac{k_\lambda}{d_\lambda} \mathrm{n}_\lambda(g v_\lambda) \tag{23}$$

are pure positive definite functions on W, where $d_\lambda = |\Phi_\lambda|$ and $k_\lambda = \dim V_\lambda$.

Proof. We will show that

$$(\pi_\lambda(g) P_\lambda \xi_\lambda, P_\lambda \xi_\lambda) = \frac{k_\lambda}{\lambda^2 d_\lambda} \mathrm{n}_\lambda(g v_\lambda)$$

$$(\pi_\lambda(g)(I - P_\lambda)\xi_\lambda, (I - P_\lambda)\xi_\lambda) = \tfrac{1}{4} d_\lambda - \ell_\lambda(g) - \frac{k_\lambda}{\lambda^2 d_\lambda} \mathrm{n}_\lambda(g v_\lambda).$$

This will yield that both functions are positive definite. Since the restrictions of π_λ to either $P_\lambda \mathcal{H}_\lambda$ or $(I - P_\lambda)\mathcal{H}_\lambda$ are irreducible, the above functions are pure positive definite functions on W.

By definition of P_λ we have

$$P_\lambda \delta_x(y) = \frac{k_\lambda}{\lambda^2 d_\lambda}(x,y).$$

Combining this, Lemma 5 and the fact that P_λ is an orthogonal projection commuting with $\pi_\lambda(g)$ gives

$$\begin{aligned}(\pi_\lambda(g)P_\lambda v_\lambda, P_\lambda v_\lambda) = (\pi_\lambda(g)P_\lambda v_\lambda, v_\lambda) &= \frac{1}{4}\sum_{x,y\in\Pi_\lambda}(P_\lambda\{\delta_{gx}-\delta_{-gx}\}, \delta_y - \delta_{-y}) \\ &= \frac{k_\lambda}{\lambda^2 d_\lambda}\sum_{x,y\in\Pi_\lambda}(gx,y) = \frac{k_\lambda}{\lambda^2 d_\lambda}\sum_{x\in\Pi_\lambda}\mathrm{n}_\lambda(gx) \\ &= \frac{k_\lambda}{\lambda^2 d_\lambda}\mathrm{n}_\lambda(gv_\lambda).\end{aligned}$$

Furthermore, since $I - P_\lambda$ is a projection commuting with $\pi_\lambda(g)$, we obtain

$$\begin{aligned}(\pi_\lambda(g)(I-P_\lambda)\xi_\lambda, (I-P_\lambda)\xi_\lambda) &= (\pi_\lambda(g)(I-P_\lambda)\xi_\lambda, \xi_\lambda) \\ &= (\pi_\lambda(g)\xi_\lambda, \xi_\lambda) - (\pi_\lambda(g)P_\lambda\xi_\lambda, \xi_\lambda) \\ &= \tfrac{1}{4}d_\lambda - \ell_\lambda(g) - \frac{k_\lambda}{\lambda^2 d_\lambda}\mathrm{n}_\lambda(gv_\lambda).\end{aligned}$$

3.3 Examples

(a) Groups of type A_n.

We have

$$V = \{x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i = 0\} \qquad \text{and} \qquad \Phi = \{e_i - e_j \mid i \neq j\}.$$

The roots have equal length $\lambda = \sqrt{2}$. The positive roots and simple roots are as follows.

$$\begin{aligned}\Pi &= \{e_i - e_j \mid i < j\}, \\ \Delta &= \{e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_n - e_{n+1}\}.\end{aligned}$$

We also have

$$\begin{aligned}d_\lambda &= n(n+1), \\ k_\lambda &= n, \\ v_\lambda &= \sum_{i<j}(e_i - e_j), \\ \mathrm{n}_\lambda(e_i - e_j) &= j - i.\end{aligned}$$

The group W can be identified with the permutation group S_{n+1}. In this case we have

$$\varphi_\lambda(\sigma) = \sum_{i<j}[\sigma(j) - \sigma(i)]$$
$$\psi_\lambda(\sigma) = \frac{n(n+1)}{4} - \ell(\sigma) - \frac{1}{2(n+1)}\sum_{i<j}[\sigma(j) - \sigma(i)]$$

By Theorem 5 both functions are pure positive definite functions on S_{n+1}.

