Attraction principle for nonlinear integral operators of the Volterra type

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1. Introduction

We are studying the integral equation of the form

$$u(x) = \int_0^x a(x,y)\varphi\left(u(y)\right) dy. \tag{1}$$

All function appearing here are nonnegative and defined for $0 \le y \le x$. The Eq.(1) has the trivial solution $u(x) \equiv 0$. It can have also other solutions. We prove, using the method due to Okrasiński, that under certain conditions upon $a(x,y)$ and $\varphi(x)$ there can be at most one solution which does not vanish identically in a neighborhood of 0. Our main result is the attraction property of this nonnegative solution, provided that it exists. Namely we show that the iterations $T^n u$ of the operator

$$Tu(x) = \int_0^x a(x,y)\varphi\left(u(y)\right) dy$$

tend to the unique nonnegative solution for every function u, strictly positive in a neighborhood of 0.

A similar equation was studied in [3], under the conditions that $a(x,y)$ is invariant and $\varphi(x)$ is concave.

2. The results

We will deal with the integral operators T of the form

$$Tu(x) = \int_0^x a(x,y)\varphi(u(y))\,dy.$$

The functions u and φ are assumed to be nonnegative and strictly increasing on the half-axis $[0,+\infty)$ and $u(0)=0,\ \varphi(0)=0$. Let the kernel $a(x,y)$, $x>y$, be positive and satisfy the following conditions.

$$\begin{aligned} \frac{\partial a}{\partial x} &\geq 0 \\ \frac{\partial a}{\partial x} + \frac{\partial a}{\partial y} &\geq 0. \end{aligned} \tag{2}$$

We also assume that $a(x,x)=0$. If not specified otherwise all the functions we introduce are smooth on the open half-axis $(0,+\infty)$ and continuous on $[0,+\infty)$. The kernel $a(x,y)$ is to be smooth for $x>y$ and continuous for $x \geq y$. The task we are going to take up is to study the equation

$$Tu(x) = u(x),$$

where u is nonnegative, strictly increasing and $u(0)=0$. Obviously the conditions (2) imply that if $u(x)$ is strictly positive for $x>0$ and satisfies (1), then u is strictly increasing . Observe that the conditions (2) are equivalent to the following.

$$a(x,y) \geq a(s,t) \qquad \text{for} \quad \begin{array}{l} 0 \leq s \leq x,\ 0 \leq t \leq y, \\ y \leq x \text{ and } x-y > s-t. \end{array} \tag{3}$$

Lemma 1 *Let u and h be increasing functions on $[0,+\infty)$ such that $u(0) = h(0) = 0$. Assume also that $h(x)$ is a continuous and piecewise smooth function on $[0,+\infty)$. Put $\tilde{u}(x) = u(h(x))$.*

(i) If $Tu(x) \geq u(x)$ and $h'(x) \leq 1$, then $T\tilde{u}(x) \geq \tilde{u}(x)$.

(ii) If $Tu(x) \leq u(x)$ and $h'(x) \geq 1$, then $T\tilde{u}(x) \leq \tilde{u}(x)$.

Proof. We will only prove the first part of the lemma. The proof of the second part is similar. Observe that if $0<y<x$ then

$$a(h(x),h(y)) \leq a(x,y). \tag{4}$$

Indeed, since $h' \leq 1$ and $h(0) = 0$ we have $h(x) \leq x$, $h(y) \leq y$ and $h(x) - h(y) \leq x - y$ for $0 < y < x$. Applying (3) we get the inequality (4). Therefore

$$\begin{aligned} T\tilde{u}(x) &= \int_0^x a(x,y)\varphi\left(u(h(y))\right) dy \\ &\geq \int_0^x a(x,y)\varphi\left(u(h(y))\right)\, h'(y)dy \\ &= \int_0^{h(x)} a\left(x, h^{-1}(s)\right)\varphi\left(u(s)\right) ds \\ &\geq \int_0^{h(x)} a\left(h(x), s\right)\varphi\left(u(s)\right) ds \\ &= Tu\left(h(x)\right) \;\geq\; u\left(h(x)\right) \;=\; \tilde{u}(x). \end{aligned}$$

By applying Lemma 1 with

$$h(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq c \\ x - c & \text{if } c < x \end{cases}$$

we get the following.

Corollary 1 *Assume that u satisfies $Tu\,(x) \geq u(x)$. For a given $c > 0$ let*

$$u_c(x) = \begin{cases} 0 & \textit{if } 0 \leq x \leq c \\ u(x - c) & \textit{if } c < x \end{cases}$$

Then $Tu_c(x) \geq u_c(x)$.

Example Let $f(x)$ be an increasing function such that $f(0) = 0$. Then the invariant kernel

$$a(x, y) = f(x - y)$$

satisfies the conditions (1). Observe, that if $Tu = u$ then $Tu_c = u_c$ in this case.

