Attraction principle for nonlinear integral operators of the Volterra type

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1. Introduction

We are studying the integral equation of the form

\[ u(x) = \int_0^x a(x, y) \varphi(u(y)) \, dy. \] (1)

All function appearing here are nonnegative and defined for \(0 \leq y \leq x\). The Eq.(1) has the trivial solution \(u(x) \equiv 0\). It can have also other solutions. We prove, using the method due to Okrasiński, that under certain conditions upon \(a(x, y)\) and \(\varphi(x)\) there can be at most one solution which does not vanish identically in a neighborhood of 0. Our main result is the attraction property of this nonnegative solution, provided that it exists. Namely we show that the iterations \(T^n u\) of the operator

\[ Tu(x) = \int_0^x a(x, y) \varphi(u(y)) \, dy \]

tend to the unique nonnegative solution for every function \(u\), strictly positive in a neighborhood of 0.
A similar equation was studied in [3], under the conditions that \( a(x, y) \) is invariant and \( \varphi(x) \) is concave.

2. The results

We will deal with the integral operators \( T \) of the form

\[
Tu(x) = \int_0^x a(x, y) \varphi(u(y)) \, dy.
\]

The functions \( u \) and \( \varphi \) are assumed to be nonnegative and strictly increasing on the half-axis \([0, +\infty)\) and \( u(0) = 0, \varphi(0) = 0 \). Let the kernel \( a(x, y), x > y, \) be positive and satisfy the following conditions.

\[
\begin{align*}
\frac{\partial a}{\partial x} &\geq 0 \\
\frac{\partial a}{\partial x} + \frac{\partial a}{\partial y} &\geq 0.
\end{align*}
\]

We also assume that \( a(x, x) = 0 \). If not specified otherwise all the functions we introduce are smooth on the open half-axis \((0, +\infty)\) and continuous on \([0, +\infty)\).

The kernel \( a(x, y) \) is to be smooth for \( x > y \) and continuous for \( x \geq y \). The task we are going to take up is to study the equation

\[
Tu(x) = u(x),
\]

where \( u \) is nonnegative, strictly increasing and \( u(0) = 0 \). Obviously the conditions (2) imply that if \( u(x) \) is strictly positive for \( x > 0 \) and satisfies (1), then \( u \) is strictly increasing. Observe that the conditions (2) are equivalent to the following.

\[
a(x, y) \geq a(s, t) \quad \text{for} \quad 0 \leq s \leq x, \ 0 \leq t \leq y,
\]

\[
y \leq x \text{ and } x - y > s - t.
\]

**Lemma 1** Let \( u \) and \( h \) be increasing functions on \([0, +\infty)\) such that \( u(0) = h(0) = 0 \). Assume also that \( h(x) \) is a continuous and piecewise smooth function on \([0, +\infty)\). Put \( \tilde{u}(x) = u(h(x)) \).

(i) If \( Tu(x) \geq u(x) \) and \( h'(x) \leq 1 \), then \( T\tilde{u}(x) \geq \tilde{u}(x) \).

(ii) If \( Tu(x) \leq u(x) \) and \( h'(x) \geq 1 \), then \( T\tilde{u}(x) \leq \tilde{u}(x) \).

**Proof.** We will only prove the first part of the lemma. The proof of the second part is similar. Observe that if \( 0 < y < x \) then

\[
a(h(x), h(y)) \leq a(x, y).
\]

(4)
Indeed, since $h' \leq 1$ and $h(0) = 0$ we have $h(x) \leq x$, $h(y) \leq y$ and $h(x) - h(y) \leq x - y$ for $0 < y < x$. Applying (3) we get the inequality (4). Therefore

$$T\tilde{u}(x) = \int_0^x a(x, y)\varphi(u(h(y))) \, dy$$

$$\geq \int_0^{h(x)} a(x, y)\varphi(u(h(y))) \, h'(y) \, dy$$

$$= \int_0^{h(x)} a\left(x, h^{-1}(s)\right) \varphi(u(s)) \, ds$$

$$\geq \int_0^{h(x)} a(h(x), s) \varphi(u(s)) \, ds$$

$$= Tu(h(x)) \geq u(h(x)) = \tilde{u}(x).$$

By applying Lemma 1 with

$$h(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq c \\ x - c & \text{if } c < x \end{cases}$$

we get the following.

**Corollary 1** Assume that $u$ satisfies $Tu(x) \geq u(x)$. For a given $c > 0$ let

$$u_c(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq c \\ u(x - c) & \text{if } c < x \end{cases}$$

Then $Tu_c(x) \geq u_c(x)$.

**Example** Let $f(x)$ be an increasing function such that $f(0) = 0$. Then the invariant kernel

$$a(x, y) = f(x - y)$$

satisfies the conditions (1). Observe, that if $Tu = u$ then $Tu_c = u_c$ in this case.