It can be easily determined that the irreducible representation of S_{n+1} corresponding to the function φ_λ has the diagram $(2,1,1,\dots,1)$ int he Young tableaux, while the representation corresponding to ψ_λ has the diagram $(3,1,1,\dots,1)$.

(b) Groups of type B_n.

In that case $V = \mathbb{R}^n$ and

$$\Phi = \{\pm e_i \pm e_j \mid i \neq j = 1,2,\dots,n\} \cup \{\pm e_i \mid i = 1,2,\dots,n\}.$$

The positive roots and simple roots are

$$\Pi = \{e_i \pm e_j \mid i < j\} \,\cup\, \{e_i \mid i = 1,2,\dots,n\},$$
$$\Delta = \{e_1 - e_2,\ e_2 - e_3,\ \dots,\ e_{n-1} - e_n, e_n\}.$$

There are two lengths 1 and $\sqrt{2}$. Let $\lambda = \sqrt{2}$. Then

$$\Pi_\lambda = \{e_i \pm e_j \mid i < j\},$$
$$\Delta_\lambda = \{e_1 - e_2,\ e_2 - e_3,\ \dots,\ e_{n-1} - e_n\}.$$

We thus have

$$d_\lambda = 2n^2,$$
$$k_\lambda = n,$$
$$v_\lambda = 2\sum_{i=1}^{n-1}(n-i)e_i,$$
$$\mathrm{n}_\lambda(e_i) = n - i.$$

The group W can be identified with $\mathbb{Z}_2^n \rtimes S_n$. An element (τ,σ), where $\tau \in \mathbb{Z}_2^n$ and $\sigma \in S_n$, acts on V by the rule

$$(\tau,\sigma)e_i = (-1)^{\tau(i)} e_{\sigma(i)}.$$

We have

$$\varphi_\lambda(\tau,\sigma) = 2\sum_{i=1}^{n-1}(-1)^{\tau(i)}(n-i)[n-\sigma(i)]$$

$$\psi_\lambda(\tau,\sigma) = \frac{n^2}{2} - \ell(\tau,\sigma) - \frac{1}{2n}\sum_{i=1}^{n-1}(-1)^{\tau(i)}(n-i)[n-\sigma(i)].$$

(c) Groups of type D_n.

In that case $V = \mathbb{R}^n$ and

$$\Phi = \{\pm e_i \pm e_j \mid i \neq j = 1,2,\ldots,n\}.$$

The roots have equal length $\lambda = \sqrt{2}$. The positive roots and simple roots are

$$\begin{aligned} \Pi &= \{e_i \pm e_j \mid i < j\}, \\ \Delta &= \{e_1 - e_2,\ e_2 - e_3,\ \ldots,\ e_{n-1} - e_n, e_{n-1} + e_n\}. \end{aligned}$$

We have

$$\begin{aligned} d_\lambda &= 2n(n-1), \\ k_\lambda &= n, \\ v_\lambda &= 2\sum_{i=1}^{n-1}(n-i)e_i, \\ \mathrm{n}_\lambda(e_i) &= n - i. \end{aligned}$$

The group W can be identified with the semidirect product of S_n and the subgroup A of $\mathbb{Z}_2^n$ consisting of $\tau = (\tau(1), \tau(2), \ldots, \tau(n))$ such that $\sum \tau(i)$ is an even number. The subgroup A is normal in W. Then

$$\varphi_\lambda(\tau,\sigma) = 2\sum_{i=1}^{n-1}(-1)^{\tau(i)}(n-i)[n-\sigma(i)]$$

$$\psi_\lambda(\tau,\sigma) = \frac{n(n-1)}{2} - \ell(\tau,\sigma) - \frac{1}{2(n-1)}\sum_{i=1}^{n-1}(-1)^{\tau(i)}(n-i)[n-\sigma(i)].$$