Before stating the main result about attraction principle for the equation

$$Tu(x) = u(x) \tag{5}$$

we need some auxiliary lemmas.

Lemma 2 *Assume that the function $u(x)$ satisfies $Tu(x) \geq u(x)$ and let*

$$v(x) = \begin{cases} u(x) & \textit{if } 0 \leq x \leq c \\ u(c) & \textit{if } c < x \end{cases}$$

Then there exists $\varepsilon > 0$ such that

$$\liminf_{n\to\infty} T^n v\,(x) \geq u(x),$$

for $c < x < c + \varepsilon$.

Proof. Assume that $\varepsilon < 1$. Let

$$\begin{aligned} c_a &= \sup_{y\le x\le c+1} a(x,y), \\ c_\varphi &= \sup_{u(c)\le x\le u(c+1)} \varphi'(x), \\ c_u &= \sup_{c\le x\le c+1} u'(x). \end{aligned}$$

Then for $c < x < c+1$ we have

$$\begin{aligned} u(x) - Tv\,(x) &\le Tu\,(x) - Tv\,(x) \\ &= \int_0^x a(x,y)\,[\varphi(u(y)) - \varphi(v(y))]\,dy \\ &= \int_c^x a(x,y)\,[\varphi(u(y)) - \varphi(u(c))]\,dy \\ &\le c_a c_\varphi [u(x) - u(c)](x-c) \\ &\le c_a c_\varphi c_u (x-c)^2. \end{aligned}$$

Similarly we get

$$\begin{aligned} u(x) - T^n v\,(x) &\le T^n u\,(x) - T^n v\,(x) \\ &= \int_c^x a(x,y)\left[\varphi(u(y)) - \varphi(T^{n-1}v(y))\right] dy \\ &\le c_a c_\varphi (x-c) \sup_{c<y<c+1} \left[\varphi(u(y)) - \varphi(T^{n-1}v(y))\right]. \end{aligned}$$

Thus by induction we can prove that

$$u(x) - T^n v(x) \le c_u (c_a c_\varphi)^n (x-c)^{n+1}.$$

This implies

$$\liminf_{n\to\infty} T^n v\,(x) \ge u(x),$$

if $x - c < c_a^{-1} c_\varphi^{-1}$ and $x - c < 1$.

Lemma 3 *Assume that* $Tu\,(x) = u(x)$ *and let*

$$v(x) = \begin{cases} u(x) & \text{if } 0 \le x \le c \\ u(c) & \text{if } c < x \end{cases}$$

Then there is $\varepsilon > 0$ *such that*

$$\lim_{n\to\infty} T^n v\,(x) = u(x),$$

for $c < x < c + \varepsilon$.

Proof. From the preceding lemma we have that $\liminf_{n\to\infty} T^n v\,(x) \geq u(x)$, for $c < x < c + \varepsilon$. for some $\varepsilon > 0$. On the other hand

$$\limsup_{n\to\infty} T^n v\,(x) \leq u(x).$$

This is because $u(x) \geq v(x)$ and T is monotonic.

The idea of the proof of the next proposition is due to Okrasiński.

Proposition 1 *The equation (1) can have at most one positive solution .*

Proof. Suppose $u(x)$ and $v(x)$ are two different positive solution of (1). Without loss of generality we may assume that $u \not\leq v$. Then there is $c > 0$ such that $u(x - d) > v(x)$ for some $x > 0$. If not, then we would have $u(x-d) \leq v(x)$ for every x and d, which would imply $u \leq v$. Thus let $u(x - d) > v(x)$. This can be written as $u_d(x) > v(x)$. Let c be the lower bound of the numbers x for which $u_d(x) > v(x)$. Thus $u_d(x) \leq v(x)$ for $0 \leq x \leq c$. Define the function $\tilde{u}(x)$ as follows.

$$\tilde{u}(x) = \begin{cases} u_d(x) & \text{if } 0 \leq x \leq c \\ u_d(c) & \text{if } c < x \end{cases}$$

By Corollary 1 we have $Tu_d(x) \geq u_d(x)$. Moreover $\tilde{u}(x) \leq v(x)$. Therefore

$$\limsup_{n\to\infty} T^n \tilde{u}(x) \leq v(x).$$

On the other hand by Lemma 2

$$\liminf_{n\to\infty} T^n \tilde{u}(x) \geq u_d(x).$$

for $c < x < c + \varepsilon$. This implies that $u_d(x) \leq v(x)$ for $c < x < c + \varepsilon$. The latter contradicts the choice of c.

We are now ready to prove the attraction principle for the equation (1).

Theorem 1 *Let $u(x)$ be a positive solution of (1) and let $a(x, y)$ satisfy (2). Assume $v(x),\ x > 0$ is a positive function satisfying $v(0) = 0$. Then*

$$\lim_{n\to\infty} T^n v\,(x) = u(x),$$

for $x \geq 0$. The convergence is uniform on every bounded interval.