Before stating the main result about attraction principle for the equation

$$Tu(x) = u(x) \quad (5)$$

we need some auxiliary lemmas.

**Lemma 2** Assume that the function $u(x)$ satisfies $Tu(x) \geq u(x)$ and let

$$v(x) = \begin{cases} u(x) & \text{if } 0 \leq x \leq c \\ u(c) & \text{if } c < x \end{cases}$$

Then there exists $\varepsilon > 0$ such that

$$\liminf_{n \to \infty} T^n v(x) \geq u(x),$$

for $c < x < c + \varepsilon$. 

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Proof. Assume that \( \varepsilon < 1 \).

Let

\[
c_a = \sup_{y \leq x \leq c+1} a(x, y),
\]

\[
c_\varphi = \sup_{u(c) \leq x \leq u(c+1)} \varphi'(x),
\]

\[
c_u = \sup_{c \leq x \leq c+1} u'(x).
\]

Then for \( c < x < c+1 \) we have

\[
u(x) - Tv(x) \leq Tu(x) - Tv(x)
\]

\[
= \int_0^x a(x, y) [\varphi(u(y)) - \varphi(v(y))] \, dy
\]

\[
= \int_c^x a(x, y) [\varphi(u(y)) - \varphi(u(c))] \, dy
\]

\[
\leq c_a c_\varphi [u(x) - u(c)](x - c)
\]

\[
\leq c_a c_\varphi c_u (x - c)^2.
\]

Similarly we get

\[
u(x) - T^n v(x) \leq T^n u(x) - T^n v(x)
\]

\[
= \int_c^x a(x, y) [\varphi(u(y)) - \varphi(T^{n-1}v(y))] \, dy
\]

\[
\leq c_a c_\varphi (x - c) \sup_{c \leq y \leq c+1} [\varphi(u(y)) - \varphi(T^{n-1}v(y))].
\]

Thus by induction we can prove that

\[
u(x) - T^n v(x) \leq c_n (c_a c_\varphi)^n (x - c)^{n+1}.
\]

This implies

\[
\lim_{n \to \infty} T^n v(x) \geq u(x),
\]

if \( x - c < c_a c_\varphi^{-1} \) and \( x - c < 1 \).

**Lemma 3** Assume that \( Tu(x) = u(x) \) and let

\[
v(x) = \begin{cases} 
  u(x) & \text{if } 0 \leq x \leq c \\
  u(c) & \text{if } c < x
\end{cases}
\]

Then there is \( \varepsilon > 0 \) such that

\[
\lim_{n \to \infty} T^n v(x) = u(x),
\]

for \( c < x < c + \varepsilon \).
Proof. From the preceding lemma we have that \( \lim \inf_{n \to \infty} T^n v(x) \geq u(x) \), for \( c < x < c + \varepsilon \), for some \( \varepsilon > 0 \). On the other hand

\[
\lim \sup_{n \to \infty} T^n v(x) \leq u(x).
\]

This is because \( u(x) \geq v(x) \) and \( T \) is monotonic.

The idea of the proof of the next proposition is due to Okrasiński.

**Proposition 1** The equation (1) can have at most one positive solution .

Proof. Suppose \( u(x) \) and \( v(x) \) are two different positive solution of (1). Without loss of generality we may assume that \( u \not\leq v \). Then there is \( c > 0 \) such that \( u(x - d) > v(x) \) for some \( x > 0 \). If not, then we would have \( u(x - d) \leq v(x) \) for every \( x \) and \( d \), which would imply \( u \leq v \). Thus let \( u(x - d) > v(x) \). This can be written as \( u(x - d) > v(x) \). Let \( c \) be the lower bound of the numbers \( x \) for which \( u(x - d) > v(x) \). Define the function \( \tilde{u}(x) \) as follows.

\[
\tilde{u}(x) = \begin{cases} 
  u(x) & \text{if } 0 \leq x \leq c \\
  u(c) & \text{if } c < x
\end{cases}
\]

By Corollary 1 we have \( Tu_d(x) \geq u_d(x) \). Moreover \( \tilde{u}(x) \leq v(x) \). Therefore

\[
\lim \sup_{n \to \infty} T^n \tilde{u}(x) \leq v(x).
\]

On the other hand by Lemma 2

\[
\lim \inf_{n \to \infty} T^n \tilde{u}(x) \geq u_d(x).
\]

for \( c < x < c + \varepsilon \). This implies that \( u_d(x) \leq v(x) \) for \( c < x < c + \varepsilon \). The latter contradicts the choice of \( c \).

We are now ready to prove the attraction principle for the equation (1).

**Theorem 1** Let \( u(x) \) be a positive solution of (1) and let \( a(x, y) \) satisfy (2). Assume \( v(x), x > 0 \) is a positive function satisfying \( v(0) = 0 \). Then

\[
\lim_{n \to \infty} T^n v(x) = u(x),
\]

for \( x \geq 0 \). The convergence is uniform on every bounded interval.