References

1. Berg, C., Christensen, J. P.: Harmonic Analysis on Semigroups: Theory of Positive Definite Functions. Graduate Texts in Mathematics, **100**, Springer Verlag, New York, (1984)
2. Bourbaki, N.: Groupes et algèbres de Lie, Ch. 4–6. Hermann, Paris, (1968)
3. Bożejko, M.: Positive and negative kernels on discrete groups. Lectures at Heidelberg University, (1987)

4. Bożejko, M., Januszkiewicz, T., Spatzier, R. J.: Infinite Coxeter groups do not have Kazhdan's property. J. Operator Theory, **19**, 63–68 (1988)
5. Bożejko, M., Picardello, M. A.: Weakly amenable groups and amalgamated products. Proc. Amer. Math. Soc., **117**, 1039–1046 (1993)
6. Bożejko, M., Speicher, R.: An example of a generalized Brownian motion. Comm. Math. Phys., **137**, 519–531 (1991)
7. Bożejko, M., Speicher, R.: Completely positive maps on Coxeter groups, deformed commutation relations and operator spaces. Math. Ann., **300**, 97–120 (1994)
8. Bożejko, M., Speicher, R.: Interpolation between bosonic and fermionic relations given by generalized Brownian motions. Math. Z., **222**, 135–160 (1996)
9. Bożejko, M., Kümerer, B., Speicher, R.: q-Gaussian processes: nonclassical and classical aspects. Comm. Math. Phys. (to appear)
10. Carter, R.: Simple Groups of Lie Type. Wiley–Interscience, London (1972)
11. Coxeter, H. S. M.: Regular Complex Polytopes. Cambridge University Press, Cambridge, (1974)
12. Coxeter, H. S. M.: Discrete groups generated by reflections. Annals of Math., **35**, 588–621 (1934)
13. Coxeter, H. S. M., Moser, W. O.: Generators and Relations for Discrete Groups, 2nd ed. Springer–Verlag, New York, (1965)
14. de Canniere, J., Haagerup, U.: Multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups. Amer. Math. Soc., **107**, 455–500 (1983)
15. de la Harpe, P.: Groupes de Coxeter infinites non affines. Expo. Math., **5**, 91–96 (1987)
16. Dykema, K., Nica, A.: On the Fock representation of the q-commutation relations. J. Reine Angew. Math., **440**, 201–212 (1993)
17. Greenberg, O. W.: Particles with small violations of Fermi or Bose statistics. Phys. Rev. D, **43**, 4111–4120 (1991)
18. Grove, L. C., Benson, C. T.: Finite Reflection Groups, 2nd ed. Springer-Verlag, New York (1985)
19. Hiller, H.: Geometry of Coxeter Groups. Pitman, London, (1982)
20. Humphreys, J. E.: Reflection groups and Coxeter groups. Cambridge University Press, Cambridge, (1990)
21. Januszkiewicz, T.: For right angled Coxeter groups $z^{|g|}$ is a coefficient of uniformly bounded representation. Proc. Amer. Math. Soc., **119**, 1115–1119 (1993)
22. Jorgensen, P. E. T., Schmitt, L M., Werner, R. F.: q-canonical commutation relations and stability of the Cuntz algebra. Pacific Math. J., **165**, 131–151 (1994)
23. Pytlik, T., Szwarc, R.: An analytic series of uniformly bounded representations of free groups. Acta Math., **157**, 287–309 (1986)
24. Szwarc, R.: Groups acting on trees and approximation properties of the Fourier algebra, J. Funct. Anal., **95**, 320–343 (1991)
25. Szwarc, R.: Structure of geodesics in the Cayley graph of infinite Coxeter groups. Preprint (1996)
26. Tits, J.: Buildings of Spherical Type and Finite BN-pairs. Lecture Notes in Math. **386**, Springer-Verlag, (1974)
27. Valette, A.: Weak amenability of right-angled Coxeter groups. Proc. Amer. Math. Soc., **119**, 1331–1334 (1993)
28. Zagier, D.: Realizabilty of a model in infinite statistics. Comm. Math. Phys., **147**, 199–210 (1992)