Proof. Suppose first that

$$Tv\,(x) \geq v(x)$$

and $0 \leq v(x) \leq u(x)$. Then the sequence of functions $\{T^n v\,(x)\}$ is increasing and bounded by $u(x)$. Thus the limit

$$\tilde{u}(x) = \lim_{n\to\infty} T^n v\,(x)$$

defines the solution $\tilde{u}(x)$ of (1). By Proposition 1 we have $\tilde{u}(x) = u(x)$. This proves the theorem in the case when $Tv \geq v$.

A similar reasoning shows that if

$$Tv\,(x) \leq v(x)$$

and $0 \leq u(x) \leq v(x)$, then

$$\lim_{n\to\infty} T^n v\,(x) = u(x),$$

for $x \geq 0$.

We will complete the proof by showing that there exist increasing positive functions w_1 and w_2 such that

$$w_1(x) \leq v(x) \leq w_2(x), \quad w_1(x) \leq u(x) \leq w_2(x),$$

and

$$Tw_1(x) \geq w_1(x), \quad Tw_2(x) \leq w_2(x).$$

We can assume that $v(x)$ is a strictly increasing function. If not, then $Tv\,(x)$ is so. Obviously the solution $u(x)$ is strictly increasing. Introduce the increasing function $w_1(x)$ by

$$w_1^{-1}(x) = v^{-1}(x) + u^{-1}(x).$$

Then

$$0 \leq w_1(x) \leq v(x) \text{ and } w_1(x) \leq u(x).$$

Since the functions $u^{-1},\ v^{-1},\ w_1^{-1}$ are increasing

$$\left(w_1^{-1}\right)' \geq \left(u^{-1}\right)'. \tag{6}$$

Write w_1 in the form $w_1(x) = u\left(h_1(x)\right)$. Then $h_1(x) = u^{-1}(w_1(x))$ and by (6)

$$h_1'(x) = \left(u^{-1}\right)'(w_1(x))\, w_1'(x) \leq 1.$$

By Lemma 1 we then have

$$Tw_1\,(x) \geq w_1(x).$$

Define the function $w_2(x)$ as

$$w_2^{-1}(x) = \int_0^x \min\left\{\left(u^{-1}\right)'(y),\ \left(v^{-1}\right)'(y)\right\} dy.$$

Then

$$w_2^{-1}(x) \leq \int_0^x \left(v^{-1}\right)'(y)\, dy = v^{-1}(x),$$

$$w_2^{-1}(x) \le \int_0^x \left(u^{-1}\right)'(y)\, dy = u^{-1}(x),$$

Thus $w_2(x) \ge \max\{u(x), v(x)\}$. Moreover,

$$\left(w_2^{-1}\right)' \le \left(u^{-1}\right)'. \tag{7}$$

Thus w_2 can be written as $w_2(x) = u\left((h_2(x)\right)$, where $h_2(x) = u^{-1}\left(w_2(x)\right)$. By (7) we have

$$\left(h_2\right)'(x) = \left(u^{-1}\right)'\left(w_2(x)\right)\ w_2'(x)\ \ge\ 1.$$

Again by Lemma 1

$$Tw_2(x) \ge w_2(x).$$

Summarizing we proved that there are w_1 and w_2 such that

$$w_1(x) \le v(x) \le w_2(x),$$

$$\lim_{n\to\infty} T^n w_i(x) = u(x), \quad i = 1,\, 2.$$

Thus

$$\lim_{n\to\infty} T^n v\,(x) = u(x).$$

Furthermore, the sequences $T^n w_1$ and $T^n w_2$ are increasing and decreasing respectively. Hence by Dini's theorem both converge to $u(x)$ uniformly on bounded intervals. So does $T_n v$ as

$$T^n w_1\,(x) \le T^n v\,(x) \le T^n w_2\,(x).$$

This completes the proof.
Remark. By Theorem 1 we can get an estimate for the nonzero solution $u(x)$, if it exists. Assume that the function $v(x)$ satisfies

$$Tv(x) \le v(x), \text{ for } 0 \le x \le c.$$

Then

$$u(x) \le v(x) \text{ for } 0 \le x \le c.$$

In particular we have the following.

Corollary 2 *Let $\{v_n(x)\}_{n=1}^{\infty}$ be a sequence of positive increasing functions such that*

$$\lim_{n\to\infty} v_n(x) = 0, \textit{ for } x \ge 0.$$

and

$$Tv_n\,(x) \le v_n(x), \textit{ for } x \ge 0.$$

Then the equation $Tu(x) = u(x)$ has no positive solutions.

In a forthcoming paper we will use Corollary 2 to prove that if $\varphi(x) = \sqrt{x}$ and $a(x, y) = f(x - y)$ is an invariant kernel given by the function

$$f(x) = e^{-e^{1/x}},$$

then the equation (2) admits no nonzero solutions.

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