Proof. Suppose first that

\[
Tv(x) \geq v(x)
\]

and \( 0 \leq v(x) \leq u(x) \). Then the sequence of functions \( \{T^n v(x)\} \) is increasing and bounded by \( u(x) \). Thus the limit

\[
\tilde{u}(x) = \lim_{n \to \infty} T^n v(x)
\]
defines the solution $\tilde{u}(x)$ of (1). By Proposition 1 we have $\tilde{u}(x) = u(x)$. This proves the theorem in the case when $Tv \geq v$.

A similar reasoning shows that if

$$Tv(x) \leq v(x)$$

and $0 \leq u(x) \leq v(x)$, then

$$\lim_{n \to \infty} T^n v(x) = u(x),$$

for $x \geq 0$.

We will complete the proof by showing that there exist increasing positive functions $w_1$ and $w_2$ such that

$$w_1(x) \leq v(x) \leq w_2(x), \quad w_1(x) \leq u(x) \leq w_2(x),$$

and

$$Tw_1(x) \geq w_1(x), \quad Tw_2(x) \leq w_2(x).$$

We can assume that $v(x)$ is a strictly increasing function. If not, then $Tv(x)$ is so. Obviously the solution $u(x)$ is strictly increasing. Introduce the increasing function $w_1(x)$ by

$$w_1^{-1}(x) = v^{-1}(x) + u^{-1}(x).$$

Then

$$0 \leq w_1(x) \leq v(x) \text{ and } w_1(x) \leq u(x).$$

Since the functions $u^{-1}$, $v^{-1}$, $w^{-1}_1$ are increasing

$$\left(\frac{1}{w_1^{-1}}\right)' \geq (u^{-1})', \quad (v^{-1})' \geq \left(\frac{1}{w_1^{-1}}\right)'.$$  \hfill (6)

Write $w_1$ in the form $w_1(x) = u(h_1(x))$. Then $h_1(x) = u^{-1}(w_1(x))$ and by (6)

$$h_1'(x) = (u^{-1})'(w_1(x)) w_1'(x) \leq 1.$$

By Lemma 1 we then have

$$Tw_1(x) \geq w_1(x).$$

Define the function $w_2(x)$ as

$$w_2^{-1}(x) = \int_0^x \min \left\{(u^{-1})'(y), (v^{-1})'(y)\right\} \, dy.$$

Then

$$w_2^{-1}(x) \leq \int_0^x (v^{-1})'(y) \, dy = v^{-1}(x),$$

and
\[ w_2^{-1}(x) \leq \int_0^x (u^{-1})'(y) \, dy = u^{-1}(x), \]

Thus \( w_2(x) \geq \max\{u(x), v(x)\} \). Moreover,

\[ (w_2^{-1})' \leq (u^{-1})'. \tag{7} \]

Thus \( w_2 \) can be written as \( w_2(x) = u(h_2(x)) \), where \( h_2(x) = u^{-1}(w_2(x)) \). By (7) we have

\[ (h_2)'(x) = (u^{-1})'(w_2(x)) \quad w_2'(x) \geq 1. \]

Again by Lemma 1

\[ Tw_2(x) \geq w_2(x). \]

Summarizing we proved that there are \( w_1 \) and \( w_2 \) such that

\[ w_1(x) \leq v(x) \leq w_2(x), \]

\[ \lim_{n \to \infty} T^nw_i(x) = u(x), \quad i = 1, 2. \]

Thus

\[ \lim_{n \to \infty} T^nv(x) = u(x). \]

Furthermore, the sequences \( T^nw_1 \) and \( T^nw_2 \) are increasing and decreasing respectively. Hence by Dini’s theorem both converge to \( u(x) \) uniformly on bounded intervals. So does \( T^n v \) as

\[ T^nw_1(x) \leq T^n v(x) \leq T^nw_2(x). \]

This completes the proof.

**Remark.** By Theorem 1 we can get an estimate for the nonzero solution \( u(x) \), if it exists. Assume that the function \( v(x) \) satisfies

\[ Tv(x) \leq v(x), \quad 0 \leq x \leq c. \]

Then

\[ u(x) \leq v(x) \quad 0 \leq x \leq c. \]

In particular we have the following.

**Corollary 2** Let \( \{v_n(x)\}_{n=1}^{\infty} \) be a sequence of positive increasing functions such that

\[ \lim_{n \to \infty} v_n(x) = 0, \quad \text{for } x \geq 0. \]

and

\[ T v_n(x) \leq v_n(x), \quad \text{for } x \geq 0. \]

Then the equation \( Tu(x) = u(x) \) has no positive solutions.

In a forthcoming paper we will use Corollary 2 to prove that if \( \varphi(x) = \sqrt{x} \) and \( a(x, y) = f(x - y) \) is an invariant kernel given by the function

\[ f(x) = e^{-x^{1/\alpha}}, \]

then the equation (2) admits no nonzero solutions.
References


